Problem Sheet 11

(Due: January 16, 2013)

Problem 11.1:
Our aim is to prove the formula

$$S_{S^2}(k_1,\dots,k_n) = \langle \prod_{i=1}^{n} :e^{ik_i\cdot X(z_i,\bar z_i)}:\rangle = \text{const}\times \delta^{(D)}\Big(\sum_{i=1}^{n} k_i\Big)\prod_{i<j}|z_i-z_j|^{\alpha' k_i\cdot k_j} \tag{1}$$

for the X-part of the tree-level S-matrix describing the scattering of n closed string tachyons propagating in D dimensions.

We do so by evaluating more generally the functional integral

$$Z_{S^2}[J] = \int \mathcal{D}X^\mu \exp\Big(\int d^2z\,\Big(\frac{1}{2\pi\alpha'}\,X^\mu\partial\bar\partial X_\mu\Big) + iJ^\mu X_\mu\Big), \tag{2}$$

where the fields $X^\mu(z,\bar z)$ couple to the source $J^\mu(z,\bar z)$.

(a) Starting from the path integral of the X-CFT, give the form of $J^\mu(z,\bar z)$ relevant to the S-matrix (1) and convince yourself that indeed the problem reduces to the one defined in (2). Important: At the moment ignore the normal ordering prescription in (1).

(b) This Gaussian integral is best performed by expanding the fields into eigenmodes of the operator $\partial\bar\partial$:

$$\partial\bar\partial X_I(z,\bar z) = -\omega_I^2 X_I(z,\bar z).$$

These form a complete set:

$$X^\mu(z,\bar z) = \sum_I x_I^\mu\, X_I(z,\bar z), \qquad x_I^\mu = \int d^2z X^\mu(z,\bar z)X_I(z,\bar z),$$

$$\int d^2z\, X_I(z,\bar z)X_{I'}(z,\bar z) = \delta_{II'}, \qquad \sum_I X_I(z,\bar z)X_I(z',\bar z') = \delta^{(2)}(z-z',\bar z-\bar z').$$

Use this to arrive at

$$Z_{S^2}[J] = \prod_{I,\mu}\int dx_I^\mu \exp\Big(-\frac{\omega_I^2 x_I^\mu x_{\mu\,I}}{2\pi\alpha'} + ix_I^\mu J_{I\mu}\Big), \quad J_I^\mu = \int d^2z\, J^\mu(z,\bar z)X_I(z,\bar z).$$

(c) By considering the zero mode $I=0$ with $\omega_0=0$ and the non-zero modes $I\neq 0$ separately argue that

$$Z_{S^2}[J] = i(2\pi)^D\,\delta^{(D)}(J_0)\prod_{I\neq 0}\Big(\frac{2\pi^2\alpha'}{\omega_I^2}\Big)^{D/2}\exp\Big(-\frac{\pi\alpha' J_I^\mu J_{I\,\mu}}{2\omega_I^2}\Big). \tag{3}$$

(d) This can be rewritten as

$$Z_{S^2}[J] = i(2\pi)^D\,\delta^{(D)}(J_0)\,\det{}'\Big(-\frac{\partial\bar\partial}{2\pi^2\alpha'}\Big)^{-D/2}\times \exp\Big(-\frac{1}{2}\int d^2z d^2z' J^\mu(z,\bar z)J_\mu(z',\bar z')G'(z,\bar z,z',\bar z')\Big) \tag{4}$$

where the the notation det′ indicates that we omit zero modes in the functional determinant. Determine the Green's function $G'(z,\overline{z},z',\overline{z}')$ from (3) and (4) and argue that it satisfies

$$-\frac{1}{\pi\alpha'}\partial\overline{\partial}G'(z,\overline{z},z',\overline{z}') = \sum_{I\neq 0} X_I(z,\overline{z})X_I(z',\overline{z}') = \delta^{(2)}(z-z',\overline{z}-\overline{z}') - X_0^2 \tag{5}$$

Recall that, until now, we have not accounted for the fact that we forgot about the normal ordering prescription in the original expression (1) so far. Implementing this is tricky, see e.g. Polchinski Vol. 1, page 170ff. Without proof we state that one gets the correct result by using the full Green's function

$$G(z,\overline{z},z'\overline{z}') = -\frac{\alpha'}{2}\log|z-z'|^2$$

and omitting self-contractions (i.e. in the exponent of (4) one omits terms with $k_i^\mu k_{\mu,i}$).

