13 Consistent superstring theories

13.1 GSO projection (Gliozzi, Scherk, Olive)

- Quite generally, a "projection operator" P ($P^2 = P$) commuting with H ($PH = HP$) allows one to consistently "project" a QM-System to a sub-system:

$$\mathcal{H}_{new} \equiv \mathrm{Im}\, \underbrace{P}_{\text{viewed as operator on } \mathcal{H}.} \qquad (\text{or, equivalently, } \ker(P-1))$$

(i.e. one one only keeps states $|\phi\rangle$ satisfying $P|\phi\rangle = +|\phi\rangle$)

- For the open superstring, let us define $P = \frac{1}{2}(1 + (-1)^F)$ [$F \equiv$ "fermion number"] and hence <u>keep only states with even F</u>. This is the GSO projection.

- F is defined by

$$(-1)^F |0,k\rangle = -|0,k\rangle \qquad (NS)$$
$$(-1)^F |\alpha,k\rangle = |\beta\rangle \Gamma_\beta{}^\alpha \qquad (R)$$

(where $\Gamma = \Gamma^{11} \equiv \Gamma^0 \Gamma^1 \dots \Gamma^9$)

together with: $(-1)^F X^\mu = X^\mu (-1)^F$ & $(-1)^F \psi^\mu = -\psi^\mu (-1)^F$

[By this, $F = e^{\pi i F}$, is obviously only defined "mod 2", which is sufficient for us.]

- Applying the GSO projection to the open superstring we thus find:

NS: tachyon gone ; massless vector survives

R: $u(k)$ (taken to be Majorana) $\xrightarrow[GSO]{}$ $\frac{1}{2}(1+\Gamma)u(k)$ (Maj.-Weyl)
($\not{k} u(k) = 0$)

Thus, the spectrum is
$$\boxed{\begin{matrix} \text{massless vector} & + & \text{Majorana-Weyl spinor} \\ 8 & + & 8 \text{ d.o.f.} \end{matrix}}$$

- This suggests 10d (space-time rather than world-sheet) SUSY.
[<u>Deriving</u> space-time from world-sheet SUSY is possible but not trivial.]

- Indeed, assuming 10d SUSY there is precisely one 10d FT with our spectrum:

10d Super Yang-Mills (SYM) Theory

$$S = \int d^{10}x \left(-\frac{1}{4} F^2 + \frac{i}{2} \bar{\psi} \not{D} \psi \right) \quad ; \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$$(D_\mu \psi)^a = \partial_\mu \psi^a + g \underbrace{f^{abc}}_{\sim\, i (T^b)^{ac}} A^b_\mu \psi^c$$

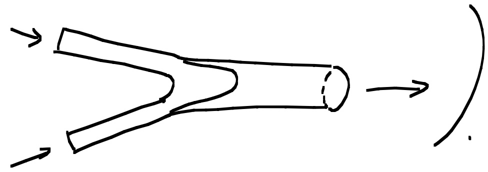
ψ is a Majorana-Weyl fermion in the adjoint representation

Counting of d.o.f.: A_μ – $(D-2) = 8$ d.o.f.

ψ – 32 complex $\xrightarrow[\text{Maj.}]{}$ 32 real $\xrightarrow[\text{Weyl}]{}$ 16 real $\xrightarrow[\text{on-shell}]{}$ 8 real ✓

- Consult the App. of Polch. II for an analysis of spinors in various dims., esp. Maj. & Weyl conditions

Note: Theories of open strings only can not be consistent

(cf. the amplitude ).

Indeed, in consistent superstring theories the above open string sector will only appear as part of a specific SUGRA + SYM theory. It is not consistent by itself. For us, it was a useful toy model to introduce GSO.

13.2 Type II superstrings

(closed strings only ; $\underbrace{\text{two gravitini in 10d}\ \equiv\ N=2\ \text{10d SUSY}}_{\Rightarrow \text{ type "II"}}$)

$$\left[g_{\mu\nu} \begin{array}{l} \xrightarrow{Q_1} \psi^1_\mu \\ \xrightarrow[Q_2]{} \psi^2_\mu \end{array} \right]$$

- Based on our experience with the bosonic string, we will directly build the closed string theory by combining left-moving (~) and right-moving sectors of the open string, respecting level matching.

