

## PROBLEM SHEET 12

(Due: January 23, 2013)

**Problem 12.1:**

Consider a torus  $T^2$  (which is topologically the same as  $S^1 \times S^1$ ), parametrised by

$$(\sigma^1, \sigma^2) \simeq (\sigma^1, \sigma^2) + 2\pi(m, n) \quad \text{for } m, n \in \mathbb{Z}. \quad (1)$$

- (a) Convince yourself that, up to an overall rescaling, the most general metric is given by

$$ds^2 = |d\sigma^1 + \tau d\sigma^2|^2, \quad \text{for } \tau \in \mathbb{C}. \quad (2)$$

- (b) Argue that Weyl transformations and diffeomorphisms can be used to define new coordinates  $\xi^i$  in terms of which the metric takes the standard form  $ds^2 = |d\xi^1 + id\xi^2|^2$ .

*Hint:* Use that  $\chi = \frac{1}{4\pi} \int_{T^2} d^2\xi \sqrt{h} R = 2 - 2g = 0$ .

Argue that the new periodicities are given by

$$\xi^1 \simeq \xi^1 + 2\pi(m + nv^1), \quad \xi^2 \simeq \xi^2 + 2\pi(nv^2),$$

for some vector  $(v^1, v^2)$ .

- (c) Now define the complex variable  $w = \sigma^1 + \tau\sigma^2$  such that in the first picture  $ds^2 = dw d\bar{w}$ . Convince yourself that the identification (1) implies the identification

$$w \simeq w + 2\pi(m + n\tau), \quad (3)$$

i.e. a  $T^2$  is a lattice in  $\mathbb{C}$ .

*Remark:* The upshot is the following: We can parametrize the torus in two equivalent ways: In the first picture (1), the coordinates  $\sigma^1, \sigma^2$  have standard periodicity  $2\pi$ , but there appears a modulus  $\tau$  - the so-called Teichmüller parameter - in the metric (2). Alternatively, we can choose coordinates  $w, \bar{w}$  such that the metric has the standard flat form  $ds^2 = dw d\bar{w}$ , but now in general the periodicity (3) contains a modulus  $\tau$  that cannot be removed by conformal transformations.

- (d) Consider now the transformations

$$T : \tau \rightarrow \tau + 1 \quad S : \tau \rightarrow -\frac{1}{\tau}, \quad U : \tau \rightarrow \frac{\tau}{\tau + 1}.$$

Determine their action on the parameters  $(m, n)$  in (3) and use this to argue that  $T, S$  and  $U$  leave the torus invariant. Argue that  $T, S$  and  $U$  generate the group  $SL(2, \mathbb{Z})$ , whose action on  $\tau$  is

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}.$$

Argue that the full modular group of  $T^2$  is  $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$ .

*Note:* In fact,  $S$  and  $T$  suffice to generate  $SL(2, \mathbb{Z})/\mathbb{Z}$ , but this is non-trivial to show.

- (e) Use  $PSL(2, \mathbb{Z})$  invariance to show that  $\tau$  can always be brought to the *fundamental domain*

$$F_0 : -\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2}, \quad |\tau| \geq 1.$$

Represent  $F_0$  graphically and discuss the various identifications of its boundaries.

**Problem 12.2:**

In this exercise we want to compute the one-loop vacuum amplitude in the oriented closed string theory. The corresponding worldsheet is a torus and there are no vertex operator insertions. The moduli space of the torus was subject of the discussion in problem 12.1. The upshot of this discussion was that (unlike in the case of the tree-level amplitude on  $S^2$  discussed in the lecture, which has no metric moduli) conformally inequivalent tori are characterized by a complex parameter  $\tau$  which takes values in the fundamental domain  $F_0$ . Thus, the Amplitude involves an integration over  $\frac{1}{2} \int_{F_0} d^2\tau$  where the factor  $\frac{1}{2}$  appears due to an extra  $\mathbb{Z}_2$  symmetry which sends the complex coordinate  $w$  on the torus to its negative value,  $w \rightarrow -w$ .

The conformal Killing group of the torus is  $U(1) \times U(1)$ , corresponding to constant shifts along the coordinate axes  $\sigma^1$  and  $\sigma^2$ .

(a) Show that the volume of the conformal Killing group of the torus is  $(2\pi)^2 \mathfrak{V}(\tau) \equiv (2\pi)^2 \tau_2$ .

