String Theory and String Phenomenology
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Problem Sheet 3

(Due: November 7, 2012)

Problem 3.1:
Compute

$$\langle\phi|\phi\rangle = \frac{2c_1^2}{25}(D-1)(26-D)$$

for

$$|\phi\rangle = \left\{c_1\alpha_{-1}\cdot\alpha_{-1} + c_2 p\cdot\alpha_{-2} + c_3\left(p\cdot\alpha_{-1}\right)^2\right\}|0,p\rangle.$$

Hint: Given $(L_0-1)|\phi\rangle = L_1|\phi\rangle = L_2|\phi\rangle = 0$, determine the relation between c_1, c_2 and c_3 defining $|\phi\rangle$. Then compute $\langle\phi|\phi\rangle$.

Problem 3.2:
Consider the boundary term

$$\int \mathrm{d}\tau\partial_\sigma X^\mu\delta X_\mu\Big|_{\sigma=0}^{\sigma=\pi} = 0. \tag{1}$$

Take for simplicity $\mu = 1, 2$ ($D = 2$ case). Discuss the most general situation for which the boundary terms in (1) at $\sigma = 0$ and $\sigma = \pi$ vanish separately.
Hint: Think of the possibility that branes are not aligned with the coordinate axes.

Problem 3.3:
Quantize the antiperiodic boson

$$X(\sigma+\pi,\tau) = -X(\sigma,\tau).$$

Proceed as follows:

(i) Using the mode expansion for the above situation from problem 2.4, and, given $[\Pi(\sigma,\tau), X(\sigma',\tau)] = -i\delta(\sigma-\sigma')$, find the commutator between the modes.

(ii) Find L_0 and $\tilde{L}_0$.

(iii) Try to understand the physical meaning of the situation at hand.

Problem 3.4:
Following problem 2.5, quantize the open string with the boundary conditions

$$\begin{cases} \partial_\sigma X(\sigma,\tau)|_{\sigma=0} = 0, \\ X(\pi,\tau) = 0. \end{cases}$$

Problem 3.5 (*):

(a) Consider the Nambu-Goto string

$$S_{NG} = -T\int d^2\xi(-\det A)^{\frac{1}{2}} = -T\int d^2\xi\sqrt{(\dot{X}X')^2 - \dot{X}^2X'^2} = \int d^2\xi\,\mathcal{L}.$$

Compute the canonical momentum $\Pi_\mu(\tau,\sigma) = \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}$ and derive from it the following two non-trivial constraints:

$$\Phi_1 := \Pi_\mu X'^\mu = 0, \qquad \Phi_2 := \Pi^2 + T^2 X'^2 = 0.$$

These are **primary constraints**, i.e. constraints which follow without use of equations of motion from the very definition of the canonical coordinates. Compare this to the situation of the point particle discussed at the beginning of the lecture. What was the primary constraint there?

(b) In systems with primary constraints $\Phi_i = 0$ we must distinguish between the canonical Hamiltonian

$$H_{can} = \Pi_\mu \dot{X}^\mu - \mathcal{L}$$

and the total Hamiltonian

$$H_{tot} = H_{can} + \sum_i c_i \Phi_i$$

for c_i independent of the canonical coordinates. Different choices of c_i correspond to different gauges (here: different meanings of (τ,σ)). The τ-variation of a function $f(X,\Pi,\tau)$ is governed by the Poisson-bracket involving H_{tot}, not H_{can}:

$$\frac{df}{d\tau} = \frac{\partial f}{\partial\tau} + \{H_{tot}, f\}_{P.B.}. \tag{2}$$

The complete dynamics is to be understood as follows: One first computes (2) to get the equations of motion. Then, in addition to these e.o.m., one has to impose the constraints $\Phi_i = 0$.

Show that for the Nambu-Goto string H^{NG}_{can} vanishes identically so that the τ-involution is governed exclusively by the constraints appearing in H_{tot}. Furthermore show that for the Nambu-Goto string the equations of motion for X and Π follow from (2) as

$$\dot{X}^\mu = c_1 X'^\mu + 2c_2\Pi^\mu, \qquad \dot{\Pi}^\mu = \partial_\sigma(c_1\Pi^\mu + 2T^2 c_2 X'^\mu).$$

Show that for the choice $c_1 = 0$ and $c_2 = \frac{1}{2T}$ you recover the free wave equation.

Note: In addition, we must impose the two constraints Φ_1 and Φ_2 on the solution of the free wave equation. The fact that the total Hamilton vanishes (because it is just given by the constraints) does not mean that the τ-evolution is trivial.

(c) Now consider the Polyakov string. For the full Polyakov action

$$S_{\mathrm{P}} = -\frac{T}{2}\int \mathrm{d}^2\xi\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

there are no primary constraints. We can (partially) fix diffeomorphism and Weyl invariance to rewrite the action in flat coordinates, where it looks like a free scalar theory. However, the gauge fixed action is equivalent to the original Polyakov action only provided we impose the equations of motion for the metric h_{ab} as constraints. **The full string theory is not merely a free scalar theory**, but we have to take into account the constraints as remnants of the equations of motion of the WS metric.

With this in mind, in flat gauge, the canonical Hamiltonian takes the form

$$H^{Pol.,flat}_{can} = \frac{T}{2}\int d\sigma((\dot{X})^2 + (X')^2).$$

Relate $H_{can}^{Pol.,flat}$ to H_{tot}^{NG} and interpret the result.

Note: The upshot is: $H_{can}^{Pol.,flat}$ is just one of the constraints that must be imposed in flat gauge as the remnant of the equations of motion for the metric. Again, the fact that $H_{can}^{Pol.,flat}$ is one of the constraints does not imply triviality of the τ-involution.

(d) Derive the (classical) mass-shell condition for the closed string

$$M^2 = -p^\mu p_\mu = \frac{2}{\alpha'} \sum_{n=1}^{\infty} (\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n)$$

from the constraint $H_{can}^{Pol.,flat} = 0$.

Important: We must distinguish between the notion of energy in the 2-dim. worldsheet theory and energy in spacetime. $H_{can}^{Pol.,flat}$ is the precise analogue of the the Hamiltonian for a free boson field. It does represent the correct notion of energy on the 2-dimensional worldsheet. Unlike for the free Klein-Gordon field, this energy vanishes because we must impose the constraints in addition. However, the physical spacetime energy of a string solution is defined not as the Hamiltonian of the WS-theory, but rather as the quantity $E^2 = M^2 + \vec{p}^2$, with $\vec{p}$ the spatial momentum in the ambient space. $H_{can}^{Pol.,flat} = 0$ relates this to the oscillator modes of the string.