3 (Covariant) Canonical Quantization

3.1 Closed String

- We will quantize the Polyakov action in flat gauge:

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\dot{X}^2 - X'^2\right) = -\frac{T}{2}\int d^2\sigma\, \partial^a X_\mu \partial_a X^\mu \quad ; \quad d^2\sigma = d\tau\, d\sigma$$

- This is just a 2d QFT with D scalars X^μ (note however the wrong-sign kinetic term of X^0!).
- Coordinates: X^μ ; Conj. momenta: $\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = T\dot{X}_\mu$
- Equal-time commut. relations:

$$[\hat{\Pi}_\mu(\tau,\sigma), \hat{X}^\nu(\tau,\sigma')] = -i\delta(\sigma-\sigma')\delta_\mu{}^\nu \quad ; \quad [\hat{X}, \hat{X}] = [\hat{\Pi}, \hat{\Pi}] = 0$$

(equivalently: $[\hat{\Pi}^\mu(\tau,\sigma), \hat{X}^\nu(\tau,\sigma')] = -i\delta(\sigma-\sigma')\eta^{\mu\nu}$)

- Promote our previous expression $X^\mu = X^\mu(x, p, \alpha, \tilde{\alpha})$ to the operator level ($x \to \hat{x}$; $\alpha \to \hat{\alpha}$; etc.). Do the same for $\Pi^\mu = T\dot{X}^\mu$. Insert them into the commutations relations above & Fourier transform. Find (let $l \equiv 1$ for simplicity):

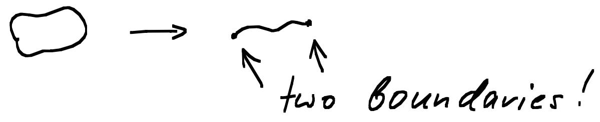
$$[\hat{p}^\mu, \hat{x}^\nu] = -i\eta^{\mu\nu} \quad ; \quad [\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] = m\,\delta_{m+n}\eta^{\mu\nu} \quad ; \quad [\hat{\tilde{\alpha}}_m^\mu, \hat{\tilde{\alpha}}_n^\nu] = m\,\delta_{m+n}\eta^{\mu\nu}$$

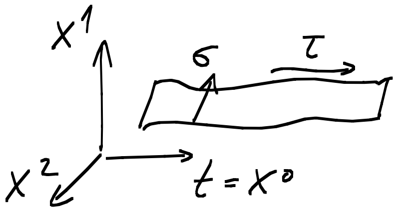
$$(\delta_{m+n} \equiv \delta_{m+n,0})$$

$$\left[\text{Use} \quad \int_0^\pi d\sigma \int_0^\pi d\sigma'\, e^{2im\sigma} e^{2in\sigma'}\delta(\sigma-\sigma') = \pi\,\delta_{m+n} \quad \text{etc.}\right]$$

- So far, the only difference w.r.t. usual QFT is the appearance of $\hat{x}^\mu$ & $\hat{p}^\nu$. Normally, working in infinite space, the "average value of the field", x^μ, is non-dynamical.

3.2 Open string

- Now let 



WS parameterized by τ & $\sigma \in (0,\pi)$. Again, we use the Polyakov action & go to flat gauge.

- $S = -\frac{T}{2}\int d^2\sigma \, (\partial^a X^\mu)(\partial_a X_\mu)$; $\delta S = -T\int d^2\sigma \, (\partial^a X^\mu)(\partial_a \delta X_\mu) \overset{!}{=} 0$

Crucially, when we now derive EOMs, a boundary term appears. We need to choose boundary conditions such that it vanishes.

$$\delta S = \underbrace{T\int d^2\sigma \, (\partial^2 X^\mu)\,\delta X_\mu}_{\Rightarrow \text{ usual EOMs}} - \underbrace{\int d\tau \int_0^\pi d\sigma \, \partial_\sigma (\partial^\sigma X^\mu \, \delta X_\mu)}_{}$$

