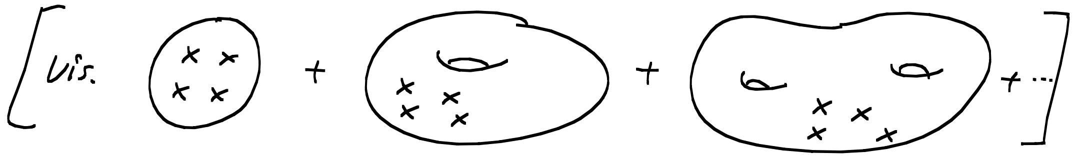
10 Scattering amplitudes

10.1 Sum over Riemann surfaces

- Let us make the basic formula motivated at the beginning of Sect. 6 more precise:

$$\mathcal{A}_n = \sum_{g=0}^{\infty} \int \frac{Dh\, DX}{\mathrm{Vol}_{\mathrm{Diff.\times Weyl}}} \cdot e^{-S[X,h]} \int d^2z_1 \cdots d^2z_n\, V_1(z_1,\bar{z}_1) \cdots V_n(z_n,\bar{z}_n)$$



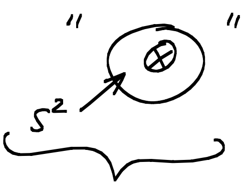
- "g" is the genus (# of handles).
- The $\frac{1}{\mathrm{Vol}}$-factor will be (mostly) compensated by the corresponding divergent factors extracted in the FP-gauge fixing procedure.
- $$S[X,h] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{h}\, (\partial X)^2 + \phi \underbrace{\left[\frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{h}\, R + \frac{1}{2\pi} \int_{\partial\Sigma} ds\, k \right]}_{= \chi(\Sigma) = 2 - 2g - b - c}$$

naively a dimensionless parameter, but really the "background VEV" (on equal footing with $\eta_{\mu\nu} \to g_{\mu\nu}$) of the dilaton (a scalar field ϕ in the 26d-theory) found earlier

R: Ricci scalar (is a total derivative in $d=2$)

k: extrinsic curvature (needed to cancel boundary terms in variation)

b, c: # of boundaries & crosscaps



≡ real projective plane

- We let $g_s \equiv e^{\phi}$ (string coupling), focus on just oriented, closed strings and get an extra factor

$$\boxed{g_s^{-\chi} = g_s^{-2+2g}}$$

from the new terms $\sim \phi$.

- Our V's are precisely as follows from Sect. 9.6, but with an extra factor g_s. E.g. for the closed-string tachyon:

$$V(k, z, \bar{z}) = g_s : e^{ik^\mu X_\mu (z,\bar{z})} :$$

This is the correct normalization to get a tree-level (i.e. $g=0$) 2-point-fct. $\sim g_s^0$; a tree-level 4-point-fct. $\sim g_s^2$; etc.

10.2 Gauge fixing

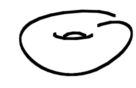
- In slightly schematic notation:

now without $\mathcal{R}$-term

$$\mathcal{A}_n = \sum_g g_s^{-2+2g} \int \frac{DX Dh}{Vol_{Diff \times Weyl}} \cdot e^{-S_X} \prod_{i=1}^n \int_{z_i} V_i$$

- In Sect. 6, we basically used

$$\int Dh \cdots = \int D\zeta \, \Delta_{FP}[\hat{h}] \underbrace{\cdots}_{\text{with } h \to \hat{h}} = Vol_{Diff \times Weyl} \cdot \Delta_{FP}[\hat{h}] \cdots$$

- Here, we need to be more careful since
  a) On a compact Riem. surface, there are variations of the metric which are not just gauge freedom.
  (vis.  $\longleftrightarrow$  for T^2)
  They are parametrized by μ "moduli" t^k; $\hat{h} = \hat{h}(t)$.
  b) There is a (κ-dimensional) space of gauge-trfs. which do not change h, but can be traded for κ of the z_i-integrations. They are called "conf. killing vectors". (vis. $SL(2,\mathbb{C})$ trfs. on S^2)