- We label our building blocks (sectors) by NS/R and +/- [for the eigenvalue of $(-1)^F$]:

Sector	SO(8)-repres.	
NS −	1	– tachyon
NS +	8_v (vector)	massless
R −	8' (l.h. & r.h. spinor)	massless
R +	8	massless

- We have $L_0 = \frac{\alpha'}{4}p^2 + N - \nu$; $\tilde{L}_0 = \frac{\alpha'}{4}p^2 + \tilde{N} - \tilde{\nu}$ with $\overset{(\sim)}{\nu} = \begin{cases} 0 & (R) \\ 1/2 & (NS) \end{cases}$

- Mass-shell + Level-matching $\Rightarrow$ $\boxed{\alpha' m^2 = \frac{4}{\alpha'}(N-\nu) = \frac{4}{\alpha'}(\tilde{N}-\tilde{\nu})}$
  $(L_0 + \tilde{L}_0)$ $\quad$ $(L_0 - \tilde{L}_0)$

(Note that the spacing differs by a factor-of-4 w.r.t. open string.)

- $\overset{(\sim)}{N} - \overset{(\sim)}{\nu}$ is integer for R+, R−, NS+
- $\overset{(\sim)}{N} - \overset{(\sim)}{\nu}$ is half-integer for NS− (where the tachyon sits at $-\frac{1}{2}$)

Can not be combined!

$\Rightarrow$ Unprojected spectrum:

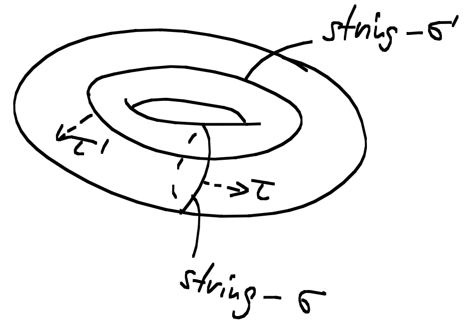
		Sector	SO(8)-repr.	
10 pairings		(NS−, NS−)	1	← tachyon
	$3\times 3 = 9$ pairings	NS+, NS+	$8_v \times 8_v$	all massless
		NS+, R−	$8_v \times 8'$	
		NS+, R+	$8_v \times 8$	
		R−, NS+	$8' \times 8_v$	
		...	...	

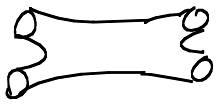
- In the open case, the GSO projection simply selected 2 out of 4 sectors.
- We can in principle try the same here, but there are clearly 2^{10} possibilities (a priori any of the 10 sectors could or could not be selected)
- It can be shown that by demanding:

1) no tachyon
2) modular invariance
3) consistency of the interacting theory

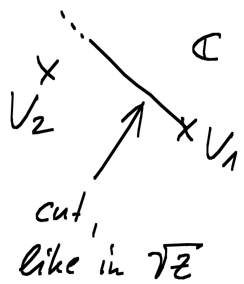
only two inequivalent choices are left: type IIA & type IIB.

- We don't explain in detail but just state the ideas:
  1) obvious
  2) possibility of T^2 automorphisms not linked to identity, i.e. reading the same T^2 with $\tau \leftrightarrow \sigma$.



  3) an excluded state should not be produced in scattering (together with 2) this is related to the existence of particular spin bundles on "interacting" world sheets like  etc.

(At CFT-level:



This should not arise since then one can't integrate V_1, V_2 over Σ.)

<u>Result</u>: (→ Polch.)

IIB: $(-1)^F = 1$; $(-1)^{\tilde F} = 1$

IIA: $\underbrace{(-1)^F = 1}_{\text{removes tachyon}}$; $\underbrace{(-1)^{\tilde F} = \begin{cases} 1 & (NS) \\ -1 & (R) \end{cases}}_{\text{same (IIB) or other (IIA) chirality of spinor in R-sector chosen between right-movers \& leftmovers}}$ ← This "−" means choose 8'

(exist IIA'/IIB' with $8 \leftrightarrow 8'$; phys. equivalent)

In other words: $\mathrm{IIB},r \,/\, \mathrm{IIA},r$: $NS+$, $R+$

IIB,ℓ : $NS+$, $R+$

IIA,ℓ : $NS+$, $R-$

$\Downarrow$

Field content IIA	SO(8)		tensor/spinor		repr. (dim. of)
$(NS+, NS+)$	$8_v \times 8_v$	=	$[0]_\phi + [2]_{B_2} + (2)_G$	=	$1 + 28 + 35$
$(NS+, R-)$	$8_v \times 8'$	=	spinor$_\lambda$ + vector-spinor$_{\chi'_\mu}$	=	$8 + 56'$
$(R+, NS+)$	$8 \times 8_v$	=	spinor$_{\lambda'}$ + vector-spinor$_{\chi_\mu}$	=	$8' + 56$
$(R+, R-)$	$8 \times 8'$	=	$[1]_{C_1} + [3]_{C_3}$	=	$8_v + 56_t$

