12 Quantization of the Superstring

12.1 Constraints

bosonic gauge symms.
- Diff. (2 d.o.f.)
- Lokal Lorentz (1 d.o.f.)
- Weyl (1 d.o.f.)

enough freedom to set $e_a{}^m = \delta_a{}^m$

fermionic gauge symms.
- local SUSY ; δ_ϵ (2 d.o.f.)
- super-Weyl ; δ_η (2 d.o.f.)

... to set $\chi_a = 0$
(cf. BLT for details)

$\Rightarrow$ "flat" action: $$-\frac{1}{2\pi}\int d^2\sigma \left[(\partial_a X^\mu)(\partial^a X_\mu) - i\bar\psi^\mu \gamma^a \partial_a \psi_\mu \right]$$

- Exactly as before: EOMs of eliminated fields become constraints

1) $\frac{\delta S}{\delta e_a{}^m} \overset{!}{=} 0 \iff \frac{\delta S}{\delta h_{ab}} \overset{!}{=} 0$; As a result, the "Boson" constraint remains unchanged:

$\boxed{T_{ab} = 0}$ with $T_{ab} = \partial_a X \cdot \partial_b X + \frac{i}{2}\bar\psi \cdot \gamma_{(a}\partial_{b)}\psi - \frac{1}{2}\delta_{ab}\left\{ \begin{matrix} \text{trace of} \\ \text{prev. express.} \end{matrix} \right\}$

$\equiv$ "symmetrization"

2) $$0 \overset{!}{=} \frac{\delta}{\delta \chi^a}\Big(S_2[e,X,\psi] + \underset{\text{linear}}{S_3[e,X,\psi,\chi]} + \underset{\text{quadratic}}{S_4[e,\psi,\chi]} \Big)$$

$$\iff 0 \overset{!}{=} \frac{\delta}{\delta\chi^a} S_3 = -\frac{2e}{\pi} J_a \, ,$$ where J is the supercurrent

(The appearance of J is natural since we defined S_3 as $S_3 \sim \int \bar\chi J$ to cancel $\delta_\epsilon S_2$.)

$\boxed{(J_a)_\alpha = 0}$

12.2 Mode expansion

$S = S_B + S_F$; focus on S_F ; use light-cone coordinates:

$$\gamma^\pm = \gamma^0 \pm \gamma^1 = 2i \begin{cases} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \text{for "+"} \\ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & \text{for "−"} \end{cases}$$

Recall notation $\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$; $\Rightarrow$ $\boxed{S_F = \frac{i}{\pi}\int d^2\sigma \left(\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+\right)}$

[This justifies our use of +/- for spinor indices since we now discover that ψ_- & ψ_+ are right & left-movers respectively.]

<u>Closed string</u>

$$X(\tau, \sigma+\pi) = X(\tau,\sigma) \quad \text{vs.} \quad \psi(\tau,\sigma+\pi) = \pm \psi(\tau,\sigma)$$

(naively, one would think that even a phase $e^{i\alpha}$ would be OK, but the Majorana cond. prevents this)

Since ψ_+ & ψ_- decouple, we have 4 choices:

$\psi_+(\sigma+\pi) = +\psi_+(\sigma)$; $\psi_-(\sigma+\pi) = +\psi_-(\sigma)$: R-R ("Ramond-Ramond")
$\psi_+(\sigma+\pi) = +\psi_+(\sigma)$; $\psi_-(\sigma+\pi) = -\psi_-(\sigma)$: R-NS ("Ramond-Neveu/Schwarz")
$-$ $+$: NS-R
$-$ $-$: NS-NS

• This fixes the Fourier decomposition. E.g., in the R-NS sector we have:

$$\psi_+^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}} \tilde{\psi}_r^\mu \cdot e^{-2ir(\tau+\sigma)} \; ; \; \psi_-^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu \cdot e^{-2ir(\tau-\sigma)}$$

(and analogously for the other 3 sectors)

<u>Open string</u>

• We already know that we can "derive" the open from the closed string by going from S^1 to $S^1/\mathbb{Z}_2$ (with $\mathbb{Z}_2 : \sigma \to -\sigma$).

