

## 11 World Sheet Supersymmetry

Motivation: • need fermions (target space spinors)  
• need to avoid tachyon

### 11.1 Global SUSY on the WS

• let us add fermions at the most fundamental level:

$\sigma^a$  - WS coords.  $\oplus$   $\theta_\alpha$  - fermionic WS coords.  
(Grassmann variables with spinor index  $\alpha$ )

Recall basic facts about spinors, in particular in 2 dims.:

$\sigma^a \rightarrow \Lambda^a_b \sigma^b$  - WS Lorentz group; vector repr.

$\theta_\alpha \rightarrow S_\alpha^\beta \theta_\beta$  - " - ; spinor repr.

as usual:  $\Lambda^a_b = (e^{i\varepsilon^{cd}} \gamma_{cd})^a_b$

$$S_\alpha^\beta = (e^{i\varepsilon^{cd} \{i[\gamma_c, \gamma_d]/4\}})_\alpha^\beta$$

e.g.  $\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ;  $\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  fulfil  $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$   
( $\eta^{ab} = \text{diag}(-1, 1)$ )

•  $\gamma$ 's imaginary  $\Rightarrow$   $S$  real  $\Rightarrow$  can demand reality of  $\theta$ ,  
i.e.  $\theta^* = \begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix}^* = \begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix} = \theta$

[This is just the particularly simple 2d-version of the familiar Majorana condition  $\psi = \psi^c \equiv C\bar{\psi}^T$

- in short: We have added a 2d Majorana spinor  $Q_\alpha$  to our bosonic coords.  $\sigma^a$  and thus introduced superspace.

- We also have to generalize our fields:

$$X^\mu(\sigma) \rightarrow Y^\mu(\sigma, \theta) \quad (\text{"superfields"})$$

- Taylor expansion:

$$Y^\mu(\sigma, \theta) = X^\mu(\sigma) + \bar{\theta} \psi^\mu(\sigma) + \frac{1}{2} \bar{\theta} \theta B^\mu(\sigma)$$

(Note: -  $\bar{\theta} \equiv \theta^\dagger \gamma^0$  - as in 4d

- e.g., a linear term in  $\theta$  does not need to be introduced since  $\theta^\dagger = \theta^T$ .)

### Symmetries of superspace:

- 2d Poincare for  $\sigma^a$
- could try to add translations in  $\theta$  (generated by  $\frac{\partial}{\partial \theta}$ ), but that would be "boring" since it doesn't mix  $\theta$  &  $\sigma$ .

- better:  $\frac{\partial}{\partial \bar{\theta}^\alpha} \rightarrow \boxed{\frac{\partial}{\partial \bar{\theta}^\alpha} + i(\gamma^a \theta)_\alpha \partial_a \equiv Q_\alpha}$

"SUSY generator" (also a Majorana-spinor)

- the symm. algebra now includes the relation

$$\{Q_\alpha, \bar{Q}^\beta\} = -2i(\gamma^a)_\alpha{}^\beta \partial_a \quad (*)$$

Note:  $\{Q, \bar{Q}\}$  &  $\{\bar{Q}, Q\}$  are not independent  
 since  $(Q_\alpha)^* = \bar{Q}^\alpha$ .

To check (\*), let us first determine  $\bar{Q}^\alpha$ :

$$1) \overline{i(\gamma^a \theta)_\alpha \partial_a} = -i \overline{(\gamma^a \theta)^\alpha} \partial_a = -i (\theta^\dagger \gamma^a \gamma^0)^\alpha \partial_a = \\ = -i (\theta^\dagger \gamma^0 \gamma^0 \gamma^a \gamma^0)^\alpha \partial_a = -i (\bar{\theta} \gamma^a)^\alpha \partial_a$$