Comments:
- $[m]\,/\,(m)$ just means rank-m tensor, symmetric/antisymmetric
- λ & λ' are dilatinos
- χ_μ & χ'_μ are gravitinos

} of opposite chirality (Both partners of $G_{\mu\nu}$: $G_{\mu\nu} \xrightarrow{SUSY_1} \psi_\mu$, $G_{\mu\nu} \xrightarrow{SUSY_2} \psi'_\mu$ $\Rightarrow$ "$N=2$ SUSY" "type II")

- Counting of d.o.f.:

- ϕ : 1 (obvious)
- NS-2-form-potential B_2: $B_2 = (B_2)_{\mu\nu}\, dx^\mu \wedge dx^\nu$;

$\Rightarrow$ 3-form field-strength $H_3 = dB_2$; As for the photon, only transverse components "count" (the others are lost because of phys.-state-condition + gauge-freedom)

$$\Rightarrow (B_2)_{ij} \text{ with } i,j \in \{1,..,D-2\} \Rightarrow \#(d.o.f.) = \binom{D-2}{2} \overset{D=10}{=} \underline{\underline{28}}$$

- metric $G_{\mu\nu} \to G_{ij}$, but now symmetric. Also: traceless (because of extra gauge freedom of GR)

$$\Rightarrow \#(d.o.f.) = \binom{D-2}{2} + (D-3) = \underbrace{\frac{D(D-3)}{2}}_{} = \underline{\underline{35}}$$

This is of more general interest since it gives the #(d.o.f.) of a D-dim. metric.

- dilatino λ, #(d.o.f.) = 8 as explained for spinor in general
- gravitino $\psi_\mu \to \psi_i$ $(i = 1,\dots,D-2)$ due to gauge-inv. (as for photon)

Novel feature: The extra condition $\gamma^\mu \psi_{\mu} = 0$ can and must be imposed $\Rightarrow$ further restriction of d.o.f. by "equivalent of one spinor"

$$\Rightarrow \#(\text{d.o.f.}) = \#(\text{spinor d.o.f.}) \times (D-3) = 8 \cdot 7 = \underline{\underline{56}}$$

– RR-1/3-form-potentials C_1 / C_3, e.g. $C_3 = (C_3)_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$

$$F_4 = dC_3$$

$\underline{C_3:}$

$$\#(\text{d.o.f.}) = \binom{D-2}{3} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = \underline{\underline{56}}$$

$\underline{C_1:}$

$$\#(\text{d.o.f.}) = D-2 = \underline{\underline{8}}$$

Analogously:

Field content IIB

(NS+, NS+)	$(8_v \times 8_v) =$	$[0]_\phi + [2]_{B_2} + (2)_G$	$= 1 + 28 + 35$
(NS+, R+)	$(8_v \times 8) =$	spinor + vector-spinor	$= 8' + 56$
(R+, NS+)	$(8 \times 8_v) =$	spinor + vector-spinor	$= 8' + 56$
(R+, R+)	$(8 \times 8) =$	$[0]_{C_0} + [2]_{C_2} + [4]_{+, C_4}$	$= 1 + 28 + 35_+$

Comments: • The "+" of $[4]_+$ means self-duality, i.e. (assuming $\eta_{ij} = \delta_{ij}$)

$$t_{i_1 \dots i_4} = \varepsilon_{i_1 \dots i_8}\, t^{i_5 \dots i_8}$$

• In the corresponding FT, self-duality is imposed on $F_5 = dC_4$ (using $\epsilon_{\mu_1 \dots \mu_{10}}$);

$$\#(\text{d.o.f.}) = \frac{1}{2}\binom{8}{4} = \underline{\underline{35}}$$

• By contrast to IIA, the IIB-theory is <u>chiral</u> (i.e. one of the two possible chiralities, namely that of the gravitini, is preferred)