• This $\mathbb{Z}_2$ (Parity!) of course exchanges ψ_+ & ψ_-, making R-NS & NS-R inconsistent. Thus, only R-R & NS-NS remain (but without independent $\tilde{\psi}_r^\mu$-oscillators):

$$\underbrace{\psi_\pm^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}} \psi_r^\mu e^{-ir(\tau\pm\sigma)}}_{\text{"R"}} \Bigg/ \underbrace{\psi_\pm^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu e^{-ir(\tau\pm\sigma)}}_{\text{"NS"}}$$

convention

- The oscillator anticommut. relations now follow in standard fashion:

$$\Pi_+^\mu = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_{+\mu}} = \frac{i}{\pi}\psi_+^\mu \quad \Rightarrow \quad \{\psi_+^\mu(\sigma), \Pi_+^\nu(\sigma')\} = i\delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\Rightarrow \quad \{\psi_+^\mu(\sigma), \psi_+^\nu(\sigma')\} = \pi\,\delta(\sigma-\sigma')\eta^{\mu\nu}$$

(at equal τ)

... and analogously for ψ_-.

$$\Rightarrow \boxed{\begin{aligned}[][\alpha_m^\mu, \alpha_n^\nu] &= m\,\delta_{m+n}\,\eta^{\mu\nu} \\ \{\psi_r^\mu, \psi_s^\nu\} &= \delta_{r+s}\,\eta^{\mu\nu}\end{aligned}}$$

– as before

with $\begin{cases} r,s \in \mathbb{Z} & (R) \\ r,s \in \mathbb{Z}+\frac{1}{2} & (NS)\end{cases}$

12.3 Old covariant quantization

- Focus on the open string (just T_{++} & J_+ as constraints); Fourier-expand:

$$L_m \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma\, e^{im\sigma}\, T_{++}$$

$$G_r \equiv \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma\, e^{ir\sigma}\, J_+$$

& find: $L_m = \frac{1}{2} : \left\{ \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\alpha_{m+n} + \sum_{r\in\mathbb{Z}+\nu}\left(r+\frac{m}{2}\right)\psi_{-r}\cdot\psi_{m+r}\right\} :$

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_{-n}\psi_{r+n}$$

$$\boxed{\nu \equiv \begin{cases} 0 & - R \\ 1/2 & - NS \end{cases}}$$

- These operators generate the "Super-Virasoro algebra"

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\,\delta_{m+n}$$

$$[L_m, G_r] = \left(\tfrac{1}{2}m - r\right)G_{m+r}$$

$$\{G_r, G_s\} = 2L_{r+s} + B(r)\,\delta_{r+s}$$

where:

$$A(m) = \begin{cases} \frac{1}{8}D(m^3-m) & NS \\ \frac{1}{8}Dm^3 & R \end{cases}$$

$$B(r) = \begin{cases} \frac{1}{2}D\left(r^2-\frac{1}{4}\right) & NS \\ \frac{1}{2}Dr^2 & R \end{cases}$$

(Note: · The "$Dm^3/12$" of the bosonic string has been enhanced by the fermions ; The asymm. between NS & R can be avoided by redef. L_0)

- As before, only the "annihilator-part" of the constraints is imposed quantum mechanically:

$$\boxed{\begin{aligned} (L_m - a\,\delta_{m,0})\,|phys\rangle &= 0 \qquad (m \geq 0) \\ G_r\,|phys\rangle &= 0 \qquad (r \geq 0) \end{aligned}}$$

← no ordering-ambiguity

<u>NS sector</u> (open string)

- Vacuum: $|0,k\rangle$; Fock-space constructed using α^μ_{-m} ; ψ^μ_{-r} $(m,r>0)$

- The lowest physical-state-condition (mass-shell condition) is:

$$0 \overset{!}{=} (L_0 - a)|0,k\rangle = (\alpha' p^2 + N^\alpha + N^\psi - a)|0,k\rangle$$

where $N^\alpha = \sum_{m=1,2,..} \alpha_{-m}\cdot\alpha_m$; $N^\psi = \sum_{r=\frac{1}{2},\frac{3}{2},..} r\,\psi_{-r}\cdot\psi_r$

$$\Rightarrow \alpha' M^2 = -\alpha' k^2 = -a$$

- As before, a & $D_{crit.}$ follow purely algebraically by requiring "extra, zero-norm states" (≡ being on the boundary of developing ghosts).

