5 Light-cone quantization

5.1 Light-cone gauge

- The flat gauge, which we continue to use, leaves a certain residual gauge freedom unfixed (cf. the null states of Sect. 4).
- To demonstrate this, recall the generic form of diffeomorphisms:

$$\sigma^a \to \sigma^{a\prime} = \sigma^{a\prime}(\sigma) \quad ; \quad h_{ab} \to h'_{ab}(\sigma') = h_{cd}(\sigma)\frac{\partial\sigma^c}{\partial\sigma^{a\prime}}\frac{\partial\sigma^d}{\partial\sigma^{b\prime}}$$

- Restrict to infinitesimal trfs.:

$$\sigma^{a\prime} = \sigma^a + \epsilon^a(\sigma) \quad ; \quad h'_{ab}(\sigma) = h_{ab}(\sigma) - \left(\epsilon^c\partial_c h_{ab} + (\partial_a\epsilon^c)h_{cb} + (\partial_b\epsilon^c)h_{ac}\right)$$

$$= h_{ab} - \left(D_a\epsilon_b + D_b\epsilon_a\right)$$

[Check these formulae!]

- Applying this to $h_{ab} = \eta_{ab}$ gives:

$$\eta_{ab} \to \eta_{ab} - \partial_a\epsilon_b - \partial_b\epsilon_a .$$

- Weyl rescaling gives $\eta_{ab} \to \eta_{ab}(1+2\omega)$. Hence, Weyl compensates Diff if

$$\boxed{2\omega\eta_{ab} = \partial_a\epsilon_b + \partial_b\epsilon_a} \quad \text{(residual gauge freedom)}$$

- Let's write these 3 eqs. explicitly using light-cone coordinates:

$$\underbrace{2\omega\eta_{+-} = \partial_+\epsilon_- + \partial_-\epsilon_+}_{\text{Can always be solved by choice of }\omega} \quad ; \quad \underbrace{0 = \partial_+\epsilon_+ \quad ; \quad 0 = \partial_-\epsilon_-}_{\text{Is equivalent to}}$$

$$\partial_+\epsilon^- = 0 \quad \& \quad \partial_-\epsilon^+ = 0,$$

which holds for any $\epsilon^- = \epsilon^-(\sigma^-)$ & $\epsilon^+ = \epsilon^+(\sigma^+)$

Note: We recognise 2 copies of the group of S^1-reparametrizations (generated by the Witt alg. and, in the quantum theory, by the Virasoro alg.). Indeed, this is what was left of the constraint

$T^{ab} = 0$ in flat gauge. This is consistent with the fact that T^{ab} in general generates Diff., of which only (S^1-reparam.)^2 are left after fixing the flat gauge.

- Main idea: Fix the residual gauge freedom classically (instead of first quantizing and then imposing $L_m = 0$). This can be realized by choosing $\tau \sim X^+$.
- In more detail: $X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^1)$.

Classically, X^+ satisfies the usual EOM: $\partial_+\partial_- X^+(\sigma^+,\sigma^-) = 0$.
$\tau = \frac{1}{2}(\sigma^+ + \sigma^-)$ satisfies the same eq. Hence the difference

$X^+ - (x^+ + p^+\tau)$ satisfies $\partial_+\partial_-(\cdot\cdot) = 0$ and can hence be written as

$$X^+ - (x^+ + p^+\tau) = f(\sigma^+) + g(\sigma^-).$$

With $\frac{p^+}{2}\sigma^+ + f(\sigma^+) \equiv \frac{p^+}{2}\sigma^{+\prime}$ (and analog. for $\sigma^{-\prime}$) we have (now dropping the prime): $\boxed{X^+ = x^+ + p^+\tau}$

light-cone gauge
(i.e. no X^+-oscillators excited)

