Institute for Theoretical Physics
Heidelberg University
Winter term 2012/13

Problem Sheet 2

(Due: November 2, 2012)

Problem 2.1:
Derive the commutation relations of the coefficients x^μ, p^μ and α_m^μ from the canonical commutation relations of X^μ and Π^μ in the case of an open string.

Problem 2.2:
Derive

$$H = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{-n}\alpha_n$$

from the Polyakov action in the case of an open string, ignoring ordering problems (i.e. classically).

Problem 2.3:
The Riemann curvature tensor, the Ricci tensor, and the Ricci scalar are defined as

$$\begin{aligned} R^\kappa{}_{\lambda\mu\nu} &= \partial_\mu \Gamma^\kappa{}_{\nu\lambda} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\eta{}_{\nu\lambda}\Gamma^\kappa{}_{\mu\eta} - \Gamma^\eta{}_{\mu\lambda}\Gamma^\kappa{}_{\nu\eta}, \\ \mathrm{Ric}_{\mu\nu} &= R^\kappa{}_{\mu\kappa\nu}, \\ \mathcal{R} &= g^{\mu\nu}\mathrm{Ric}_{\mu\nu}. \end{aligned}$$

where the Christoffel symbols are given by

$$\Gamma^\kappa{}_{\lambda\mu} = \frac{1}{2} g^{\kappa\nu}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu}\right).$$

(a) Using the symmetries of its indices, convince yourself that in two dimensions the Riemann curvature tensor has only 1 independent degree of freedom.

(b) Verify that the ansatz $R_{abcd} = \lambda\left(h_{ac}h_{bd} - h_{ad}h_{bc}\right)$ with h_{ab} the two-dimensional metric is consistent with the symmetries of the Riemann tensor. Show that $\lambda = \frac{1}{2}\mathcal{R}$ in terms of the Ricci scalar $\mathcal{R}$.

(c) Compute the Einstein tensor defined by

$$G_{\mu\nu} = \mathrm{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R}$$

for a general two-dimensional metric. Discuss the result.

(d) Show that in two dimensions under the Weyl rescaling $h_{ab} \to \exp(2\omega(x))h_{ab}$ the combination $\sqrt{-h}\mathcal{R}$ transforms as

$$\sqrt{-h}\mathcal{R} \to \sqrt{-h'}\mathcal{R}' = \sqrt{-h}(\mathcal{R} - 2\nabla^2\omega).$$

(e) Argue from the above result that the 2-dimensional Einstein-Hilbert term is indeed conformally invariant for a closed string worldsheet.

Note: By contrast, for an open string worldsheet Σ with boundary $\partial\Sigma$ only the combination

$$\chi = \frac{1}{4\pi}\int_\Sigma d^2\xi\sqrt{-h}\mathcal{R} + \frac{1}{2\pi}\int_{\partial\Sigma} ds\,\mathcal{K}$$

is conformally invariant. Here the extrinisic curvature $\mathcal{K}$ is defined as

$$\mathcal{K} = \pm t^a\, n_b\, \nabla_a\, t^b$$

with t^a a unit vector tangent to the boundary and n^a an outward pointing unit vector orthogonal to t^a. The upper/lower sign refer to timelike/spacelike boundaries. Indeed this object is the **Euler characteristic of a worldsheet with boundary**.

Problem 2.4:
Describe the mode expansion for "twisted" closed strings, i.e. strings with boundary conditions

$$X(\sigma+\pi,\tau) = -X(\sigma,\tau).$$

Problem 2.5:
Describe the mode expansion of ND open strings, i.e. open strings with boundary conditions

$$\begin{aligned} X(0,\tau) &= a, \quad \text{Dirichlet at } \sigma = 0, \\ \partial_\sigma X(\sigma,\tau)\big|_{\sigma=\pi} &= 0, \quad \text{Neumann at } \sigma = \pi. \end{aligned}$$

Problem 2.6:
Compute the anomaly term $A(m)$ in the Virasoro Algebra. Proceed as follows:

- Use the Jacobi Identity

  $$[L_m,[L_n,L_k]] + \text{cyclic permutations} = 0$$

  for $m+n+k=0$ to prove that

  $$A(n+1) = \frac{n+2}{n-1}A(n) - \frac{2n+1}{n-1}A(1).$$

- Use the recursion relation to find out $A(n) = cn + dn^3$ with $c, d \in \mathbb{R}$. (You may assume that $A(n)$ is polynomial in n.)
- Fix c and d by computing the value of $\langle 0|\,[L_m, L_{-m}]\,|0\rangle$ for "some" m.