- Consider, e.g., $|\phi\rangle \equiv G_{-1/2}|0,k\rangle = \left(\sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\psi_{-\frac{1}{2}+n}\right)|0,k\rangle$

$$= \alpha_0\cdot\psi_{-1/2}|0,k\rangle$$

$$(L_0 - a)|\phi\rangle = 0 \Rightarrow \alpha' M^2 = \frac{1}{2} - a$$

$$\langle\phi|\phi\rangle = \langle 0,k|G_{1/2}G_{-1/2}|0,k\rangle \underset{\text{Super-Virasoro-alg.}}{\overset{}{=}} \langle 0,k|2L_0|0,k\rangle = 2\alpha' k^2$$

$$\langle\phi|\phi\rangle = 0 \Rightarrow \boxed{a = \frac{1}{2}}$$

- Analogously: $|\phi\rangle \equiv (G_{-3/2} + \lambda G_{-1/2}L_{-1})|\tilde\phi\rangle$;

$|\phi\rangle$ phys. & zero-norm $\xRightarrow[\text{algebra}]{}$ $\lambda = 2$, $\boxed{D = 10}$

R-sector (open string)

- Crucial new point: The vacuum is necessarily <u>degenerate</u>!

To see this, recall:

$$(L_0 - a)|phys\rangle \overset{!}{=} 0 \;;\quad L_0 = \alpha' p^2 + N^\alpha + N^\psi \;;$$

$$N^\alpha \text{ as above} \;;\quad N^\psi = \sum_{r=0,1,2,\ldots} r\,\psi_{-r}\psi_r = \sum_{r=1,2\ldots} r\,\psi_{-r}\psi_r$$

$\Rightarrow$ ψ_0^μ doesn't appear in L_0 yet it satisfies the non-trivial algebra

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu} \quad (\text{Clifford alg. in } D \text{ dims.!})$$

$\Rightarrow$ Since $\alpha' M^2 = -\alpha' p^2 = N^\alpha + N^\psi - a$, we have $[\psi_0^\mu, M^2] = 0$

$\Rightarrow$ States at any mass-level (including the lowest!) carry repr. of Clifford algebra.

$\Rightarrow$ No "$|0,k\rangle$", instead $|\alpha, k\rangle$ with $\alpha = 1 \cdots 2^{D/2} = 32$

for $D = 10$, see below

$$\psi_0^\mu |\alpha, k\rangle = |\beta, k\rangle (\Gamma^\mu)_\beta{}^\alpha \frac{1}{i\sqrt{2}}$$

(with 32×32 matrices Γ^μ satisfying $\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu}$)

$\boxed{\text{The vacuum is a space-time spinor!}}$

- Determining a & D is simple:

$$\{G_0, G_0\} = 2L_0 \Rightarrow L_0 = G_0^2 \;;\quad G_0|\phi\rangle = 0 \Rightarrow L_0|\phi\rangle = 0 \;;$$

Together with $(L_0 - a)|\phi\rangle = 0$ this implies $a = 0$.

To fix D, consider $|\phi\rangle = G_0 G_{-1}|\tilde\phi\rangle$ with $G_1|\tilde\phi\rangle = 0$

$(L_0 + 1)|\tilde\phi\rangle = 0$

by shifting momentum, e.g. of vacuum

$|\phi\rangle$ phys. & zero-norm $\underset{\text{alg.}}{\Rightarrow}$ $\boxed{D = 10}$

(cf. GSW I)

- $a=0$ obviously implies $k^2=0$ for R-vacuum
- $G_0|\alpha,k\rangle = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\psi_n|\alpha,k\rangle = \alpha_0\cdot\psi_0|\alpha,k\rangle \sim \underbrace{k_\mu\psi_0^\mu}_{\hat{=}\,\not{k}}|\alpha,k\rangle$

$\Rightarrow |\alpha,k\rangle \sim u_\alpha(k)$ with $\underbrace{\not{k}u(k)=0}_{\text{massless Dirac eq.}}$

Mini-summary:

NS:	R:
$\|0,k\rangle$ – scalar tachyon $\psi^\mu_{-1/2}\|0,k\rangle$ – massless vector + massive	$\|\alpha,k\rangle$ – massless spinor + massive

12.4 Light cone & BRST quantization (quick review)

- Recall bosonic case: Using full gauge freedom (including residual gauge freedom) to go to flat gauge and set

$X^+(\tau,\sigma) = x^+ + p^+\tau \quad\Rightarrow X^-$ becomes dependent

$\Rightarrow$ only α^i-oscillators ($i=1\cdots D-2$) relevant.

