Institute for Theoretical Physics
Heidelberg University
Winter term 2012/13

Problem Sheet 9

(Due: December 19, 2012)

Important: There will be no tutorial on Wednesday, December 12.

Note: At some point during your QFT lecture you might have come across a problem similar to 9.1 and/or 9.2a, 9.2b. If that is the case, feel free to skip those exercises.

Problem 9.1:
Starting with a representation of the Clifford algebra in d dimensions

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu},$$

one can define Lorentz-generators $\Sigma^{\mu\nu}$ via

$$\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right].$$

Show that these generators indeed satisfy the $SO(1, d-1)$-algebra

$$i\left[\Sigma^{\mu\nu}, \Sigma^{\sigma\rho}\right] = \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\sigma}\Sigma^{\nu\rho}.$$

Problem 9.2:

(a) Given two 2d Majorana spinors

$$\chi = \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix},$$

show that $\overline{\chi}\psi = \overline{\psi}\chi$.

(b) Do these spinors correspond to an irreducible representation of the Lorentz-algebra? Define $\gamma^3 = \gamma^0\gamma^1$ and show that $\left(\gamma^3\right)^2 = \mathbb{1}$, $\left\{\gamma^3, \gamma^a\right\} = \left[\gamma^3, \Sigma^{ab}\right] = 0$ where the Σ's are defined in problem 9.1 and $a, b = 0, 1$. Compute γ^3 explicitly using γ^0, γ^1 as defined in the lecture. Observe how the operators $P_\pm = \frac{1}{2}\left(\mathbb{1} \pm \gamma^3\right)$ act on a Majorana spinor. From this you learn that $\begin{pmatrix} \chi_- \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \chi_+ \end{pmatrix}$ form two real one-dimensional *Majorana-Weyl* representations of the two-dimensional Lorentz-algebra.

(c) Given the action

$$S = \frac{i}{4\pi}\int \mathrm{d}^2\sigma\, \mathrm{d}^2\theta \left(\overline{D}^\alpha Y^\mu\right)\left(D_\alpha Y_\mu\right)$$

for the superfield Y^μ, find the component action by Taylor expanding and integrating in the θ's.

$$\begin{aligned}
Y^\mu(\sigma, \theta) &= X^\mu(\sigma) + \overline{\theta}\psi^\mu(\sigma) + \frac{1}{2}\overline{\theta}\theta B^\mu(\sigma), \\
D_\alpha &= \frac{\partial}{\partial\overline{\theta}^\alpha} - i\left(\gamma^a\theta\right)_\alpha \partial_a, \\
\overline{D}^\alpha &= -\frac{\partial}{\partial\theta_\alpha} + i\left(\overline{\theta}\gamma^a\right)^\alpha \partial_a.
\end{aligned}$$

Problem 9.3:

(a) Given $Q_\alpha = \frac{\partial}{\partial \overline{\theta}^\alpha} + i\left(\gamma^a\theta\right)_\alpha \partial_a$, show that

$$\begin{aligned} e^{\overline{\epsilon}Q} Y(\sigma,\theta) &= e^{\overline{\epsilon}^\alpha \frac{\partial}{\partial \overline{\theta}^\alpha}} e^{i\overline{\epsilon}^\alpha (\gamma^a\theta)_\alpha \partial_a} e^{\frac{i}{2}\overline{\epsilon}^\alpha (\gamma^a\epsilon)_\alpha \partial_a} Y(\sigma,\theta) \\ &= Y\left(\sigma^a + i\overline{\epsilon}\gamma^a\theta + \frac{i}{2}\overline{\epsilon}\gamma^a\epsilon, \theta+\epsilon\right). \end{aligned}$$

(b) Given

$$\begin{aligned} \delta_\epsilon Y^\mu(\sigma,\theta) &\equiv \delta X^\mu(\sigma) + \overline{\theta}\delta\psi^\mu(\sigma) + \frac{1}{2}\overline{\theta}\theta\delta B^\mu(\sigma) \quad \text{and} \\ \delta_\epsilon Y^\mu(\sigma,\theta) &= Y^\mu\left(\sigma^a + i\overline{\epsilon}\gamma^a\theta + \frac{i}{2}\overline{\epsilon}\gamma^a\epsilon, \theta+\epsilon\right) - Y^\mu(\sigma,\theta), \end{aligned}$$

derive the off-shell SUSY transformations

$$\begin{cases} \delta X^\mu = \overline{\epsilon}\psi^\mu \\ \delta\psi^\mu = -i\left(\gamma^a\epsilon\right)\partial_a X^\mu + B^\mu\epsilon \\ \delta B^\mu = -i\overline{\epsilon}\gamma^a\partial_a\psi^\mu \end{cases}$$