2) To find  $\overline{\left(\frac{\partial}{\partial \bar{\theta}}\right)}$ , recall that Majorana spinors satisfy  $\bar{\psi} \chi = \bar{\chi} \psi$  and try to enforce this on  $Q$ :

$$\bar{\epsilon} Q = \bar{Q} \epsilon \quad \text{or} \quad \bar{\epsilon} \cdot \overline{\left(\frac{\partial}{\partial \bar{\theta}}\right)} = \overline{\left(\frac{\partial}{\partial \bar{\theta}}\right)} \cdot \epsilon;$$

calculate

$$a) \left(\bar{\epsilon}^\alpha \frac{\partial}{\partial \bar{\theta}^\alpha}\right) (\bar{\theta} \psi) = \bar{\epsilon}^\alpha \delta_\alpha^\beta \psi_\beta = \bar{\epsilon} \psi$$

$$b) \left(\frac{\partial}{\partial \bar{\theta}^\alpha} \epsilon_\alpha\right) (\bar{\psi} \theta) = \epsilon_\alpha \bar{\psi}^\beta \delta_\beta^\alpha = -\bar{\psi} \epsilon = -\bar{\epsilon} \psi$$

$\Rightarrow$  We must define  $\overline{\left(\frac{\partial}{\partial \bar{\theta}}\right)} = -\left(\frac{\partial}{\partial \theta}\right)$ , or

$$\boxed{\bar{Q}^\alpha = -\frac{\partial}{\partial \theta^\alpha} - i(\bar{\theta} \gamma^a)^\alpha \partial_a}$$

• Now it is easy to check (\*):

$$\left\{ \frac{\partial}{\partial \bar{\theta}^\alpha} + i(\gamma^a \theta)_\alpha \partial_a, -\frac{\partial}{\partial \theta^\beta} - i(\bar{\theta} \gamma^a)^\beta \partial_a \right\} =$$

$$= -i(\gamma^b)_\alpha{}^\beta \partial_b - i(\gamma^a)_\alpha{}^\beta \partial_a = -2i(\gamma^a)_\alpha{}^\beta \partial_a \quad \checkmark$$

Note: The structure  $\{Q, \bar{Q}\} \sim P$  found above is (together with the usual  $P_\mu, J_{\mu\nu}$  - Poinc. alg. & the standard action of  $J_{\mu\nu}$  on  $Q_\alpha$  as appropriate for a spinor) the heart of the SUSY algebra.

The SUSY algebra is the unique (under certain conditions) extension of the Poinc. alg.

Note: This extension involves the generalization of the symmetry concept based on Lie algebras to a symmetry concept allowing for super Lie algebras (or super Lie groups). [both commutators & anti-commutators appear!]

•  $Q$  acts on superfields as a differential operator:

- infinitesimal trf.:  $\delta_\epsilon Y = (\bar{\epsilon} Q) \cdot Y(\sigma, \theta)$

- finite version: ( $\rightarrow$  problems)

$$e^{\bar{\epsilon} Q} Y = Y\left(\underbrace{\sigma^a + i\bar{\epsilon}\gamma^a\theta + \frac{1}{2}i\bar{\epsilon}\gamma^a\epsilon}_{\text{Translation in } \theta}, \underbrace{\theta + \epsilon}_{\text{Translation in } \sigma}\right)$$

Translation in  $\theta$  and corresponding translation in  $\sigma$

• Since  $Q$  is a diff. operator, it acts on products of superfields as prescribed by the Leibnitz-rule:

$$(\bar{\epsilon} Q)(Y_1 Y_2) = ((\bar{\epsilon} Q)Y_1)Y_2 + Y_1((\bar{\epsilon} Q)Y_2).$$

- To write down actions, we need supercovariant derivatives:  $D_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} - i(\gamma^a \theta)_\alpha \partial_a$

$$\bar{D}^\alpha = -\frac{\partial}{\partial \theta_\alpha} + i(\bar{\theta} \gamma^a)^\alpha \partial_a.$$

- Crucial feature:  $\{D_\alpha, Q_\beta\} = 0 \Rightarrow$

$\parallel$  If  $Y$  transforms as a superfield, then so does  $D_\alpha Y$ .  $\parallel$

Actions:  $S = \int d^2\sigma d^2\theta \mathcal{L}(Y_1, Y_2, \dots, DY_1, \dots)$

SUSY invariance is obvious since

$$\delta_\epsilon \mathcal{L} = (\epsilon Q) \mathcal{L} = \text{total derivative}$$

Of course, we need to know that  $\int d^2\theta \frac{\partial}{\partial \theta_\alpha} f(\theta) = 0$ .