- Superstring: In addition, use $\delta_{\epsilon,\eta}\chi_a = i\gamma_a\eta + \nabla_a\epsilon$ to set $\chi_a=0$ and set

$\psi^+(\tau,\sigma)=0 \Rightarrow \psi^- = \frac{1}{p_+}\sum_{i=1}^{D-2}\sum_s \alpha^i_{r-s}\psi^i_s$ becomes dependent

$\Rightarrow$ only "transverse" ψ^i-oscillators relevant.

- Hilbert space built using just α^i, ψ^i
- Lorentz-alg. anomalous unless $D=10$ & $a=\begin{cases}1/2 & (NS)\\ 0 & (R)\end{cases}$.
- Independently: Normal-ordering constant can be fixed using regularized sum of contributions of (just transverse!) oscillators:

- We already know: One boson: $a_b = -\frac{1}{2}\sum_{n=1}^{\infty} n = -\frac{1}{2}\left(\sum_{n=1}^{\infty} n^{-s}\right)_{s=-1}$

$$= -\frac{1}{2}\zeta(-1) = -\frac{1}{2}\left(-\frac{1}{12}\right) = \frac{1}{24}$$

- One periodic ($\equiv R$) fermion: $a_f^R = -\frac{1}{2}\left\langle \sum_{r=-\infty}^{\infty} r\,\psi_{-r}\psi_r \right\rangle$

$$= -\frac{1}{2}\left\langle \sum_{r=1}^{\infty}\left(r\psi_{-r}\psi_r + (-r)\psi_r\psi_{-r}\right)\right\rangle = -\frac{1}{2}\sum_{r=1}^{\infty}(-r) = -\frac{1}{24}$$

related to $\{\psi_r, \psi_{-r}\} = 1$ vs. $[\alpha_m, \alpha_{-m}] = m$

- One antiperiodic ($\equiv NS$) fermion: $a_f^{NS} = -\frac{1}{2}\sum_{r=0}^{\infty}\left(-\left(r+\frac{1}{2}\right)\right)$

$$= \frac{1}{2}\left(\sum_{r=0}^{\infty}(r+q)^{-s}\right)_{\substack{s=-1\\ q=1/2}} = \frac{1}{2}\zeta(-1,q)\Big|_{q=\frac{1}{2}}$$

$$= \frac{1}{48}$$

($\rightarrow$ Hurwitz zeta fct. as a generalization of Riemann zeta fct.)

<u>Thus:</u> Bosonic string: $a = (D-2)\frac{1}{24} = 1 \qquad (D=26)$

Superstring: R: $a = (D-2)\left(\frac{1}{24} - \frac{1}{24}\right) = 0$

NS: $a = (D-2)\left(\frac{1}{24} + \frac{1}{48}\right) = \frac{1}{2}$ ✓

(This together with the requirement that the vector should be massless fixes the critical dimensions 26 and 10!)

- Finally, in path integral quantization à la Fadeev-Popov we have:

$$\int Dh \cdots = \int D\xi \, \det\left(\frac{\delta \hat{h}^{\xi}}{\delta \xi}\right) \cdots$$

$$\int D\chi \cdots = \int D\epsilon D\eta \, \det^{-1}\left(\frac{\delta \hat{\chi}^{\epsilon,\eta}}{\delta\epsilon\,\delta\eta}\right) \cdots$$

"-1" from Grassmann integration

$$\cdots \Rightarrow \det{}^{-1}(\cdot\cdot) = \int \underset{\substack{\uparrow \\ \text{spinor}}}{D\gamma}\, \underset{\substack{\uparrow \\ \text{vector-spinor}}}{D\beta_c}\, \exp\left[-\frac{i}{2\pi}\int d^2\sigma\, e\, h^{ab}\bar{\gamma}\,\nabla_a\beta_b\right]$$

Full action: $$S = -\frac{1}{2\pi}\int d^2\sigma\, e\left\{(\partial_a X^\mu)(\partial^a X_\mu) - i\bar{\psi}^\mu\gamma^a\nabla_a\psi_\mu\right\}$$

$$-\frac{i}{2\pi}\int d^2\sigma\, e\left\{h^{ab}c^c\nabla_a b_{bc} + h^{ab}\bar{\gamma}\,\nabla_a\beta_b\right\}$$

"bc-system" (fermionic ghosts) "βγ-system" (bosonic ghosts)

Central charges:
X : D
ψ : $D/2$
bc : -26
$\beta\gamma$: $+11$

$\Rightarrow$ For the superstring we need

$$0 \overset{!}{=} D + \frac{D}{2} - 26 + 11$$

$$\Rightarrow \underline{\underline{D = 10}}$$