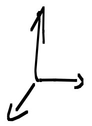
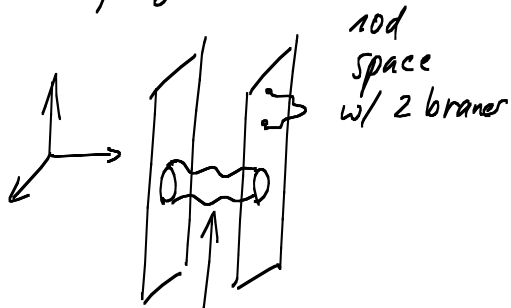
Crucial more general point: Both theories have B_2 (natural for coupling to string: $\int_\Sigma B_2$)

However: IIA has odd RR-pot.s $\longrightarrow$ e.g. $\int_{D0} C_1 \, ; \int_{D2} C_3 \, , \ldots$

IIB has even RR-pots. $\longrightarrow$ e.g. $\int_{D1} C_2 \, ; \int_{D3} C_4 \, , \ldots$

$\Rightarrow$ even / odd D-branes in IIA/B.

- Dp-branes are non-pert. objects which are part of the theory and may or may not be part of a particular solution, e.g.

 ... empty 10 d space vs. 

e.g. force between branes calculable via open-string one-loop diagram $\equiv$ closed string exchange.

... much more would have to be said...

13.3 Type I superstrings

- Our WS-theory has a $\mathbb{Z}_2$-symm: $\tau \to \tau' = \tau$, $\sigma \to \sigma' = \pi - \sigma$ (for $\sigma \in (0, \pi)$)

($\equiv$ "orientation change")

- At the quantum level, it is realized by an operator Ω: $|\psi'\rangle = \Omega |\psi\rangle$

$$\Omega^2 = 1 \; ; \quad \text{e.g. } \Omega^{-1} \hat{X}(\tau,\sigma) \Omega = \hat{X}(\tau, -\sigma) \qquad \text{(using periodicity in } \sigma\text{)}$$

$$\Omega^{-1} \alpha_n \Omega = \tilde{\alpha}_n$$

$$\& \; \Omega^{-1} \tilde{\alpha}_n \Omega = \alpha_n \quad \text{etc.}$$

- Since $\Omega^2 = 1$, the operator $P = \frac{1}{2}(1 + \Omega)$ is a projection operator ($P^2 = P$).
- We can "mod out Ω" $\equiv$ "gauge Ω" $\equiv$ "project our theory on sub-theory using P"

$\Rightarrow$ unoriented string

- Recall for (simplicity) the $m^2 = 0$ - level of the bosonic string: $\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,k\rangle$
- Note: $\Omega\, \alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,k\rangle = \tilde{\alpha}^{\mu}_{-1}\alpha^{\nu}_{-1}|0,k\rangle = \alpha^{\nu}_{-1}\tilde{\alpha}^{\mu}_{-1}|0\rangle$ ("symmetrization")

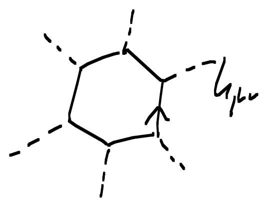
$\Rightarrow$ Projection at field level: $G_{\mu\nu}, B_{\mu\nu}, \phi \longrightarrow G_{\mu\nu}, \phi$

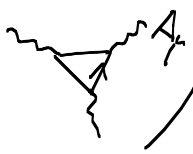
(naturally, since $\int_\Sigma B_2$ doesn't make sense for unorientable Σ)

- For superstring, Ω can only be modded out if l./r.-movers are treated symmetrically, i.e. <u>only for IIB</u>:

Projection: $\underbrace{G_{\mu\nu}, \cancel{B_{\mu\nu}}, \phi, \cancel{C_0}, C_2, \cancel{C_4}, \chi^{(1)}_{\mu}, \cancel{\chi^{(2)}_{\mu}}, \lambda^{(1)}, \cancel{\lambda^{(2)}}}_{\text{just } N=1 \text{ SUSY in 10d !}}$

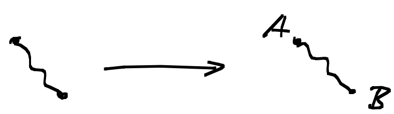
- Unfortunately, this theory is not consistent because of gravitational anomalies:



(cf. familiar case of 

)

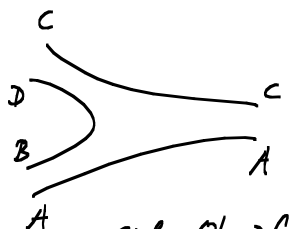
- This can be cured by adding an (also unoriented!) open (super)string sector.
- <u>More specifically</u>:

Introduce "Chan-Paton factors": 