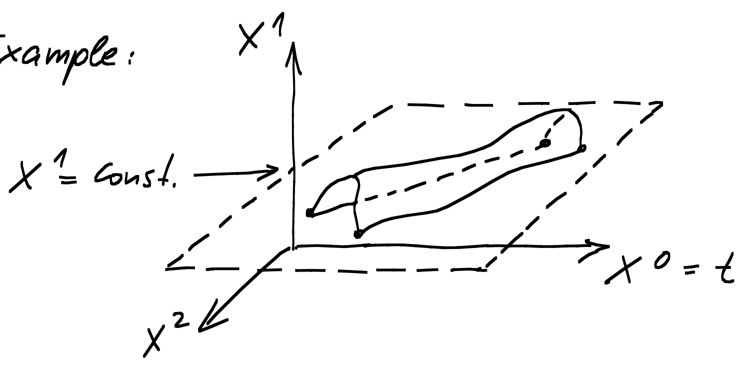
$$= \int d\tau \, (\partial_\sigma X^\mu)\, \delta X_\mu \Big|_{\sigma=0}^{\sigma=\pi}$$

1) $\partial_\sigma X^\mu = 0$ (Neumann) 2) $\delta X_\mu = 0$ (Dirichlet)

1) It can be shown that, in this case, there is no energy loss at the end of the string. Thus, we have a "freely moving string end".

2) In this case, we have $X^\mu = \text{const.}$ $\Rightarrow$ Transl. inv. in X^μ-direction broken; string-end is confined to lie in hyperplane $X^\mu = \text{const.}$

Example:



Open string with Neumann-bc in X^0, X^2 & Dirichlet-bc in X^1. Its ends are confined to the "D1-Brane" at $X^1 = \text{const.}$

- More generally, a Dp-brane has p spatial dimensions and a string ending on it has p+1 Neumann b.c.'s (the rest Dirichlet.)

- We now focus on Neumann (the modification to Dirichlet is straightforward): $X^\mu = x^\mu + l^2 p^\mu \tau + il \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma$

[$\hat{=}$ half of the d.o.f. of the closed string. This is intuitive since α & $\tilde{\alpha}$ are tied to each other by the b.c.'s]

- Canonical quantization as above: $[\hat{p}^\mu, \hat{x}^\nu] = -i\eta^{\mu\nu}$

$$[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n}\,\eta^{\mu\nu}$$

- Note: Dirichlet b.c.s with two different branes for each end or mixed, DN-b.c.'s are also possible:

3.3 The Fock space

- We found: 1) Set of oscillators $\hat{\alpha}_m^\mu$, $\hat{\alpha}_m^{\mu +} = \hat{\alpha}_{-m}^\mu$ (just like QFT, $m \hat{=} \vec{k}$)
  [It is doubled by $\hat{\tilde{\alpha}}$'s in the closed case]
  2) Hermitian operators $\hat{p}^\mu, \hat{x}^\nu$ commuting with the α's.

- As usual, we are looking for a Hilbert space representation.
- If we find one, we will always be able to diagonalize either $\hat{p}$ or $\hat{x}$. We pick $\hat{p}$ since we want to describe particles with fixed momentum:

$$\mathcal{H} = \bigoplus_p \underbrace{\mathcal{H}(p)}_{\text{eigenspace of } \hat{p}^0 \cdots \hat{p}^{D-1} \text{ with eigenvalues } p^0 \cdots p^{D-1}}$$

- Focus on <u>one</u> of these eigenspaces (with one fixed $p \in \mathbb{R}^D$) and try to build a repres. of the $\hat{\alpha}, \hat{\tilde{\alpha}}$ algebra.

$$[\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] = m\,\delta_{m+n}\,\eta^{\mu\nu} \quad \hat{=} \quad [\hat{\alpha}_m^\mu, \hat{\alpha}_n^{\nu +}] = |m|\,\delta_{m,n}\,\eta^{\mu\nu}$$

- Thus, as in QM, define <u>vacuum</u> $|0,p\rangle$ by

$$\hat{\alpha}_m^\mu |0, \underset{\uparrow}{p}\rangle = 0 \quad \text{all } m > 0 \text{ \& all } \mu$$

fixed set of 4 real numbers $\{p^\mu\}$.