- We also minimally change notation, $\prod_i \int d^2 z_i \to \int d^{2n}\sigma$, and write:

$$\frac{1}{\mathrm{Vol.}}\int \mathcal{D}h\, d^{2n}\sigma \cdots = \frac{1}{\mathrm{Vol.}}\int \mathcal{D}h\, d^{2n}\sigma\, \mathcal{D}\zeta\, d^\mu t\, \delta[h-\hat{h}(t)^\zeta]\prod_{i=1}^{k}\delta^2(\sigma_i-\hat{\sigma}_i^\zeta)\cdot \Delta_{FP}\cdots$$

$$= \frac{1}{\mathrm{Vol.}}\int d^{2n-k}\sigma\, \mathcal{D}\zeta\, d^\mu t\, \Delta_{FP}\cdots$$

$$= \int d^{2n-k}\sigma\, d^\mu t\, \Delta_{FP}\cdots$$

where

$$\Delta_{FP}^{-1} = \int d^\mu t \int \mathcal{D}\zeta\, \delta(\underbrace{h-\hat{h}(t)^\zeta}_{})\prod_i \delta^2(\sigma_i-\hat{\sigma}_i^\zeta)$$

can replace this by δh, due to <u>both</u> ζ and δt:

$$\delta h_{ab} = -(P\epsilon)_{ab} + (2\omega - D_c\epsilon^c)h_{ab} + \sum_{k=1}^{\mu}\delta t^k \partial_{t^k}\hat{h}_{ab}$$

- analogously, we can replace

$\sigma_i - \hat{\sigma}_i^\zeta$ by $\delta\hat{\sigma}_i^\zeta = \epsilon(\hat{\sigma}_i)$ (ϵ characterizes our diffeomorphism, as before)

and write

$$\delta^2(\sigma_i-\hat{\sigma}_i^\zeta) \sim \int dx\, e^{2\pi i\, x\cdot\epsilon(\hat{\sigma}_i)}.$$

- Thus, we have

$$\Delta_{FP}^{-1} = \int d^\mu\delta t\, d^k x\, \mathcal{D}\beta\, \mathcal{D}\epsilon \cdot \exp 2\pi i\Big[\int \beta\cdot(-P\epsilon + \delta t^k \partial_k \hat{h}) + x_i\,\epsilon(\hat{\sigma}_i)\Big]$$

- To get Δ_{FP}, we replace

$$\begin{aligned} \epsilon &\to c \\ \beta &\to b \\ x &\to \eta \\ \delta t &\to \xi \end{aligned}$$

$\underbrace{}_{}$ grassmann fields / variables.

- Since η & ξ appear just linearly in the exponent, the corresponding integrals can easily be done. The result is:

$$\boxed{\begin{aligned}\mathcal{A}_n = &\sum_g g_s^{-2+2g} \int d^\mu t\, DX\, Db\, Dc\; e^{-S_X - S_g} \prod_{\substack{\text{unfixed}\\ z_i}} \int d^2 z_i \prod_{k=1}^{\mu} \frac{1}{4\pi} \int b\cdot \partial_k h \\ &\cdot \prod_{\text{fixed } z_i} c^1(\hat{\sigma}_i)\, c^2(\hat{\sigma}_i) \prod_{i=1}^{n} \sqrt{h(\sigma_i)}\; V_i(\sigma_i)\end{aligned}}$$

10.3 The Virasoro-Shapiro amplitude (4-Tachyon-scattering at tree-level, i.e. on S^2)

$$\mathcal{A}_4 = g_s^2 \int d^2 z_4 \left\langle :e^{ik_4\cdot X(z_4,\bar{z}_4)}: \prod_{i=1}^{3} :\underbrace{c\bar{c}\, e^{ik_i X}}_{\text{all fields at } z_i, \bar{z}_i}: \right\rangle$$

- This splits into a product of independent X & $c, \bar{c}$-parts. The X-part is (ignoring normal ordering):

$$\int DX \exp\left[\frac{1}{2\pi\alpha'} \int d^2 z\, X\cdot \partial\bar{\partial} X + i \int d^2 z\, J\cdot X\right]$$

$$\text{where } J(z,\bar{z}) = \sum_{i=1}^{4} k_i\, \delta^2(z - z_i, \bar{z} - \bar{z}_i).$$

- We no split $\displaystyle\int \prod_{\mu=1}^{26} DX^\mu \to \int \underbrace{\prod_{\mu=1}^{26} D\tilde{X}^\mu}_{\substack{\text{without}\\ \text{zero-mode}}} \int \underbrace{d^{26}x}_{\text{zero-mode (on } S^2)}$

finding: $$\int d^{26}x \exp\left(i \sum_{i=1}^{4} k_i \cdot x\right) \sim \delta^{26}\left(\sum_{i=1}^{4} k_i\right).$$

- The non-zero-mode-part of the DX-integration is easily evaluated since the exponent is a Gaussian with an invertible operator in the quadratic part:

$$\Rightarrow \sim \exp\left(\frac{\pi\alpha'}{2} \int d^2 z\, d^2 z'\, J(z,\bar{z}) \frac{1}{\partial\bar{\partial}} \cdot J(z', \bar{z}')\right)$$

- $1/(\partial\bar\partial)$ is just the Green's fct., which we know. Thus, we find:

$$\sim \exp\Big(\frac{\alpha'}{2}\sum_{i,j} k_i\cdot k_j \ln|z_i - z_j|\Big).$$

[In fact, more care is required since we excluded the zero-mode. Cf. Polch. I for details.]

