Problem Sheet 6

(Due: November 28, 2012)

Problem 6.1:

(a) Given the bc-ghost, quantized as

$$\{c_m, b_n\} = \delta_{m+n} \quad \text{with} \quad L_0 = \sum_{n\in\mathbb{Z}} -n b_n c_{-n}$$

and the identification $c_n, b_n \equiv$ creation operators for $n < 0$, find the normal ordered $: L_0 :$ and, via ζ-function regularization, the normal-ordering constant.

(b) Given $L_0 = L_0^X + L_0^{\text{ghost}} - a$, compute the normal-ordering constant a using ζ-function regularization for a system of D X-bosons and a bc-ghost.

- Compare the result with the one obtained in light-cone quantization.
- What is the 'net'-effect of the ghosts, regarding a?

Problem 6.2:

(a) Check that the BRST charge is such that $Q^2 = 0$, i.e. take

$$\begin{cases} \delta_\epsilon \phi_i = -i\epsilon c^\alpha \delta_\alpha \phi \\ \delta_\epsilon B_A = 0 \\ \delta_\epsilon b_A = \epsilon B_A \\ \delta_\epsilon c^\alpha = \frac{i}{2}\epsilon c^\beta c^\gamma f_{\beta\gamma}{}^\alpha \end{cases}$$

and show that $\delta_\epsilon(\delta_{\epsilon'}\phi) = \delta_\epsilon(\delta_{\epsilon'} b_A) = \delta_\epsilon(\delta_{\epsilon'} c^\alpha) = 0$.
Hint: Remember the Jacobi identity for $f_{\alpha\beta}{}^\gamma$.

(b) Show that $\delta_\epsilon \left(b_A F^A\right) = i\epsilon \left(S_{gf} + S_g\right)$.

Problem 6.3:
For some $n \times n$ matrix M, show that

$$\int \left(\prod_{i=1}^n \mathrm{d}\psi_i \, \mathrm{d}\theta_i\right) e^{\theta^T M \psi} = \det M,$$

where θ_i, ψ_i, $i = 1, \dots, n$ are Grassmann variables.
Hint: Use the power series of e^x and the fact that $\int \mathrm{d}^n\theta \, \theta_{i_1}\theta_{i_2}\dots\theta_{i_n} = \epsilon_{i_1 i_2 \dots i_n}$.

Problem 6.4:
A conformal transformation is a diffeomorphism $x \to x'$ that changes the metric only by an overall prefactor. In flat d-dimensional space this amounts to

$$\eta_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} = \Lambda \, \eta_{\alpha\beta}, \qquad \Lambda = e^{w(x)}.$$

(a) Show that an infinitesimal diffeomorphism

$$x'^{\rho} = x^{\rho} - \epsilon^{\rho}(x) + \mathcal{O}(\epsilon^2)$$

has to satisfy

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \omega\ \eta_{\mu\nu}\,, \qquad \omega = \frac{2}{d}(\partial \cdot \epsilon), \tag{1}$$

in order to be a *conformal* transformation.

(b) Take the derivative ∂^ν of (1) and sum over ν. Take the derivative ∂_ν of the resulting equation. You should find

$$\partial_\mu \partial_\nu (\partial \cdot \epsilon) + (\partial \cdot \partial)\partial_\nu \epsilon_\mu = \frac{2}{d}\partial_\mu \partial_\nu (\partial \cdot \epsilon)\,. \tag{2}$$

Interchange $\mu \leftrightarrow \nu$ in (2), add the resulting equation to (2) and use (1) to eliminate the term multiplying $\partial \cdot \partial$. Contract with $\eta^{\mu\nu}$ to find the result

$$(d-1)(\partial \cdot \partial)(\partial \cdot \epsilon) = 0\,, \qquad \text{i.e. } (d-1)\partial^2 \omega = 0 \tag{3}$$

(c) Take the derivative ∂_ρ of (1). From the resulting equation produce two more equations by relabelling $(\rho, \mu, \nu) \to (\nu, \rho, \mu)$ and $(\rho, \mu, \nu) \to (\mu, \nu, \rho)$. Subtract the original equation from the sum of the last two to obtain

$$2\partial_\mu \partial_\nu \epsilon_\rho = \frac{2}{d}(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)(\partial \cdot \epsilon), \tag{4}$$

$$\text{i.e. } (\eta_{\mu\nu}\partial^2 + (d-2)\partial_\mu \partial_\nu)\,\omega = 0. \tag{5}$$

(d) This shows that the cases $d = 2$ and $d \geq 3$ are clearly different. Let us now consider the case $d \geq 3$. Assuming a polynomial expansion for ω and ϵ_μ and using (1), (3), and (5), show that ω is at most linear and ϵ_μ is at most quadratic in x^ν, such that we can make the ansatz

$$-\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho\,, \tag{6}$$

where $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho} \ll 1$ are constants and the latter is symmetric in the last two indices.

(e) We now want to see how (1) constrains these constants.

- It is easy to see that a_μ is not constrained and that it corresponds to translations $x'^\mu = x^\mu + a^\mu$, for which the generator is the momentum operator $P_\mu = -i\partial_\mu$.
- $b_{\mu\nu}$ can be decomposed into an antisymmetric piece and a symmetric piece. Argue that the antisymmetric part of $b_{\mu\nu}$ is unconstrained and thus describes rotations with the angular momentum operator as the generator. Argue further that the symmetric part of $b_{\mu\nu}$ describes scale transformations: $x \to x' = (1+\alpha)x$ for some α.

(f)* For $c_{\mu\nu\rho}$ insert (6) into (1) and take the derivative ∂_ν of the resulting equation to obtain

$$\partial_\nu(\partial \cdot \epsilon) = -2c^{\mu}_{\mu\nu}\,.$$

Use this equation in (4) to show that the $c_{\mu\nu\rho}$ can be written as

$$c_{\mu\nu\rho} = \frac{1}{d}(\eta_{\mu\rho}c^{\alpha}_{\alpha\nu} + \eta_{\mu\nu}c^{\alpha}_{\alpha\rho} - \eta_{\nu\rho}c^{\alpha}_{\alpha\mu})\,.$$

The resulting transformations are called *Special Conformal Transformations* (SCT) and have the following infinitesimal form:

$$x'^\mu = x^\mu + 2(x \cdot b)x^\mu - (x \cdot x)b^\mu\,.$$

The corresponding generator is written as $K_\mu = -i(2x_\mu x^\nu \partial_\nu - (x \cdot x)\partial_\mu)$. The finite version of these transformations is

$$x'^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2(b \cdot x) + b^2 x^2}, \qquad \text{i.e. } \frac{x'^\mu}{x' \cdot x'} = \frac{x^\mu}{x \cdot x} - b^\mu.$$

Show that indeed these transformations can be thought of as the successive application of inversion $x^\mu \to \frac{x^\mu}{x \cdot x}$, translation and inversion.