4 (Covariant) Canonical Quantization – continued

4.1 Explicit construction of physical states – open string

0) Vacuum: $|0,p\rangle$; $\hat{p}^\mu |0,p\rangle = p^\mu |0,p\rangle$

- Need to restrict to those p for which $(L_m - a\delta_m)|0,p\rangle = 0 \; \forall m \geq 0$.
- For $m > 0$, $L_m |0,p\rangle = \frac{1}{2}\sum_n \alpha_{m-n} \cdot \alpha_n |0,p\rangle = 0 \quad \forall p$

$\uparrow$ either $n > 0$ or $(m-n) > 0$.

- For $m = 0$, $\left(L_m - a\delta_m\right)|0,p\rangle$

$$= \left(\frac{\hat{p}^2}{2} + \sum_{n>0} \alpha_{-n} \cdot \alpha_n - a\right)|0,p\rangle \overset{!}{=} 0$$

"undoing" $\alpha' = \frac{1}{2}$.

$$\Rightarrow \quad p^2 = 2a \quad \left(\text{or } M^2 = -p^2 = -2a \overset{\downarrow}{=} -\frac{a}{\alpha'}\right)$$

$\Rightarrow$ <u>One scalar particle with $M^2 = -2a$.</u>

1) First excited level

- These states are of the form $\zeta_\mu \alpha_{-1}^\mu |0,p\rangle$. $\Rightarrow$ They are characterized by a "polarization vector" ζ^μ.
- The "$L_0 - a$" or "mass shell condition" gives:

$$0 \overset{!}{=} (L_0 - a)\,\zeta_\mu \alpha_{-1}^\mu |0,p\rangle = \left(\frac{\hat{p}^2}{2} + \alpha_{-1} \cdot \alpha_1 - a\right)\zeta_\mu \alpha_{-1}^\mu |0,p\rangle$$

$$= \left(\frac{p^2}{2} + 1 - a\right)\zeta_\mu \alpha_{-1}^\mu |0,p\rangle \quad \Rightarrow \quad \underline{M^2 = 2(1-a)}$$

- The "L_1 condition" is now also non-trivial:

$$0 \overset{!}{=} L_1\, \zeta \cdot \alpha_{-1} |0,p\rangle = \left(\frac{1}{2}\sum_n \alpha_{1-n} \cdot \alpha_n\right)\zeta \cdot \alpha_{-1} |0,p\rangle$$

Now $(1-n)$ is "too positive" for $n \leq -1$; n is "too positive" for $n \geq 2$. In both cases the state is annihilated trivially. Thus, we only need to consider the terms $n \in \{0, 1\}$:

$$0 \overset{!}{=} \tfrac{1}{2}(\alpha_1 \cdot \alpha_0 + \alpha_0 \cdot \alpha_1)\, \zeta \cdot \alpha_{-1} |0,p\rangle = \zeta \cdot \alpha_0 |0,p\rangle = \zeta \cdot p |0,p\rangle$$

$\Rightarrow \underline{\zeta \cdot p = 0}$ (polarization must be transverse)

- We also need to know the norm:

$$\langle 0,p| (\zeta_\mu \alpha^\mu_{-1})^+ (\zeta_\nu \alpha^\nu_{-1}) |0,p\rangle = \langle 0,p|0,p\rangle\, \zeta_\mu \zeta^\mu$$

- This already allows for <u>First steps towards a classification:</u>

a) $\underline{\underline{a>1}} \Rightarrow M^2 < 0 \Rightarrow p$ space-like $\Rightarrow \exists$ time-like ζ with $\zeta \cdot p = 0$ ...

... $\Rightarrow \exists$ neg.-norm states $\Rightarrow$ <u>excluded</u>

b) $\underline{\underline{a=1}} \Rightarrow M^2 = p^2 = 0 \Rightarrow (D-1)$ independent ζ's with $\zeta \cdot p = 0$

one longitudinal, zero-norm state ($\zeta \parallel p$)

D-2 transverse, positive-norm states

[e.g. $p \sim (1,1,0,\cdots,0)$

$\zeta = (\underbrace{0,0}_{0,1},\underbrace{1,0,\cdots,0}_{2\cdots D-1})$]

This is precisely what one finds in Gupta-Bleuler Quantization of QED.

Consistent with a gauge theory of massless vectors.

$\Rightarrow$ <u>"Critical string theory"</u>

c) $\underline{\underline{a<1}} \Rightarrow M^2 > 0 \Rightarrow p$ time-like $\Rightarrow$ all ζ's space-like

$\Rightarrow$ Consistent with D-1 polarizations of a massive vector

This is so far Ok, but will turn out to have problems (maybe solvable? ...) when loops & interactions are included.