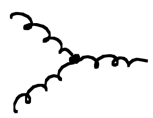
with $A, B \in \{1 \dots N\}$

(for all BCs Neumann, this corresponds to a stack of N D9-branes filling all of 10d space-time)

- scattering requires matching Chan-Paton-factors:



only OK if $D = B$

$\Rightarrow$ $U(N)$ (super-) Yang-Mills theory (cf. )

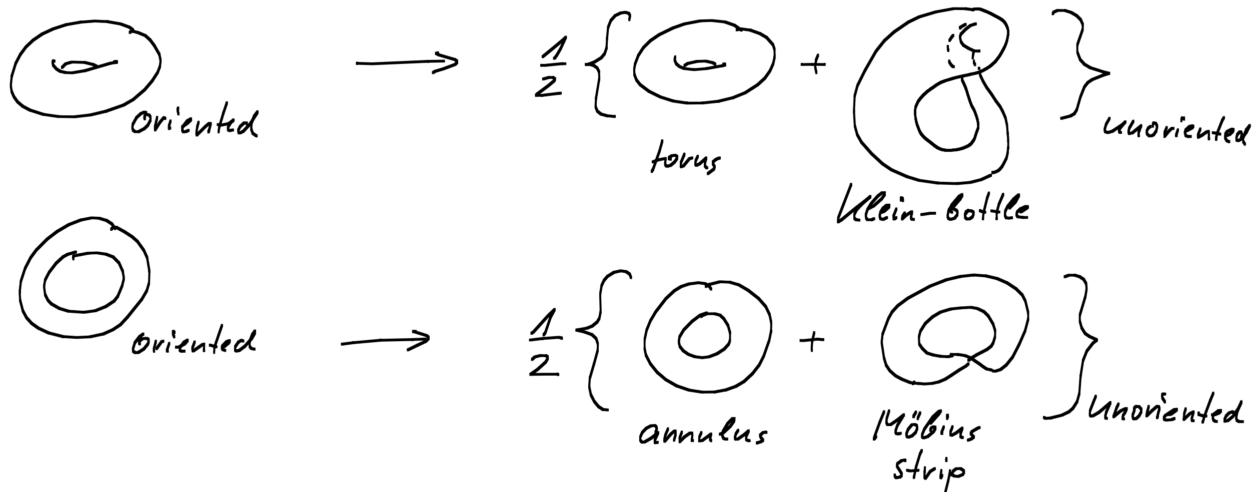
$$\underset{\substack{\uparrow \\ \text{string states} \;\sim\; \underbrace{\lambda^{AB}}_{N\times N \text{ matrix}} |A,B,k\rangle}}{}$$

- Ω-projection also applies to open-string sector : $U(N) \longrightarrow SO(N)$
- The correct choice for completing the "type IIB orientifold" introduced earlier is $N = 32$. <u>Thus:</u>

$$\underline{\text{Type I superstring:}} \quad (1_\phi + 28_{C_2} + 35_G)_B + (8' + 56)_F$$

$$+ \underset{\substack{\uparrow \\ \frac{N(N-1)}{2} \\ \text{(adjoint of } SO(32))}}{496}\,(\underset{\substack{\uparrow \\ \text{gauge fields}}}{8_v} + \underset{\substack{\uparrow \\ \text{gauginos}}}{8})$$

- <u>Loops in unoriented theory</u>



- <u>Intermediate summary:</u>

All 10d consistent theories: I, IIA, IIB, $\underbrace{\text{heterotic } SO(32), \text{ het. } E_8\times E_8}_{\text{so far still open...}}$

<u>13.4 Heterotic string</u>

- l.m. & r.m. sectors are independent
- WS-SUSY + GSO projection excludes tachyon
- Because of level-matching, it is sufficient to exclude tachyon <u>on one side</u>!