To avoid overcounting we have to divide by this volume factor. It is clear that, in analogy to the tree-level amplitude computed in the lecture, the one-loop amplitude will factorize into a product of independent  $X$ - and  $(bc)$ -parts which are, in the end, integrated over the moduli space of the torus. More precisely, the amplitude takes the form

$$Z_{T^2} := \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} \langle 1 \rangle_{T^2(\tau), X} \cdot \langle \text{ghost insertions} \rangle_{T^2(\tau), bc}$$

where  $\langle 1 \rangle_{T^2(\tau), X}$  denotes the path integral  $\int \mathcal{D}X e^{-S_X}$  on a worldsheet given by a torus with modulus  $\tau$ , while  $\langle \text{ghost insertions} \rangle_{T^2(\tau), bc}$  denotes the corresponding path integral in the  $(bc)$ -ghost sector on the same torus with some ghost field insertions. The latter can be computed to be (cf. Polchinski, vol. 1, page 212ff)

$$\langle \text{ghost insertions} \rangle_{T^2(\tau), bc} = \langle b(0) \tilde{b}(0) \tilde{c}(0) c(0) \rangle_{T^2(\tau)}$$

In this expression the two  $b$  insertions come from the one (complex) metric modulus. In order to understand the appearance of the  $c$  insertions, think of first computing a one-loop amplitude with an arbitrary number of vertex operator insertions. Then, as you know, the existence of the two conformal Killing vectors discussed above allows to fix the position of one vertex operator. This gauge fixing introduces the  $c$  fields. However, since the volume of the conformal Killing group is finite, one can alternatively reintroduce the integration over the position of the vertex operator insertion (which we fixed previously) and add a compensating volume factor in the denominator. All vertex operators are now on equal footing. To obtain the vacuum amplitude one then just takes this expression with the  $c$  fields and omits the vertex operator insertions. The fact that all ghosts are inserted at the origin comes about because the ghost path integral turns out to be independent of the position of the ghost fields.

We will now only be concerned with the computation of

$$Z(\tau) := \langle 1 \rangle_{T^2(\tau), X}.$$

One can think of the path integral  $\int \mathcal{D}X e^{-S_X}$  on the torus with modulus  $\tau$  as being the same as the sum over all states of the  $X^\mu$ -field theory on a circle which are evolved for an Euclidean time  $2\pi\tau_2$ , translated in  $\sigma^1$  direction by  $2\pi\tau_1$  and then identified with themselves.

(b) Show that, in the canonical formalism, this translates into

$$\begin{aligned} Z(\tau) &= \text{Tr} e^{2\pi i \tau_1 P - 2\pi \tau_2 H} \\ &= (q\bar{q})^{-\frac{d}{24}} \text{Tr} q^{L_0} \bar{q}^{\bar{L}_0}, \quad q \equiv e^{2\pi i \tau}, \end{aligned}$$

where, as usual,  $P = L_0 - \bar{L}_0$  generates translations along  $\sigma^1$ , while  $H = L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}$  generates time translations.

- (c) Express  $L_0$  and  $\bar{L}_0$  in terms of momentum and number operators. Furthermore, recall that  $\sum_k \xrightarrow{\text{continuum limit}} V_d (2\pi)^{-d} \int d^d k$  where  $V_d \equiv$  ‘spacetime volume’ to arrive at

$$Z(\tau) = V_d (q\bar{q})^{-\frac{d}{24}} \int \frac{d^d k}{(2\pi)^d} e^{-\pi\tau_2 \alpha' k^2} \text{Tr}' q^N \bar{q}^{\bar{N}},$$

where  $\text{Tr}'$  denotes the trace over the oscillator part of the string states only.

- (d) Show that

$$\text{Tr}' q^N \bar{q}^{\bar{N}} = \prod_{n=1}^{\infty} (1 - q^n)^{-d} (1 - \bar{q}^n)^{-d}$$

- (e) Perform a Wick rotation  $k^0 \rightarrow ik^0$  to render the integral  $\int d^d k$  finite. Compute the integral in order to obtain the final result

$$\begin{aligned} Z(\tau) &= iV_d (Z_X(\tau))^d, \\ Z_X(\tau) &= (4\pi^2 \alpha' \tau_2)^{-\frac{1}{2}} \cdot |\eta(\tau)|^{-2} \quad \text{with} \\ \eta(\tau) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{the Dedekind } \eta\text{-function.} \end{aligned}$$

The calculation of the ghost contribution to the one-loop amplitude can be found e.g. in Polchinski, vol. 1, page 212ff. As expected, this contribution cancels the oscillator contributions from the two non-transverse polarizations, i.e. it yields a factor  $|\eta(\tau)|^4$ . Thus, setting  $d = 26$ , the one-loop vacuum amplitude reads

$$Z_{T^2} = iV_{26} \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} (4\pi^2 \alpha' \tau_2)^{-13} \cdot |\eta(\tau)|^{-48}.$$