$\Rightarrow$ <u>"Non-critical string theory"</u>

2) <u>Second excited level</u>

By this we mean states of type $(\epsilon_{\mu\nu} \alpha^\mu_{-1} \alpha^\nu_{-1} + \epsilon_\mu \alpha^\mu_{-2}) |0,p\rangle$.

$$(L_0 - a)|\psi\rangle = 0 \quad \Rightarrow \quad M^2 = 2(\underset{\substack{\uparrow \\ \text{"second" level}}}{2} - a)$$

- As in 1), we should now go on and determine which polarizations are allowed, what the norms of the states are etc. We will not do this but but focus on the <u>critical case</u> ($a=1$) and on one particular type of state:

$$|\phi\rangle = \{c_1\, \alpha_{-1}\cdot\alpha_{-1} + c_2\, p\cdot\alpha_{-2} + c_3\,(p\cdot\alpha_{-1})^2\}\,|0,p\rangle$$

- One easily shows: $(L_0-1)|\phi\rangle = L_1|\phi\rangle = L_2|\phi\rangle \Rightarrow c_2$ & c_3 expressible through c_1

$$\Rightarrow \langle\phi|\phi\rangle = \frac{2c_1^2}{25}(D-1)(26-D)$$

$\Rightarrow D \leq 26$ required by positivity; $D=26$ special because of extra zero-norm, physical states

- Recall: $a = \dfrac{D-2}{24}$ from our Casimir-energy calculation.

  Thus, $D=26$ indeed "belongs" to $a=1$. It is interesting to note that our present purely algebraic discussion "knows" about this relation.

The above <u>illustrates</u> the following situation:

<u>"Summary"</u>

- $a>1$, $D>26$ – excluded (ghosts)
- $a<1$, $D<26$ – so far ok (non-critical string)
- $a=1$, $D=26$ – very special (critical string) in the following sense:

  - many "extra" zero-norm states; we saw examples of this at level 1 & 2; at level 1 we recognize these as longit. polarizations of massless vector $\equiv$ "gauge freedom".

  - We expect, from extra such "null" states at higher levels, a much higher gauge symmetry (conformal symmetry – see later) in the critical case. This makes $D=26$ special (in particular "solvable").

<u>More formally</u>:

- $(L_0 - a)|\psi\rangle = 0$ & $\langle\psi|\text{phys}\rangle = 0$ ($\forall\, |\text{phys}\rangle$) $\Rightarrow$ $|\psi\rangle$ is called "spurious".
- $|\psi\rangle$ is spurious & physical $\Rightarrow$ $|\psi\rangle$ is called "null" (it obviously has zero norm).
- One can show quite generally that the number of states grows enormously in the "critical case" $D=26$ $(a=1)$. $\rightarrow$ GSW

- The "proper" Hilbert space is defined as the space of equivalence classes:

$$\underset{\sim}{\mathcal{H}} \equiv \mathcal{H}_{phys} / \mathcal{H}_{null} \qquad (\underline{\text{recall}}: \mathcal{H}_{null} \subset \mathcal{H}_{phys} \subset \mathcal{H})$$

- It is common to rewrite the mass-shell condition "$L_0 - 1 = 0$" as (now $a=1$!)

$$M^2 = -p^2 = 2(N-1)$$

here $N \equiv \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n$ is called the "level"

- There are $\underbrace{\text{phys. states}}_{\substack{\text{satisfying also} \\ L_m |\text{phys}\rangle = 0\,;\ m \geq 1}}$ at $N = \underset{\substack{\uparrow \\ \text{scalar} \\ \text{tachyon}}}{0}, \underset{\substack{\uparrow \\ \text{massless} \\ \text{vector}}}{1}, 2, \cdots$

<u>4.2 Explicit construction of phys. states – closed string</u>

- $D=26$, $a=1$ is again special – for the same reasons. We directly focus on this critical case.
- Oscillators & Constraints are now doubled: $L_m \rightarrow L_m, \tilde{L}_m$.

- $(L_0 - a)|phys\rangle = 0$ & $(\tilde{L}_0 - a)|phys\rangle = 0$ $\Leftrightarrow$ $(L_0 - \tilde{L}_0)|phys\rangle = 0$, $(L_0 + \tilde{L}_0 - 2a)|phys\rangle = 0$

[Recall: $L_0 = \frac{\alpha_0^2}{2} + N = \frac{p^2}{8} + N$ since $\alpha_0 = \frac{p}{2}$ for the closed string]

- Thus, the open-string mass-shell condition is replaced by the following two conditions in the closed case:

(I) $(N - \tilde{N})|phys\rangle = 0$ "level matching"

(II) $(\frac{p^2}{4} + N + \tilde{N} - 2a)|phys\rangle = 0$ or, with $M^2 = -p^2$, $a = 1$

$$M^2 = 4(N + \tilde{N} - 2)$$ "mass-shell condition"

- Now we can proceed level-by-level, as before:

0) Vacuum: $|0,p\rangle$; $M^2 = -8$; tachyon

1) First excited level: Level-matching is ensured if and only if we apply both α_{-1} & $\tilde{\alpha}_{-1}$ to the vacuum:

$\xi_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0,p\rangle$; $M^2 = 4(1+1-2) = 0$
(Recall that the "-2" is really a "$-2a$", so we get $M^2 = 0$ just for $a = 1$.)