- $\mathcal{H} = \text{Span}\{\, \hat{\alpha}_m^\mu \hat{\alpha}_n^\nu \cdots |0,p\rangle \;;\;$ any number of $\hat{\alpha}$'s ; any $\mu, \nu, \ldots$ any $m, n, \ldots > 0 \,\}$

- Immediate problem: While this works for "most" μ's, it fails for $\mu = 0$. Let e.g. $\hat{\alpha}_1^0 \equiv a$. Then we have

$$\underset{!}{[a, a^+] = -1} \Rightarrow |a^+|0\rangle|^2 = \langle 0|a a^+|0\rangle = \langle 0|(-1 + a^+ a)|0\rangle$$
$$= -\langle 0|0\rangle \Rightarrow a^+|0\rangle \text{ is a neg.-norm state, i.e. a "ghost"}$$

($\neq$ Fadeev-Popov ghost).

- We can't just redefine $\hat{\alpha}_1^0$ to be the creation operator since this would destroy the Lorentz-invariance of our construction.

- In QED, the same problem arose from: $\hat{A}_\mu \to a_\mu, a_\mu^+$ $\to$ neg. norm states created by a_0^+. It was removed by implementing the constraint $\partial_\mu A^\mu = 0$ (in this case the gauge condition) quantum-mechanically. Specifically:

$$|\psi\rangle \in \mathcal{H}_{phys.} \subset \mathcal{H} \quad \Leftrightarrow \quad (\partial_\mu \hat{A}^\mu)\Big|_{\substack{\text{only}\\\text{annihilator}\\\text{part.}}} |\psi\rangle = 0$$

- Here, analogously, we need to implemen the constraint $T_{ab} = 0$ (originally the EOM for h, which is now gone) at the q.m. level:

- It is convenient to use light-coordinates & observe that

$$T_a{}^a = 0 \iff T_{+-} = 0.$$

Hence we just need to impose $\underline{\underline{T_{++} = T_{--} = 0}}$.

- $$T_{ab} = 2\pi T\left(G_{ab} - \tfrac{1}{2} h_{ab} h^{cd} G_{cd}\right) \xrightarrow[\substack{\text{flat gauge,} \\ \text{light-cone coord.}}]{\ell = 1} \begin{aligned} T_{++} &= (\partial_+ X)\cdot(\partial_+ X) \\ T_{--} &= (\partial_- X)\cdot(\partial_- X) \end{aligned}$$

- Since $\partial_+ X_R = \partial_- X_L = 0$, this implies

$$T_{++} = (\partial_+ X_L)\cdot(\partial_+ X_L)$$
$$T_{--} = (\partial_- X_R)\cdot(\partial_- X_R).$$

- In Fourier modes (using $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{\ell}{2} p^\mu$):

$$L_m \equiv \frac{T}{2}\int_0^\pi e^{-2im\sigma}\, T_{--}\, d\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n$$
$$\tilde{L}_m \equiv \frac{T}{2}\int_0^\pi e^{2im\sigma}\, T_{++}\, d\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n$$

closed string

$$L_m \equiv T\int_{-\pi}^{\pi} e^{im\sigma}\, T_{++}\, d\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n$$

open string

Note: Here and below we use the fact that, in many ways, the open string corresponds to "half the degrees of freedom of the closed string. To see this, recall the closed string mode expansion, but on a double-size interval ($\sigma \in (-\pi,\pi)$):

$$X^\mu = \cdots + \sum \frac{i}{2n}\left(\tilde{\alpha}_n^\mu e^{-in\sigma^+} + \alpha_n^\mu e^{-in\sigma^-}\right).$$

Now enforce symmetry under $\sigma \to -\sigma$. This realizes N-b.c.'s at 0 & π and clearly means $\alpha_n = \tilde{\alpha}_n$. We find

$$X^\mu = \cdots + \sum \frac{i}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma\,,$$

exactly as originally defined in the open string case. This explains the definition of L_m given above and the fact that imposing a constraint on T_{++} is sufficient.

- In particular, we have $H = L_0 + \tilde{L}_0$ for the closed string.
  $H = L_0$ open

(Calculate H from S and check this!)

- In the following, will always give only open-string formulae to save writing. Including $\tilde{\alpha}$'s is mostly straightforward.
- Thus, after quantization:

$$\hat{L}_m \equiv \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{\alpha}_{m-n} \cdot \hat{\alpha}_n \quad \text{for } n \neq 0$$

$$\hat{L}_0 \equiv \frac{1}{2} \hat{p}^2 + \sum_{n>0} \hat{\alpha}_{-n} \cdot \hat{\alpha}_n$$

$\leftarrow$ Crucial: For L_0, the transition from class. to quantum is ambiguous. We define $\hat{L}_0$ to be the normal-ordered operator expression.