- Normal ordering, as it appears in the original expression, implies a restriction to $i \neq j$, i.e.

$$\sim \prod_{i<j} |z_i - z_j|^{\alpha' k_i\cdot k_j}$$

- Finally, the ghost part further factorizes into two 3-point fcts.:

$$\langle c(z_1)c(z_2)c(z_3)\rangle \quad \& \quad \langle \bar c(\bar z_1)\bar c(\bar z_2)\bar c(\bar z_3)\rangle.$$

- Here we can use the general fact that, for primary fields, conf. symmetry completely fixes the space-time dependence of 2- & 3-point fcts. In particular, for weights $h_i, \bar h_i$:

$$G^{(3)}(z_i,\bar z_i) = C_{123}\,\frac{1}{z_{12}^{h_1+h_2-h_3}\, z_{23}^{h_2+h_3-h_1}\, z_{13}^{h_1+h_3-h_2}}\cdot\frac{1}{(\text{same with } \bar z_i \,\&\, \bar h_i)}$$

where $z_{ij} = z_i - z_j$

[Cf. P. Ginsparg's lecture notes on the Web, Sect. 2.
The general idea is that conf. symm. on S^2 is sufficient to move 3 arbitrary points to, e.g., $z_1 = 0$, $z_2 = \infty$, $z_3 = 1$. In this way any 3-point-fct. can be related to just one number, C_{123}.]

- Since we have $h = -1$ for c, we get

$$\langle c(z_1)c(z_2)c(z_3)\rangle \sim (z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$$

& analogously for $\bar c$.

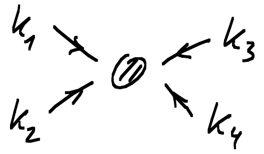
- Combining all z-dependent parts, we find

$$\sim |z_{12}|^{\alpha' k_1\cdot k_2 + 2}\,|z_{13}|^{\alpha' k_1\cdot k_3 + 2}\,|z_{23}|^{\alpha' k_2\cdot k_3 + 2}\,|z_{34}|^{\alpha' k_3\cdot k_4}\cdot|z_{14}|^{\alpha' k_1\cdot k_4}\cdot|z_{24}|^{\alpha' k_2\cdot k_4}$$

- As discussed before, conf. symm. on S^2 (i.e. $SL(2,\mathbb{C})$) allows us to move 3 points to 3 arbitrary positions. We choose

$$(z_1, z_2, z_3) = (0, 1, \infty).$$

- It is now convenient to work with the Mandelstam variables familiar from QFT or even exp. particle physics:



;
$$s = -(k_1+k_2)^2 = -2k_1\cdot k_2 + 2m^2$$
$$t = -(k_1+k_3)^2 = -2k_1\cdot k_3 + 2m^2$$
$$u = -(k_1+k_4)^2 = -2k_1\cdot k_4 + 2m^2$$

$\left(m^2 = -\frac{4}{\alpha'} \text{ in our case}\right)$

- It is easy to show that
$$s+t+u = \underbrace{\sum_{i=1}^{4} m_i^2}_{\text{general}} = \underbrace{4m^2}_{\text{case of 4 identical particles}}$$

- Hence
$$\alpha'(k_1\cdot k_3 + k_2\cdot k_3 + k_3\cdot k_4) = \frac{\alpha'}{2}(-s-t-u+6m^2)$$
$$= \frac{\alpha'}{2}\cdot 2m^2 = -4.$$

- It follows that the product of the 3 terms involving z_3 approaches 1 as $z_3 \to \infty$.

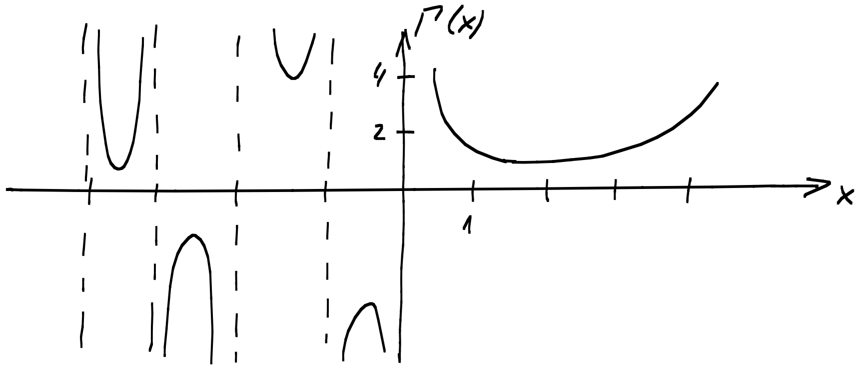
- Thus, renaming $z_4 \to z$, we find in total
$$\mathcal{A}_4 = i g_s^2 C_{S_2} (2\pi)^{26} \delta^{26}\Big(\sum_{i=1}^{4} k_i\Big) \int d^2z\, |z|^{-\frac{\alpha'}{2}u - 4} |1-z|^{-\frac{\alpha'}{2}t - 4}$$
normalization factor