- At the first level, the $L_1/\tilde{L}_1$-constraints are also non-trivial. In complete analogy to the open string, they give:

$$\xi_{\mu\nu}p^{\mu} = 0 \quad \& \quad \xi_{\mu\nu}p^{\nu} = 0.$$

- These $2D$ linear constraints on $\xi_{\mu\nu}$ are not all independent. Indeed, $(\xi_{\mu\nu}p^{\mu})\cdot p^{\nu} = 0$ is lin. combination of the first D equs. It is automatically satisfied because of the second D eqs. Hence, in total we have only $2D-1$ constraints.

$\Rightarrow$ We have $D^2 - (2D-1)$ phys. states at level 1.

- In analogy to the open string, one can show that $\langle \text{phys}|\text{phys}\rangle \sim \xi_{\mu\nu}\xi^{\mu\nu}$.

Hence, states with $\xi^{(1)}_{\mu\nu} = \alpha_\mu p_\nu$ & $\xi^{(2)}_{\mu\nu} = p_\mu \beta_\nu$

(where $\alpha \cdot p = 0$ & $\beta \cdot p = 0$)

are "null"! (They correspond to gauge-degrees-of-freedom, i.e.

$\xi \to \xi' = \xi + \xi^{(1)} + \xi^{(2)}$ is a gauge trf.)

- Let us count this gauge freedom: α: $D-1$ d.o.f.

β: $D-1$ d.o.f.

$\Rightarrow$ $2D-2$. Furthermore, we overcounted by <u>one</u> since $\xi_{\mu\nu} = p_\mu p_\nu$ belongs to <u>both</u> $\xi^{(1)}$ & $\xi^{(2)}$. Thus, finally: $\underline{\underline{2D-3}}$.

$\Rightarrow$ There are $D^2 - (2D-1) - (2D-3) = \underline{\underline{(D-2)^2}}$ physical d.o.f. at level one.

- Let us choose a frame where $p = (1, 1, 0, \cdots, 0)$. Consider ξ of the form

$$\xi = \left(\begin{array}{cc|c} 0 & 0 & \cdots \; 0 \\ 0 & 0 & 0 \\ \hline \vdots & & \\ 0 & 0 & \xi_t \end{array}\right).$$

$\uparrow$

"transverse" $(D-2) \times (D-2)$ matrix

Such ξ obviously fullfill our $(2D-1)$ constraints. It is also easy to show that our gauge freedom is sufficient to take any physical ξ to this form (demonstrate this!).

- ξ_t transforms under $SO(D-2)$, the group of rotations in the spatial hyperplane transverse to $\vec{p}$ (i.e. in indices $\mu, \nu = 2, 3, \cdots, D-1$).

[This is called the "little group" – the $SO(1, D-1)$ subgroup leaving p invariant.]

- $SO(D-2)$ - decomposition:

$$(D-2)^2 = \underbrace{\binom{D-2}{2}}_{\text{antisymm.}} + \underbrace{1}_{\substack{\text{trace} \\ (\sim \delta^{ij})}} + \underbrace{\binom{D-2}{2} + (D-2) - 1}_{\text{symm., traceless}}$$

$$= \underbrace{\tfrac{1}{2}(D-2)(D-3)}_{\substack{\text{antisymm. tensor} \\ B_{\mu\nu}}} + \underbrace{1}_{\substack{\text{dilaton} \\ \phi}} + \underbrace{\tfrac{1}{2}(D-1)(D-2) - 1}_{\substack{\text{graviton} \\ G_{\mu\nu}}}$$

This is a 2-form gauge field, $B \sim B_{\mu\nu}\, dx^\mu \wedge dx^\nu$, with a 3-form field strength $H = dB$, in analogy to our familiar $F = dA$ of electrodyn.

- We will not go on to level 2 (which is, course, massive) but summarize:

<u>Closed string</u>:
- level matching: $N = \tilde{N}$
- mass-shell: $M^2 = 8(N-1)$ $\left[= \frac{4}{\alpha'}(N-1) \right]$
- <u>levels</u>:

$$N = (\underset{\substack{\uparrow \\ \text{tachyon}}}{0}, \underset{\substack{\uparrow \\ G_{\mu\nu},\, \phi,\, B_{\mu\nu}}}{1}, 2, \cdots)$$