- With some work ($\rightarrow$ problems) one finds the

Virasoro algebra: $$[\hat{L}_m, \hat{L}_n] = \underbrace{(m-n)\hat{L}_{m+n}}_{\text{class. part}} + \underbrace{A(m)\,\delta_{m+n}}_{\text{anomaly term}}$$

where $A(m) = \frac{1}{12}(m^3 - m)\,D$

$\uparrow$ "central charge"

Comment: The classical part ("Witt algebra") appears for a simple reason:
- As in QFT, T^{ab} are the currents associated with translations (momenta P^a).
- Integrated with fcts. $f(\sigma,\tau)$, T^{ab} generates

position-dependent translations, i.e. reparametrizations.

- more specifically, all Fourier modes of $T_{++}(\sigma^+)/T_{--}(\sigma^-)$ generate all reparametrizations $X_L(\sigma^+) \to X_L(\varphi(\sigma^+))/X_R(\sigma^-) \to X_R(\varphi(\sigma^-))$ [with φ periodic].

- Indeed, it is easy to check that the Witt algebra arises as the algebra of reparametrizations on a circle:

Consider fcts. $f(\theta) = f(\theta + 2\pi n)$ on a circle.
$f(\theta) \to f(\theta + a(\theta))$ is generated by $D[a] = a(\theta)\frac{d}{d\theta}$
with periodic a. A basis for $a(\theta)$ is given by $i\exp[in\theta]$, $n \in \mathbb{Z}$. Hence

$$D_n = i\, e^{in\theta}\frac{d}{d\theta}$$

[need to take appropriate lin. combinations to get real fcts. $a(\theta)$]

generate all reparametrizations. Their algebra is:

$$[D_m, D_n]f = ie^{im\theta}(ie^{in\theta}f')' - \{m \leftrightarrow n\}$$
$$= -e^{i(m+n)\theta}\cdot in\cdot f' - \{m \leftrightarrow n\} = (m-n)\, ie^{i(m+n)\theta} f' = (m-n)D_{m+n}f$$

- We want "$\hat{T}_{ab} = 0$", which is equivalent to "$\hat{L}_m = 0$" ($m \in \mathbb{Z}$). Thus, we need to identify a subspace of "phys. states" for which "$\hat{L}_m = 0$."

- We can't demand $\hat{L}_m |phys\rangle = 0$ since this would imply

$$\langle phys|[\hat{L}_m, \hat{L}_{-m}]|phys\rangle = 0,$$ while (cf. Virasoro algebra)

$$\langle phys|\left(2m\hat{L}_0 + \frac{D}{12}(m^3-m)\right)|phys\rangle = \frac{D}{12}(m^3-m)\langle phys|phys\rangle \neq 0.$$

- As in QED, it is sufficient to demand that the "annihilator part" vanishes on phys. states:

$$\underline{\underline{(\hat{L}_m - a\delta_m)|phys\rangle = 0\ ; \quad m \geq 0.}}$$

<u>Note:</u> We have allowed for a, so far unknown, constant correction to $\hat{L}_0$. The reason is that our (normal-ordered) definition of $\hat{L}_0$ was "ad hoc". We don't know yet what the correct operator corresponding to the zero-mode of T_{++} is. We know that it equals $\hat{L}_0$ up to a constant. Hence we define it to be $\hat{L}_0 - a$ and demand $(\hat{L}_0 - a)|phys\rangle = 0$.

- It follows that $\langle phys|\hat{L}_m|phys\rangle = 0$ also for $m < 0$:

$$\langle phys|\hat{L}_m|phys\rangle \underset{\uparrow}{=} \langle phys|\hat{L}_{-m}|phys\rangle^* = 0$$

since $\hat{L}_m^+ = \hat{L}_{-m}$

<u>Summary:</u> 1) Fock space $\mathcal{H}$: Built by $\hat{\alpha}_m$'s $(m>0)$ on the direct sum of all $|0,p\rangle$. The $\hat{\alpha}_m$'s satisfy $[\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] = m\,\delta_{m+n}\,\eta^{\mu\nu}$.

