String Theory and String Phenomenology
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Problem Sheet 13

(Due: January 30, 2013)

- The graded written exam, required to obtain a Schein for this course, will be held on Friday, February 15, 2013, from 10:15 am - 12:15 pm in Großer Hörsaal, Philosophenweg 12.
- If you would like to participate in the final exam you must register by sending an email to s.kraus@thphys.uni-heidelberg.de with subject line "Registration for exam" by February 8th.
- Notes, calculator etc. are not allowed.
- The last tutorials will be held on January 23rd, 30th and February 13th.

Problem 13.1:

(a) An important property of Z_{T^2}, discussed in problem 12.2, is *modular invariance*, i.e. invariance under PSL$(2,\mathbb{Z})$ transformations $\tau \to \frac{a\tau+b}{c\tau+d}$ of the torus. Under S- and T-transformations, which generate the modular group, the Dedekind η-function transforms as

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\,\eta(\tau)\,, \quad \eta(\tau+1) = e^{i\frac{\pi}{12}}\eta(\tau)\,.$$

Use this to proof that Z_{T^2} is indeed modular invariant.

(b) We now discuss the IR- and UV-behavior of this amplitude. Argue that the corresponding one-loop partition function of a field theory describing a particle of mass m is given by

$$Z_{S^1}(m^2) = V_d \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^\infty \frac{\mathrm{d}l}{2l} e^{-\frac{1}{2}l(k^2+m^2)}. \tag{1}$$

(c) The UV-limit corresponds to the limit of a small circle, i.e. $l \to 0$. Clearly, in this limit the field theory expression (1) is divergent. Argue for the one-loop partition function discussed in problem 12.2 that this divergence is absent.

(d) In order to analyze the IR-behavior we note that the η-function can equivalently be expressed in terms of the sum

$$\eta(\tau) = q^{\frac{1}{24}} \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}, \quad q = e^{2\pi i\tau}.$$

Use this to show that the integrand of Z_{T^2} contains one piece which diverges in the IR, one finite piece, and terms which vanish in the IR-limit. The divergent piece is and artifact due to the tachyon in the spectrum of bosonic string theory. What are the sources of the finite and vanishing pieces?

Problem 13.2:

Consider the theory of a massless scalar field $\phi(x^M)$ in $d+1$ spacetime dimensions. We choose to compactify the $(d+1)^{\text{th}}$ dimension on a S^1 with radius R, meaning that we identify $x^d \cong x^d + 2\pi R$. As a consequence, $\phi(x^M)$ has to be a periodic function in x^d and can thus be expanded in terms of a complete set of exponential functions $\exp(inx^d/R)$ with coefficients $\phi_n(x^\mu)$ depending on the remaining x^μ, $\mu = 0, \dots d-1$.

(a) What are the eigenvalues of the momentum operator in the compact direction?

(b) Starting from the equation of motion in $d+1$ dimensions, show that from the d-dimensional point of view the modes $\phi_n(x^\mu)$ are an infinite tower of fields with mass-squared $m^2 = -p^\mu p_\mu = \frac{n^2}{R^2}$.

Now we turn to the string and compactify one target space dimension, such that $X \cong X + 2\pi R$. One major effect of this compactification is the generalization of the usual periodicity conditions in the closed string sector to

$$X(\sigma + 2\pi) = X(\sigma) + 2\pi R w, \quad w \in \mathbb{Z} \tag{2}$$

such that there appear new sectors in the theory which are characterized by the *winding number* w.

(c) Find the most general solution to the equations of motion $\partial_+\partial_- X(\sigma) = 0$ for the string. Concentrate on the zero-modes. The oscillator pieces will not be of importance in what follows. *Hints:* Chapter 2 of the lecture notes might be a good source of inspiration. Note that, without imposing any periodicity condition, the momenta p_R and p_L are independent!

(d) Now impose (2) and use this to constrain the mode expansion in (c).

(e) Use your knowledge gained in (a) and (b) to constrain the center of mass momentum $p_R + p_L$.

(f) From the Virasoro constraints ($(L_0 - 1)|\phi\rangle = 0$ and analogously for $\tilde{L}_0$) you can now derive an expression for the effective mass-squared in d dimensions:

$$m^2 = -p^\mu p_\mu = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \text{oscillators}$$

(g) What happens under the identification

$$n \leftrightarrow w, \quad R \leftrightarrow R' = \frac{\alpha'}{R} \ ?$$

Try to gain a physical understanding of the situation.

Problem 13.3:
Consider the Γ^μ-matrices in $d = (2k+2)$ Minkowski space, i.e.

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{d\times d}, \qquad \eta^{\mu\nu} = \text{diag}(-1, 1, \ldots, 1).$$

(a) Define

$$\begin{cases} \Gamma^{0\pm} = \frac{1}{2}\left(\pm\Gamma^0 + \Gamma^1\right) \\ \Gamma^{a\pm} = \frac{1}{2}\left(\Gamma^{2a} \pm i\Gamma^{2a+1}\right), \quad a = 1, \ldots, k \end{cases}$$

Show that

(i) $\left\{\Gamma^{a+}, \Gamma^{b-}\right\} = \delta^{ab}$

(ii) $\left\{\Gamma^{a+}, \Gamma^{b+}\right\} = \left\{\Gamma^{a-}, \Gamma^{b-}\right\} = 0$ which implies in particular $\left(\Gamma^{a+}\right)^2 = \left(\Gamma^{a-}\right)^2 = 0 \quad \forall a$.

(b) Construct a representation of this algebra, starting from a spinor ξ which satisfies $\Gamma^{a-}\xi = 0 \ \forall a$ and acting with Γ^{a+} and Γ^{a-} on this spinor. What is the dimensionality of this representation?

Comments: The representation found in this way is the *Dirac*-representation. For even d (which is the case considered here) this representation is always reducible: In analogy to problem 9.2 one can define a Γ^{d+1} with a corresponding projection operator which projects a given spinor on an invariant subspace of dimensionality 2^k. These spinors are then called *Weyl*-spinors. In many cases it is also possible to demand that the spinor is invariant under the charge conjugation operation. These spinors are known as *Majorana*-spinors. Only in $d = 2 \bmod 8$ the Majorana- and Weyl-conditions are compatible and so-called *Majorana-Weyl*-spinors exist.