$\Rightarrow$ try to combine r.m. superstring with l.m. bosonic string:

l.m.: $X_L^\mu(\sigma^+)$

r.m.: $X_R^\mu(\sigma^-)$, $\psi_-^\mu(\sigma^-)$ ($\mu = 0, \dots, 9$ because of superstring)

(+ appropriate ghost systems)

central charges: $(\tilde{c}, c) = (\underset{X_L}{10} - \underset{bc_L}{26}\,,\ \underset{X_R}{10} + \underset{\psi_-}{\tfrac{1}{2}10} - \underset{bc_R}{26} + \underset{\beta\gamma_R}{11}) = (-16, 0)$

$\Rightarrow$ need "+16" in l.m. sector,

e.g. 32 Majorana-Weyl fermions: $\lambda^A = \lambda_+^A(\sigma^+)$; $A = 1, \dots, 32$

$$S = -\frac{1}{2\pi}\int d^2\sigma \Big[\sum_{\mu=0}^{9}\left(\partial_a X^\mu \partial^a X_\mu - 2i\psi_-^\mu \partial_+ \psi_{-,\mu}\right) - 2i\sum_{A=1}^{32}\lambda_+^A \partial_- \lambda_+^A\Big]$$

- Spectrum follows from usual mass-shell + level-matching conditions
- The only novelty: normal-ordering constants on non-SUSY side

(same for all λ^A's) $\longrightarrow$ R: $a = 8\frac{1}{24} - 32\frac{1}{24} = -1$

NS: $a = 8\cdot\frac{1}{24} + 32\frac{1}{48} = 1$

- GSO-projections: $(-1)^F = (-1)^{\tilde{F}} = 1$

- Recall: IIA: $(\underset{NS}{8_v} + \underset{R}{8}) \times (\underset{NS}{8} + \underset{R}{8'}) = (1 + 28 + 35 + 8 + 56)_{\text{bosonic}} + (8 + 8' + 56 + 56')_{\text{fermionic}}$

het. SO(32): $(8_v + 8) \times ((8_v, 1) + (1, 496)) = \cdots$

SO(32) representations

$\underbrace{\lambda_{-1/2}^A \lambda_{-1/2}^B}_{\text{automatically antisymm.}}|0,k\rangle$; $496 = \binom{32}{2}$

$\cdots = (1,1)_\phi + (28,1)_{B_2} + (35,1)_G + \underbrace{(56,1) + (8',1)}_{\text{gravitino/dilatino}} + \underbrace{(8_v, 496) + (8, 496)}_{\text{gauge pol./gaugino}}$

$\Rightarrow$ SO(32) SYM coupled to 10d $\mathcal{N}=1$ SUGRA

- Now let's instead allow for independent R / NS bound. conditions for λ^A ($A = 1 \cdots 16$) & λ'^A ($A = 17 \cdots 32$). Might expect $SO(16) \times SO(16)$, but in fact find extra massless states:

$$(8_v + 8) \times \big((8_v, 1, 1) + (1, 120, 1) + (1, 1, 120) + (1, 128, 1) + (1, 1, 128) \big)$$

adjoints of $SO(16)/SO(16)'$ — spinors of $SO(16)/SO(16)'$

Fact: $E_8 \supset SO(16)$; Indeed: $E_8 \times E_8$ – SYM + SUGRA emerges!

$248 = 120 + 128$

13.5 T-duality

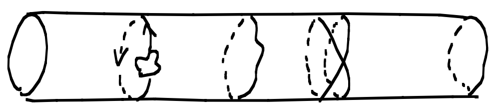
- Return to bosonic string and replace the space $\mathbb{R}^{1,25}$ where the X^M's take their values by $\mathbb{R}^{1,24} \times S^1$ (i.e. X^{25} is periodic with period $2\pi R$).

- Find mass-spectrum in 25-dimensional theory ($\rightarrow$ problems):

$$(\text{mass})^2 = \frac{2}{\alpha'} (N + \tilde{N} - 2) + \frac{m^2}{R^2} + \frac{n^2 R^2}{\alpha'^2} \quad , \quad (m,n) \in \mathbb{Z}^2$$

from $p^{25} = \frac{m}{R}$ — from "winding strings" with winding number n

- Vis.:



$n=0$ (various m's) $\quad n=1 \quad n=2$

- Observe: Symmetry under $m \leftrightarrow n$; $R \leftrightarrow R' = \frac{\alpha'}{R}$ (T-duality)
(The <u>same</u> 25d theory has <u>two</u> stringy UV-completions, one on a small ($R \ll \sqrt{\alpha'}$), one on a large ($R' \gg \sqrt{\alpha'}$) S^1)

- <u>Note:</u> The heterotic string can also be viewed as a full (26d) l.m.

bosonic string + a 10d r.m. superstring. However, the l.m. side must be compactified on a $T^{16} = (S^1)^{16}$ of a specific shape (the two allowed shapes singled out by consistency define the $SO(32)$ and $E_8 \times E_8'$ theories).

- On the CFT side, this works because a compactified boson is equivalent to a fermion ("bosonization").