2) Phys. subspace $\mathcal{H}_{phys}$: all $|\psi\rangle \in \mathcal{H}$ with $(L_m - a\,\delta_m)|\psi\rangle = 0\,,\ \forall m \geq 0$.

3.4 Normal-ordering constant vs. Casimir energy

- In spite of the operator-ordering ambiguity, it is possible to give "a" a phys. meaning and determine its value. Knowing this value will be very helpful in the following chapters.

- A direct evaluation of H from $H_{tot} = \frac{T}{2}\int_0^\pi (\dot{X}^2 + X'^2)\,d\sigma$ would have given

$$H_{tot} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{-n}\alpha_n \qquad \text{(open string!)}$$

instead of

$$H \equiv L_0 \equiv \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{-n}\alpha_n: \;=\; \frac{1}{2}p^2 + \sum_{n>0}\alpha_{-n}\alpha_n\,.$$

- H_{tot} is divergent, as expected from usual QFT:

$$\langle 0,p|H_{tot}|0,p\rangle = \frac{1}{2}p^2 + \frac{1}{2}(D-2)\sum_{n=1}^{\infty} n$$

Direct calculation gives $\eta_\mu{}^\mu = D$, but the "physically correct" prefactor is $(D-2)$, roughly speaking because of the wrong-sign X^0-contribution. This will only become technically clear in light-cone quantization (next chapter) or through Fadeev-Popov-ghosts (chapter after that).

- This divergence has to be subtracted by adding a cosmological-constant-counterterm to our original action,

$$\frac{T}{2}\int d^2\sigma\sqrt{-h}\,(\partial X)^2 \;\to\; \frac{T}{2}\int d^2\sigma\sqrt{-h}\,(\partial X)^2 - \int d^2\sigma\sqrt{-h}\,\lambda$$

- It is convenient, for the moment, to think of our WS as of a proper physical space of size πR, so that now

$$H_{tot} = \frac{1}{2R}\sum_{n=-\infty}^{\infty}\alpha_{-n}\alpha_n + \pi R\lambda$$

introduced on dim. grounds, since $\pi \to \pi R$ for size

class. contribution of counterterm.

- We need to regularize our 2d QFT by a cutoff Λ and choose $\lambda = \lambda(\Lambda)$ such that H_{tot} approaches a finite limit as $\Lambda\to\infty$:

$$H_{tot}^r = \lim_{\Lambda\to\infty}\frac{1}{2R}\left\{\sum_{n=-\infty}^{\infty}\alpha_{-n}\alpha_n\right\}_\Lambda + \pi R\lambda(\Lambda)$$

$$= \frac{1}{R}\left(\frac{1}{2}p^2 + \sum_{n>0}\alpha_{-n}\alpha_n\right) + \lim_{\Lambda\to\infty}\left[\frac{D-2}{2R}\left\{\sum_{n=1}^{\infty}n\right\}_\Lambda + \pi R\lambda(\Lambda)\right]$$

- It is <u>this</u> renormalized, physical energy which we really want to set to zero by our constraint:

$$H_{tot}^r = \frac{1}{R}(L_0 - a).$$

- Hence,

$$\underline{\underline{-a = \lim_{\Lambda\to\infty}\left[\frac{D-2}{2}\left\{\sum_{n=1}^{\infty}n\right\}_\Lambda + \pi R^2\lambda(\Lambda)\right]}}$$

- To understand how to implement the regularization by Λ, recall that

$\sum_n$ just implements the sum over all modes, each with physical momentum $k = \frac{2\pi}{\lambda} = \frac{2\pi}{(2\pi R/n)} = \frac{n}{R}$. We can suppress modes with $k \gg \Lambda$ by introducing a fct. $\exp(-k/\Lambda)$ in the sum:

$$\sum_{n=1}^{\infty} n\, e^{-n/\Lambda R} = -\frac{d}{d\alpha} \sum_{n=1}^{\infty} e^{-\alpha n} \qquad \left(\text{with } \alpha = \frac{1}{\Lambda R}\right)$$