- Let us define
$$C(a,b,c) = \int d^2z\, |z|^{2a-2} |1-z|^{2b-2} \qquad (c \equiv 1-a-b)$$
$$= \frac{2\pi\,\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-b)}$$
cf. Tong's notes + problems

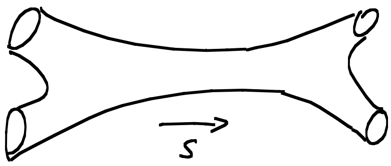
- Thus: $$\mathcal{A}_4 = i g_s^2 C_{S_2}\, 2\pi^{26}\delta^{26}(\ldots)\,\frac{\Gamma(-1-\frac{\alpha' s}{4})\,\Gamma(-1-\frac{\alpha' t}{4})\,\Gamma(1-\frac{\alpha' u}{4})}{\Gamma(2+\frac{\alpha' s}{4})\,\Gamma(2+\frac{\alpha' t}{4})\,\Gamma(2+\frac{\alpha' u}{4})}$$

Up to the normalization (to be fixed, at least in principle, shortly) this is the final answer!

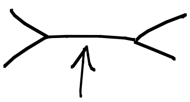
- The Γ-fct. has poles at all non-positive integers:



- Thus, our amplitude has many poles. Focus first on poles due to "s-channel exchange":



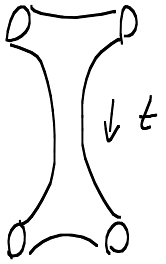
$= \sum_n$



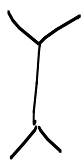
n-th excitation with mass $M_n^2 = \frac{4}{\alpha'}(n-1)$.

Indeed, $\Gamma(-1-\frac{\alpha' s}{4})$ has poles at $s = M_n^2$ $(n \geq 0)$.

- The very same amplitude has poles corresponding to "t-channel-exchange" (& u-channel exchange):



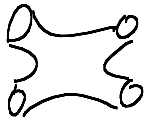
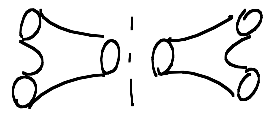
$= \sum_n$



(This time the poles come from the second Γ-fct. in the numerator.)

- Note that this is very different from QFT, where it takes <u>different</u> diagrams to get all those poles. This is a first hint that string theory encodes all field-theoretic effects (at a given loop order, here just tree-level) in one "diagram" (more precisely: one world-sheet).

- Furthermore, it suggests a way to fix C_{S^2} (assuming that it depends just on the world sheet-topology, not the number of vertex-operators): One just needs to identify the pole of

 with the product  .

$\Rightarrow$ equation where C_{S^2} appears linearly on the left & quadratically on the right. We will not work this out here.

- Finally, we briefly discuss the high-energy behaviour of amplitudes (and hence the crucial question of UV finiteness of string theory):

– One can convince oneself that $\mathcal{A}_4(s,t)$ at $s,t \to \infty$ with s/t fix describes "fixed-angle high-energy scattering".

– Use $\Gamma(x) \sim e^{x \ln x}$ at large x to show that

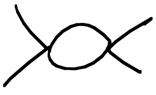
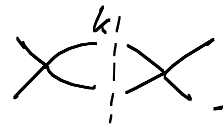
$$\mathcal{A}_4 \sim \delta(\Sigma k_i) \exp\left(-\frac{\alpha'}{2}(s \ln s + t \ln t + u \ln u)\right),$$

leading to an exponential suppression at large energies!

– This is very different from field theory, where for 4 scalars

$$\mathcal{A}_4 \sim \delta(..)\lambda \qquad (\mathcal{L}_I \sim \lambda \varphi^4)$$

(and for vectors or gravitons such amplitudes even grow due to the contraction of momenta of the particles involved)

– Through $\Big[$  $\sim \int_k$  $\Big]$ this "stringy" soft high-energy behaviour of amplitudes is at the origin of UV finiteness!