To see this, consider e.g.

$$\int d\theta_1 \frac{\partial}{\partial \theta_1} f(\theta_1) = \int d\theta_1 \frac{\partial}{\partial \theta_1} (c_0 + c_1 \theta_1) = \int d\theta_1 \cdot c_1 = 0.$$

This extends easily to many variables.

The natural action for the string WS reads:

$$\parallel S = \frac{i}{4\pi} \int d^2\sigma d^2\theta (\bar{D}^\alpha Y^{\prime n})(D_\alpha Y_n) \parallel$$

Taylor-expanding in  $\theta$  and performing the  $d^2\theta$ -integration one finds the "component action": ( $\rightarrow$  problems)

$$S = -\frac{1}{2\pi} \int d^2\sigma \left( \partial_a X^\mu \partial^a X_\mu - i \bar{\psi}^\mu \not{\partial} \psi_\mu - \underbrace{B^\mu B_\mu}_{=0, \text{ since } B_\mu = 0 \text{ by EOMs.}} \right)$$

$$\equiv \gamma^a \partial_a$$

↓  
B is an "auxiliary" field.

• In short: We have added a fermionic  $\psi^\mu$  (Maj. spinor) for every  $X^\mu$  under the requirement of WS SUSY.

• All of this could have been done without SFs (= "component formulation"). To see this, consider first how the component fields transform under SUSY:

$$- Y^\mu(\sigma, \theta) = X^\mu(\sigma) + \bar{\theta} \psi^\mu(\sigma) + \frac{1}{2} \bar{\theta} \theta B^\mu(\sigma)$$

- Calculate  $\delta_\epsilon Y$  and expand in  $\theta$ :

$$\delta_\epsilon Y^\mu(\sigma, \theta) \equiv \delta X^\mu(\sigma) + \bar{\theta} \delta \psi^\mu(\sigma) + \frac{1}{2} \bar{\theta} \theta \delta B^\mu(\sigma)$$

This defines the quantities  $\delta X, \delta \psi, \delta B$  on the

- We obtain ( $\rightarrow$  problems) r.h. side.

$$\left\| \begin{array}{l} \delta X^\mu = \bar{\epsilon} \psi^\mu \\ \delta \psi^\mu = -i(\gamma^a \epsilon) \partial_a X^\mu + B^\mu \epsilon \\ \delta B^\mu = -i \bar{\epsilon} \gamma^a \partial_a \psi^\mu \end{array} \right\| \text{"off-shell" component formulation of SUSY}$$

- Using EOMs ( $\mathcal{B}^{\dagger} = 0$  &  $\mathcal{D}\psi^{\dagger} = 0$ ) we obtain the

"on-shell" formulation: 
$$\left\| \begin{array}{l} \delta X^{\dagger} = \bar{\epsilon} \psi^{\dagger} \\ \delta \psi^{\dagger} = -i(\gamma^a \epsilon) \partial_a X^{\dagger} \end{array} \right\|$$

From this (using EOMs), the SUSY algebra can be verified by working out the action of  $[\delta_{\epsilon}, \delta_{\epsilon'}]$  on every field:

$$[\delta_{\epsilon}, \delta_{\epsilon'}] = \uparrow 2i \bar{\epsilon} \gamma^a \epsilon' \partial_a$$

The different sign compared to the  $Q$ -algebra is obtained since the  $Q$ 's act on superspace while the  $\delta_{\epsilon}$  defined above act on the fields (cf. active vs. passive description of symmetries).

Thus: One can simply start with the above algebra, realize it on  $x, \psi$  by defining appropriate  $\delta_{\epsilon} x$  &  $\delta_{\epsilon} \psi$ , and then construct an invariant Lagrangian ( $\sim (\partial X)^2 - i \bar{\psi} \mathcal{D} \psi$ ). This is all we need. In this approach, neither  $Q$  nor  $\mathcal{B}^{\dagger}$  have ever appeared.

Next step: need to make this action diff.-invariant while maintaining SUSY.

## 11.2 Supergravity on the WS

- so far: flat WS with flat SUSY; no metric
- could generalize our previous discussion to a "curved superspace" parameterized by  $\sigma$  &  $\theta$  and find metric as a component field of a "superspace metric".
- simpler (and sufficient for our purposes):

Write down diff.-inv. version of our action and extend it systematically to ensure invariance under (local) SUSY.

