Problem Sheet 7

(Due: December 5, 2012)

Problem 7.1:
The BRST-operator Q can be defined more generally as follows: Consider a physical system with symmetry operators K_i that form a closed Lie algebra G,

$$[K_i, K_j] = f_{ij}{}^k K_k \tag{1}$$

with $f_{ij}{}^k$ being the structure constants of G. Modern covariant quantization involves the introduction of ghosts c^i and antighosts b_i which obey the canonical anticommutation relations

$$\left\{c^i, b_j\right\} = \delta^i{}_j.$$

One then introduces the BRST-operator

$$Q = c^i K_i - \frac{1}{2} f_{ij}{}^k c^i c^j b_k.$$

(a) Show, using (1) and the Jacobi identity for the structure constants

$$f_{ij}{}^m f_{mk}{}^l + f_{jk}{}^m f_{mi}{}^l + f_{ki}{}^m f_{mj}{}^l = 0,$$

that $Q^2 = 0$.

(b) Now identify the K_i with the Virasoro generators L_m^X and compute Q for the bosonic string. *Note:* As you know from the lecture, in (1) as well as in the relation $Q^2 = 0$ there appears an anomaly which vanishes only in the critical case $D = 26$, $a = 1$.

Problem 7.2:
Let $\mathbb{R}^2$ be a vector (Hilbert) space. Take $\vec{v} \in \mathbb{R}^2$, $\vec{v} = (v_1, v_2)^T$. Define an operator Q via

$$Q: \quad \vec{v} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{w}.$$

(a) Check that $Q^2 = 0$.

(b) Characterize explicitly the space $\mathcal{H}_{\text{closed}}$ of vectors $\vec{v}$ such that $Q: \ \vec{v} \mapsto 0$.

(c) Characterize the space $\mathcal{H}_{\text{exact}}$ of vectors $\vec{w}$ such that $\exists\, \vec{v} \in \mathbb{R}^2$ with $\vec{w} = Q\vec{v}$.

(d) Interpret the space $\hat{\mathcal{H}} = \mathcal{H}_{\text{closed}} / \mathcal{H}_{\text{exact}}$ geometrically.

To have a more interesting example repeat the above analysis for $\vec{v} \in \mathbb{R}^3$ and an operator Q defined via

$$Q: \quad \vec{v} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{w}.$$

Problem 7.3:
Given $\ell_n = -z^{n+1}\partial_z$, $\overline{\ell}_n = -\overline{z}^{n+1}\partial_{\overline{z}}$ which operate on complex functions, show that

$$\begin{aligned} [\ell_m, \ell_n] &= (m-n)\ell_{m+n}, \\ \left[\overline{\ell}_m, \overline{\ell}_n\right] &= (m-n)\overline{\ell}_{m+n}, \\ \left[\overline{\ell}_m, \ell_n\right] &= 0. \end{aligned} \tag{2}$$

Think about this carefully, e.g. by expressing ∂_z and $\partial_{\overline{z}}$ in terms of ∂_{σ^1} and ∂_{σ^2}.

Problem 7.4:
As you know, the ℓ_n of problem 7.3, which satisfy the Witt algebra (2), generate conformal transformations on $\mathbb{C}^* \equiv \mathbb{C} \setminus \{0\}$. If we generalize this to compact spaces, only a subgroup of so-called *global* conformal transformations will be defined everywhere. This subgroup depends on the topology. Consider the example of a two-sphere $S^2 \cong \mathbb{C} \cup \infty$. The two-sphere is covered by (at least) two coordinate patches. One can choose them to be such that in one system with coordinate z the south pole is at $z = 0$, whereas in the other system with coordinate w the north pole is at $w = 0$. The corresponding coordinate transformation is given by $w = z^{-1}$.

(a) Which of the ℓ_n are well defined globally? You should find that the group of finite conformal transformations on S^2 is generated by ℓ_{-1}, ℓ_0, ℓ_1 and $\bar{\ell}_{-1}$, $\bar{\ell}_0$, $\bar{\ell}_1$.

(b) What is the geometric interpretation of these transformations?
Hints:

- The case of ℓ_{-1} should be familiar to you.
- For ℓ_0 and $\bar{\ell}_0$ work in polar coordinates $z = re^{i\phi}$ and consider the linear combinations $\ell_0 \pm \bar{\ell}_0$.
- In the case of ℓ_1 you should work out the infinitesimal version of the transformation $z \mapsto z/(cz+1)$.

(c) Combining the action of ℓ_{-1}, ℓ_0, ℓ_1 and $\bar{\ell}_{-1}$, $\bar{\ell}_0$, $\bar{\ell}_1$ you find that the globally defined conformal diffeomorphisms on S^2 are given by transformations of the form

$$z \mapsto \frac{az+b}{cz+d}. \tag{3}$$

Show that combining two transformations of the type (3) yields a new transformation of this type, where the parameters a'', b'', c'' and d'' of the resulting transformation are determined via matrix multiplication

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

Comments:

- In order for the inverse of (3) to be defined we need that the corresponding matrix is regular, i.e. $ad - bc \neq 0$. We can rescale to achieve $ad - bc = 1$. Matrices with complex entries and unit determinant form the group $SL(2,\mathbb{C})$.
- Even after rescaling there is the redundancy of going from (a,b,c,d) to $(-a,-b,-c,-d)$ without changing the transformation. Thus the group of conformal diffeomorphisms on S^2 is the Möbius group $SL(2,\mathbb{C})/\mathbb{Z}_2 \equiv PSL(2,\mathbb{C})$, where the $\mathbb{Z}_2$ action is $(a,b,c,d) \to (-a,-b,-c,-d)$.