Institute for Theoretical Physics
Heidelberg University

Problem Sheet 8

(Due: December 19, 2012)

Important: There will be no tutorial on Wednesday, December 12.

Problem 8.1:

(a) Use the path integral identity

$$0=\int \mathcal{D}X \frac{\delta}{\delta X(z,\overline{z})}\left(X(w,\overline{w})e^{-S[X]}\right)$$

to show that the propagator of the X field has to satisfy the differential equation

$$\partial_z\overline{\partial}_{\overline{z}}\langle X(z,\overline{z})X(w,\overline{w})\rangle=-\pi\alpha'\delta^2(z-w,\overline{z}-\overline{w}).$$

(b) To solve this differential equation verify

$$\partial\overline{\partial}\ln|z|^2=2\pi\delta^2(z,\overline{z})$$

by integrating this expression over a suitable disk in the complex z-plane.

(c) Show, using the same trick as in (a), that the expectation value of X satisfies the classical equation of motion, i.e.

$$\langle\partial\overline{\partial}X(z,\overline{z})\rangle=0.$$

This is known as *Ehrenfest theorem*.

Note: In the above we use the conventions of Polchinski in which $\mathrm{d}^2z=2\mathrm{d}\sigma^1\mathrm{d}\sigma^2$ and $\delta^2(z,\overline{z})=\frac{1}{2}\delta(\sigma^1)\delta(\sigma^2)$.

Problem 8.2:

An operator $\mathcal{O}(z,\overline{z})$ is called *primary* if it transforms according to

$$\mathcal{O}(z,\overline{z})\to\mathcal{O}'(z',\overline{z}')=\left(\frac{\partial z'}{\partial z}\right)^{-h}\left(\frac{\partial\overline{z}'}{\partial\overline{z}}\right)^{-\overline{h}}\mathcal{O}(z,\overline{z})$$

under conformal transformations.

(a) Show that for infinitesimal conformal transformations $z\to z+\epsilon(z)$, $\overline{z}\to\overline{z}+\overline{\epsilon}(\overline{z})$

$$\delta\mathcal{O}(z,\overline{z})=-\left(h\partial_z\epsilon+\overline{h}\partial_{\overline{z}}\overline{\epsilon}+\epsilon\partial_z+\overline{\epsilon}\partial_{\overline{z}}\right)\mathcal{O}(z,\overline{z}).$$

(b) Compare this result to the general transformation rule of an operator under infinitesimal conformal transformations

$$\delta\mathcal{O}(z_1,\overline{z}_1)=-\operatorname{Res}_{z-z_1}\left[\epsilon(z)T(z)\mathcal{O}(z_1,\overline{z}_1)\right]-\operatorname{Res}_{\overline{z}-\overline{z}_1}\left[\overline{\epsilon}(\overline{z})\overline{T}(\overline{z})\mathcal{O}(z_1,\overline{z}_1)\right]$$

to find

$$T(z)\mathcal{O}(w,\overline{w})=h\frac{\mathcal{O}(w,\overline{w})}{(z-w)^2}+\frac{\partial\mathcal{O}(w,\overline{w})}{z-w}+\text{non-singular},$$
$$\overline{T}(\overline{z})\mathcal{O}(w,\overline{w})=\overline{h}\frac{\mathcal{O}(w,\overline{w})}{(\overline{z}-\overline{w})^2}+\frac{\overline{\partial}\mathcal{O}(w,\overline{w})}{\overline{z}-\overline{w}}+\text{non-singular}.$$