- diff.-inv. action:

$$S = -\frac{1}{2\pi} \int d^2\sigma \text{Tr} \{ h^{ab} \partial_a X^\mu \partial_b X_\mu - i \bar{\Psi} \gamma^a \nabla_a \Psi \}$$

- to understand  $\nabla_a \Psi$  we need a vielbein (to be general, in  $d$  dimensions):

$e_a^m(\sigma)$  - set of  $d$  orthog. vectors at each point  $\sigma$

$$e_a^m e_b^n = e_a^m e_b^n h^{ab} = \eta^{mn} \quad (\Rightarrow h_{ab} = \eta_{mn} e_a^m e_b^n)$$

"curved" or "Einstein" indices      "frame" or "Lorentz" indices

- extra symm.: local Lorentz:  $e_a^m(\sigma) \rightarrow \Lambda_n^m(\sigma) e_a^n(\sigma)$

(Any vector  $v^a$  can be written with a Lorentz index:

$$v^m \equiv e_a^m v^a. \text{ Then it transforms as } v \rightarrow \Lambda \cdot v)$$



- spinor: local Lorentz:  $\psi \rightarrow S(\sigma)\psi$  where  $S(\sigma)$  belongs to  $\Lambda(\sigma)$  as usual.
- as in a gauge theory,  $\nabla_a$  is defined by demanding  $\nabla_a v^m \rightarrow \Lambda^m_n \nabla_a v^n$  if  $v^m \rightarrow \Lambda^m_n v^n$
- $\Rightarrow \nabla_a = \partial_a + \omega_a$  with  $\omega_a \in \text{Lie}(SO(1, d-1))$  in the appropriate representation (usually vector or spinor).
- If  $\nabla_a$  is applied to an object that also carries Einstein indices, the appropriate covariantization w.r.t. these indices (using  $\Gamma_{ab}^c$ ) must be included.
- Thus, demanding  $\nabla_a e_b^m = 0$  fixes  $\omega_a^m$  in terms of  $\Gamma_{ab}^c$ :  
 $(0 \stackrel{!}{=} \nabla_a e_b^m = \partial_a e_b^m + (\omega_a)^m_n e_b^n - \Gamma_{ab}^c e_c^m)$
- this is the natural spin connection  $\omega$  coming with the vielbein  $e_a^m$ .

- the relevant explicit forms are:

$$(\omega_a)^m_n = \omega_a^{(pq)} (i\gamma_{(pq)})^m_n \quad - \text{vector}$$

$$(\omega_a)_{\alpha\beta} = \omega_a^{(pq)} (i\frac{1}{4}[\gamma_p, \gamma_q])_{\alpha\beta} \quad - \text{spinor}$$



this is the  $\omega$  appearing in  $\nabla_a \psi$

- note also:  $\gamma^a \equiv e_m^a \gamma^m$

- Since  $\epsilon$  links  $X$  and  $\psi$  ( $\delta X = \bar{\epsilon}\psi$ ) and, under local Lorentz trfs.,  $\left\{ \begin{array}{l} X \rightarrow X \\ \psi \rightarrow S(\sigma)\psi \end{array} \right\}$ , we must allow for a  $\sigma$ -dependence of  $\epsilon$ :  $\epsilon = \epsilon(\sigma)$  ("local SUSY").

- even if we treat  $e_a^m$  just as a small perturbation around flat space ( $e_a^m = \delta_a^m$ ), it needs a superpartner:  $\delta e_a^m = -2i\bar{\epsilon}\gamma^m\chi_a$

↑  
"gravitino" or

"Rarita-Schwinger field"

( $\chi_a = (\chi_a)_\alpha$  is the smallest appropriate Lorentz-repres.)