(e) Now we return to the amplitude (1). Deduce the result (1) from the above considerations.

Problem 11.2:
$PSL(2,\mathbb{C})$ invariance of the vacuum of a 2d CFT implies that correlation functions of quasi-primary fields must satisfy

$$\langle\phi_1(z_1)\ldots\phi_n(z_n)\rangle = \langle\phi_1'(z_1)\ldots\phi_n'(z_n)\rangle,$$

under conformal diffeomorphisms, where we restrict for brevity to chiral fields (i.e. fields which depend on z_i only).

(a) For the one-point function of a chiral quasi-primary of conformal dimension h this implies

$$\langle\phi(z)\rangle = \langle\phi'(z)\rangle, \qquad \phi'(z') = \Big(\frac{\partial z'}{\partial z}\Big)^{-h}\phi\Big(z(z')\Big).$$

Exploit this for translations $z' = z + a$ to show that $\langle\phi(z)\rangle = C_\phi$ for some constant C_ϕ. Then use dilatations $z' = \lambda z$ to show that $C_\phi = 0$ unless $h = 0$.

(b) Now consider the 2-point function

$$\langle\phi_1(z)\phi_2(w)\rangle$$

for two quasi-primaries of weight h_1 and h_2. Use invariance under translations to show that

$$\langle\phi_1(z)\phi_2(w)\rangle = g(z-w)$$

for some function g and invariance under dilatations to show that

$$g(z-w) = \frac{d_{12}}{(z-w)^{h_1+h_2}}$$

for some constant $d_{12} \equiv d_{\phi_1\phi_2}$. Finally use invariance under the special conformal transformation $z' = -\frac{1}{z}$ to argue that $d_{12} = 0$ unless $h_1 = h_2$, thereby proving that

$$\langle\phi_1(z)\phi_2(w)\rangle = \delta_{h_1h_2}\frac{d_{12}}{(z-w)^{2h_1}}.$$

(c) Now we turn to the 3-point function. Show that the ansatz

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{(z_1-z_2)^a(z_2-z_3)^b(z_1-z_3)^c}, \quad a+b+c = \sum_i h_i \tag{6}$$

(for some constant $C_{123} \equiv C_{\phi_1\phi_2\phi_3}$) has the correct transformation properties under translations and dilatations. Investigate the behavior under special conformal transformations in order to determine a, b, and c in terms of the h_i. Argue that, in this way, the 3-point function is uniquely determined.

Note: This shows that global conformal invariance completely specifies the functional form of the 1-, 2- and 3-point correlators for quasi-primaries, a result that continues to hold in higher dimensions. The higher n-point functions are constrained to be of the form

$$\langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = \prod_{i<j} (z_i - z_j)^{-\nu_{ij}} F(0, \langle z_2, z_1, z_{n-1}, z_n \rangle, \dots 1, \infty)$$

where $\sum_{j \neq i} \nu_{ij} = 2h_i$ and the function F depends on the $n-3$ independent cross-ratios

$$\langle z_j, z_1, z_{n-1}, z_n \rangle \equiv \frac{(z_j - z_1)(z_{n-1} - z_n)}{(z_j - z_n)(z_{n-1} - z_1)}, \quad j = 2, \dots n-2.$$

A new feature appears in 2D CFTs: The specific form of the OPE can be used to systematically reduce all higher n-point-functions to expressions that depend only on c, h_i and C_{ijk}. In that sense the theory is in principle completely solved once these data are specified.

Problem 11.3:
Show that

$$C(a,b,c) = \int \mathrm{d}^2 z \; |z|^{2a-2} |1-z|^{2b-2} = \frac{2\pi \Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)}$$

where $a + b + c = 1$.

Hints: Use the trick $|z|^{2a-2} = \frac{1}{\Gamma(1-a)} \int_0^\infty \mathrm{d}t \; t^{-a} e^{-|z|^2 t}$. Decompose the complex coordinate via $z = x + iy$. The resulting measure for the integral is $\mathrm{d}^2 z = 2\mathrm{d}x\mathrm{d}y$. Recall the definition of the Beta-function $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ which has the representation $B(a,b) = \int_0^1 \mathrm{d}t \; t^{a-1}(1-t)^{b-1}$.