$$= -\frac{d}{d\alpha}\left(\frac{1}{1-e^{-\alpha}}\right) = \frac{e^{-\alpha}}{(1-e^{-\alpha})^2} = \frac{1-\alpha+\frac{\alpha^2}{2}}{\left(\alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{6}\right)^2} + O(\alpha)$$

$$= \frac{1}{\alpha^2} \cdot \frac{1-\alpha+\frac{\alpha^2}{2}}{\left(1-\frac{\alpha}{2}+\frac{\alpha^2}{6}\right)^2} + O(\alpha) = \frac{1}{\alpha^2} \cdot \frac{1-\alpha+\alpha^2/2}{1-\alpha+\frac{7}{12}\alpha^2} + O(\alpha)$$

$$= \frac{1}{\alpha^2} \cdot \frac{1-\alpha^2/2}{1-\frac{5}{12}\alpha^2} + O(\alpha) = \frac{1}{\alpha^2}\left(1-\frac{1}{12}\alpha^2\right) + O(\alpha) = \Lambda^2 R^2 - \frac{1}{12} + O\left(\frac{1}{\Lambda}\right)$$

$$\Rightarrow \quad -a = \lim_{\Lambda\to\infty}\left[\frac{D-2}{2}\left(\Lambda^2R^2 - \frac{1}{12} + O\left(\frac{1}{\Lambda}\right)\right) + \pi R^2 \lambda(\Lambda)\right]$$

choose this $\sim \Lambda^2$ with appropr. coeff. to cancel divergence

$$\underline{\underline{a = \frac{D-2}{24}}}$$

<u>Comments:</u>

- In phys. terms, this is just a 2d-QFT-Casimir energy. The corresp. contribution to the energy

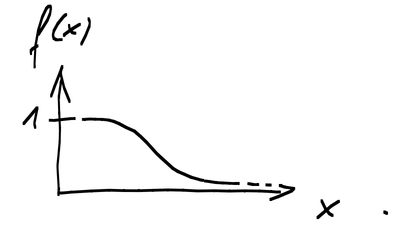
$$E_{\text{Casimir}} = -\frac{a}{R} = -\frac{D-2}{24R} \longrightarrow -\frac{1}{24R}$$

(for one Boson).

The $1/R$ behaviour is required on dim. grounds. The cutoff-procedure is normally justified by the fact that high-frequency modes don't "see" the metal plates at each end of the interval. Of course, <u>any</u>

Smoot cutoff fct. should work:

$$e^{-k/\Lambda} \longrightarrow f(k/\Lambda) \text{ with}$$



To show this is not totally trivial, see e.g. the discussion of the 4d case in Itzykson / Zuber.

- Another approach is to start from the standard QFT expression

$$E_{vac.} = V \int \frac{d^{d-1}\bar{p}}{(2\pi)^{d-1}} \cdot \frac{1}{2}\omega_{\bar{p}} \quad ; \quad \omega_{\bar{p}} = \sqrt{\bar{p}^2 + m^2} \, .$$

- Compactify one dimension on circle $2\pi R$ (the others are much large): $\quad V \longrightarrow 2\pi R \cdot V_{d-2} \quad$; also: $m = 0$

$$E_{vac.} = \frac{1}{2}\, 2\pi R \cdot V_{d-2} \cdot \frac{1}{2\pi R} \sum_n \underbrace{\int \frac{d^{d-2}\bar{p}}{(2\pi)^{d-2}} \sqrt{\left(\frac{n}{R}\right)^2 + \bar{p}^2}}_{\sim \left(\frac{n}{R}\right)^{d-1}}$$

(by "dimensional regularization, i.e., analytic continuation in d)

- It is already clear at this point (fixing the constant prefactors would nevertheless be a useful exercise!) that we will get, (since our case is $d=2$):

$$\left\{ \sum_{n=1}^{\infty} n \right\}_{reg.} = \lim_{s \to -1} \Big(\underbrace{\sum_{n=1}^{\infty} n^{-s}}_{\equiv\, \zeta(s)} \Big)$$

- Mathematical fact:

$$\zeta(-1) = -\frac{1}{12} \qquad \text{("Zeta function")}$$

(in agreement with what was found above)

[This last method of "zeta function regularization" can also be motivated without the detour via dim.-reg.; cf. many books.]