- our previous action  $S$  is not inv. under such an extended symmetry. It is just the "quadratic approximation"  $S_2$ :

$$S_2 = -\frac{1}{2\pi} \int d^2\sigma e \left\{ h^{ab} \partial_a X^\mu \partial_b X_\mu - i\bar{\psi} \gamma^a \nabla_a \psi \right\}$$

$$\uparrow$$

$$= \det(e_a^m) = \sqrt{-\det(h_{ab})}$$

- We construct the full  $S$  by the Noether method:
- calculate  $\delta_\epsilon S_2$  at quadratic order in the fields:

$$\delta_\epsilon S_2 \sim \int (\nabla_a \bar{\epsilon}) J^a \quad \text{with} \quad J^a = \underbrace{\frac{1}{2} \bar{\psi} \gamma^a \psi \partial_b X^\mu}_{\text{"supercurrent"}}$$

("supercurrent"  
(= Noether current of global SUSY trf.))

Note: - The calculation is simplified by the fact that  $\int \bar{\psi} \gamma^a \nabla_a \psi = \int \bar{\psi} \gamma^a \partial_a \psi$  in  $d=2$ .

Note also: - at this order, we do not need to take  $\delta e_a^m$  into account.

- next, we try to make the action invariant by adding a term involving the gravitino  $\chi_a$  with  $\delta\chi_a = \nabla_a \epsilon$ :

$$S_3 = -\frac{1}{\pi} \int d^2\sigma e \bar{\chi}_a \gamma^b \gamma^a \psi^i \partial_b X_i$$

$\sim J^a$  as derived above

- calculate  $\delta_\epsilon (S_2 + S_3)$  to cubic order in the fields:

-  $\delta S_2$  from  $\delta\psi$  &  $\delta X$  - compensated by  $\delta S_3$  from  $\delta\chi$

-  $\delta S_3$  from  $\delta\psi$  &  $\delta X$

-  $\delta S_2$  from  $\delta e_a^m$

} can be compensated by by

1) modified trf. of  $\psi$ :

$$\delta\psi = -i\gamma^a \epsilon (\partial_a X - \bar{\psi} \chi_a)$$

2) extra term:

$$S_4 = -\frac{1}{4\pi} \int d^2\sigma e (\bar{\psi} \psi) (\bar{\chi}_a \gamma^b \gamma^a \chi_b)$$

- finally:  $S = S_2 + S_3 + S_4$  is inv. under:

$$\delta X = \bar{\epsilon} \psi; \quad \delta\psi = -i\gamma^a \epsilon (\partial_a X - \bar{\psi} \chi_a); \quad \delta e_a^m = -2i \bar{\epsilon} \gamma^a \chi_a; \quad \delta\chi_a = \nabla_a \epsilon$$

- Note: Neither  $e_a^m$  nor  $\chi_a$  have kinetic terms

( $e\mathcal{R}$  is total derivative and the natural kinetic term for  $\chi$ ,  $\bar{\chi}_a \gamma^a \gamma^b \gamma^c \nabla_b \chi_c$ , vanishes since there is no 3rd-rank antisymm. tensor in  $d=2$ )

Note: This is different in  $d=4$ , where the symm.

$$\delta e_\mu^m = -\frac{i}{2} \bar{\epsilon} \gamma^m \chi_\mu, \quad \delta \chi_\mu = \nabla_\mu \epsilon$$

is respected by the non-trivial action

$$S = \int d^4x e \left\{ -\frac{1}{2} \mathcal{R} - \frac{i}{2} \bar{\chi}_\mu \gamma^{\mu\nu\sigma} \nabla_\nu \chi_\sigma \right\}$$

"real" fermion with 2 on-shell d.o.f.s

$$\sim \gamma [\mu \gamma^\nu \gamma^{\sigma]}$$

(antisymmetrized)

This 4d-supergravity action can be extended to couple to a 4d supersymm. theory of scalars & spinors (like  $X$  &  $\psi$  above), but this is much more complicated...

• returning to 2 dims., we find that  $S$  is inv. under

$$\delta X = 0, \quad \delta e_a^m = \omega e_a^m, \quad \delta \psi = -\frac{1}{2} \omega \psi, \quad \delta \chi_a = \frac{1}{2} \omega \chi_a$$

(Weyl rescalings)

and their fermionic counterpart:

$$\delta X = \delta e = \delta \psi = 0, \quad \delta \chi_a = i \gamma_a \eta \quad (\eta - \text{infinitesimal})$$

(problem: check that this is a symmetry!) Majorana spinor)

This makes our theory "super-Weyl-invariant".

(This is often called "superconformal" which, however, clashes with our previous use of the term conformal for flat-space field theories only.)