String Theory and String Phenomenology
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# Problem Sheet 4

(Due: November 14, 2012)

**Problem 4.1:**
Show that the polarisation tensor of any *physical* state

$$\xi_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,p\rangle$$

can be brought to the form

$$\xi_{\mu\nu} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \\ & & \xi_t & \\ 0 & 0 & & \end{pmatrix}$$

by use of the gauge freedom.
(Remind yourself of the analogous problem in Electrodynamics.)

**Problem 4.2:**
Consider the coordinate change

$$x^\mu \to x'^\mu \equiv x^{\mu'}. \tag{1}$$

The associated transformation matrices are $P^\mu{}_{\nu'} = \frac{\partial x^\mu}{\partial x^{\nu'}}$ and its inverse $P^{\mu'}{}_\nu = \frac{\partial x^{\mu'}}{\partial x^\nu}$. Recall that a tensor of type, say, $T^\mu{}_\nu$ transforms under (1) as $T^\mu{}_\nu \to T^{\mu'}{}_{\nu'} = P^{\mu'}{}_\alpha P^\beta{}_{\nu'} T^\alpha{}_\beta$. A tensor density $\widetilde{T}^\mu{}_\nu$ of weight $w$ is defined by the transformation property

$$\widetilde{T}^\mu{}_\nu \to \widetilde{T}^{\mu'}{}_{\nu'} = J^w \, P^{\mu'}{}_\alpha P^\beta{}_{\nu'} \widetilde{T}^\alpha{}_\beta$$

(and obvious generalisations for general types of tensor densities), where $J = \det(P^\mu{}_{\nu'})$.

(a) Given the tensor $S_{\mu\nu}$, convince yourself that $(\det S_{\mu\nu})^{\frac{1}{2}}$ is a scalar density of rank 1.

(b) Consider now fields of tensors and tensor densities, e.g. $T^\mu{}_\nu(x)$ etc.. Locally the transformation (1) can be expanded $x'^\mu = x^\mu - \epsilon^\mu(x) + \dots$. Show the following infinitesimal variations for a scalar field $\Phi(x)$, the metric $g_{\mu\nu}(x)$ and the associated metric density $\sqrt{-g}$:

$$\begin{aligned} &\delta\Phi = \epsilon^\mu \partial_\mu \Phi, \\ &\delta g_{\mu\nu} = \epsilon^\gamma \partial_\gamma g_{\mu\nu} + (\partial_\mu \epsilon^\gamma) g_{\gamma\nu} + (\partial_\nu \epsilon^\gamma) g_{\mu\gamma} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu, \\ &\delta\sqrt{-g} = \partial_\gamma(\epsilon^\gamma \sqrt{-g}), \end{aligned}$$

where the second equation in the second line is true for the metric connection satisfying $\nabla_\gamma g_{\mu\nu} = 0$.
*Hint:* For a scalar field the transformed object is defined via the relation $\Phi'(x') = \Phi(x)$, whereas for a tensor $T^{\mu\nu}$ the transformed object is defined via the relation $T^{\mu'\nu'}(x') = P^{\mu'}{}_\mu P^{\nu'}{}_\nu T^{\mu\nu}(x)$.

**Problem 4.3:**
Show that the constraint $T_{ab} = 0$ can equivalently be written as

$$\left(\dot{X} \pm X'\right)^2 = 0.$$

**Problem 4.4:**
Show that in the light cone gauge the modes of $X^-$ are related to the modes of the transverse directions, i.e.

$$\alpha_n^- = \frac{1}{p^+}\left(\frac{1}{2}\sum_{m=-\infty}^{\infty}\alpha_{n-m}^i\alpha_m^i\right), \quad i = 2, \ldots, D-1.$$

**Problem 4.5:**
The conserved charges associated with Poincaré invariance of the classical string are given by

$$J^{\mu\nu} = T\int d\sigma\left(X^\mu\partial_\tau X^\nu - X^\nu\partial_\tau X^\mu\right).$$

The mode expansion for $J^{\mu\nu}$ for the classical open string (NN) takes the form

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu}$$

with

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \qquad E^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu\alpha_n^\nu - \alpha_{-n}^\nu\alpha_n^\mu).$$

After quantization, the operator $E^{\mu\nu}$ has to be defined in the normal ordered form given above. Using the canonical commutation relations for the $\alpha_m^\mu$ show that the $J^{\mu\nu}$ indeed satisfy the Lorentz algebra,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\eta^{\mu\rho}J^{\nu\sigma} + i\eta^{\nu\sigma}J^{\mu\rho} - i\eta^{\mu\sigma}J^{\nu\rho} - i\eta^{\nu\rho}J^{\mu\sigma}.$$