Problem Sheet 10

(Due: January 9, 2013)

Problem 10.1:
In the complex plane (in radial quantization), the Virasoro generators L_n are given by

$$L_n = \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z) \,.$$

(a) Show that

$$[L_m, L_n] = \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} z^{m+1} w^{n+1} T(z) T(w) \,, \qquad (1)$$

where C_0 denotes a contour about $w = 0$ and C_w is a contour about $z = w$. As usual the product $T(z)T(w)$ is meant to be the radially ordered product.

Hint: Write the commutator as a difference of two double contour integrals and use a contour deformation of the dz integration for fixed w.

(b) Use (1) and the (radially ordered) operator product

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \dots \,,$$

as well as the Cauchy-Riemann formula,

$$\oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \frac{f(z)}{(z-w)^n} = \frac{1}{(n-1)!} f^{(n-1)}(w) \,,$$

to rederive the quantum Virasoro algebra:

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}.$$

(c) The Schwarzian derivative is defined as

$$S(\tilde{z}, z) = \frac{\partial^3 \tilde{z}}{\partial z^3} \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2} \left(\frac{\partial^2 \tilde{z}}{\partial z^2}\right)^2 \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} .$$

Show that the transformation

$$z \to \tilde{z} \,,$$
$$T(z) \to \tilde{T}(\tilde{z}) = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \left[T(z) - \frac{c}{12} S(\tilde{z}, z)\right]$$

gives at the infinitesimal level for $\tilde{z} = z + \epsilon(z)$

$$\delta T(z) = -\epsilon(z) \partial T(z) - 2(\partial_z \epsilon(z)) T(z) - \frac{c}{12} \partial_z^3 \epsilon(z) \,.$$

(d) Apply this to the map from the cylinder (parameterized by $w = \sigma + i\tau$) to the plane, given by $z = e^{-iw}$ to show that

$$T_{\mathrm{cyl}}(w) = -z^2 T_{\mathrm{plane}}(z) + \frac{c}{24} \,.$$

Problem 10.2:

(a) Compute the TT-OPE for the free boson and show that in this case $c = 1$.

(b) In the complex plane the α_n are given by the contour integral

$$\alpha_n = i\sqrt{\frac{2}{\alpha'}}\oint \frac{\mathrm{d}z}{2\pi i} z^n \partial X(z).$$

Use the $\partial X \partial X$-OPE to compute the commutator $[\alpha_m, \alpha_n]$, following the same logic as in problem 10.1.

(c) Compute the OPE of ∂X with $:e^{ikX}:$. Use the result to compute the eigenvalue of α_0 acting on the eigenstate $|0,k\rangle \mathrel{\widehat{=}} :e^{ikX(0)}:$.

(b) Compute the OPE of $:e^{ikX}:$ with T and $\overline{T}$ to show that $:e^{ikX}:$ is a primary with weight $h = \overline{h} = \frac{\alpha' k^2}{4}$.

Merry Christmas and a Happy New Year!