- We still have the constraint $T_{ab} = 0$, which can equivalently be written as

$$(\dot{X} \pm X')^2 = 0. \qquad \text{[Check this!]}$$

$$\Rightarrow \; -2\underbrace{(\dot{X} \pm X')^+}_{\equiv p^+ \text{ because of l.c.-gauge}}(\dot{X} \pm X')^- + (\dot{X} \pm X')^i(\dot{X} \pm X')^i = 0$$

$[i = 2,3,\cdots,D-1]$

$$\Rightarrow \; (\dot{X} \pm X')^- \equiv (\dot{X} \pm X')_\perp^2 / 2p^+$$

This translates, in Fourier modes, to

$$\boxed{\alpha_n^- = \frac{1}{p^+}\left(\frac{1}{2}\sum_{m=-\infty}^{\infty}\alpha^i_{n-m}\,\alpha^i_m\right)}$$

- Thus: all α_n^+ $(n \neq 0)$ vanish by gauge choice

  all α_n^- ($\underbrace{\text{including } \alpha_0^- \sim p^-}$) are not independent

  $\Rightarrow x^{+}$ is just a constant of integration

- The canonical pairs of variables are: X^i, Π^i & x^-, p^+

5.2 Canonical quantization

Poiss. Brackets $\rightarrow$ $\frac{1}{i}$ Commutators, i.e. $[p^+, x^-] = i$

$[\alpha_m^i, \alpha_n^j] = m\,\delta^{ij}\,\delta_{m+n}$

(after Fourier trf.)

The quantum-definition of α_n^- is now affected by operator ordering:

$$\alpha_n^- = \frac{1}{p^+}\left(\frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i\right) = \frac{1}{p^+}\left(\frac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha_{n-m}^i\alpha_m^i: - a\,\delta_n\right)$$

This is the a we already determined to be $\frac{D-2}{24}$.
(We only now properly see that the Casimir energies of $D-2$ (not of D) free bosons contribute.)

The mass shell condition now reads (open string):

$$M^2 = 2p^+p^- - p_\perp^2 = 2p^+\alpha_0^- - p_\perp^2$$

$$= \sum_{n=-\infty}^{\infty} :\alpha_{-n}^i\alpha_n^i: - 2a - \alpha_0^i\alpha_0^i = 2\left(\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i - a\right)$$

$$\boxed{M^2 = 2\,(N-a) \quad \text{with} \quad N_\perp = \sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i}$$

(Note the different status of this condition: It is not a constraint but just the result of the def. of p^-)

(By contrast to the covariant approach, only transverse oscillators appear)

We can thus construct the Fock space: (which also the Hilbert space – – no neg. / zero-norm – states!)

Level 0: $|0,p\rangle$; $M^2 = -2a = -2$

Level 1: $\alpha^i_{-1}|0,p\rangle$; $M^2 = 2(1-a) = 0$ $\left\{ \begin{array}{l} \text{Lorentz symm. requires } a=1 \\ \text{since otherwise we would have} \\ \text{a massive vector with } (D-2) \text{ d.o.f.} \end{array} \right.$

Level 2: $\alpha^i_{-1}\alpha^j_{-1}|0,p\rangle$ & $\alpha^i_{-2}|0,p\rangle$; $M^2 = 2(2-a) = 2$

...

- For the closed string, one important extra point needs attention:

  - We had derived from $T_{ab}=0$ (for both open & closed):

$$(\dot{X} \pm X')^- = (\dot{X} \pm X')^2_\perp / 2p^+$$

  - The difference of these two constraints gives

$$2(X^-)' = 2\, \dot{X}_\perp \cdot X'_\perp / p^+$$

$$\underbrace{0 = \int_0^\pi (X^-)' d\sigma}_{\text{Intuitively, this is just periodicity!}} = \int_0^\pi d\sigma\, \dot{X}_\perp \cdot X'_\perp / p^+ \underset{\substack{\text{write this out in oscillators} \\ \alpha^i_n \;\&\; \tilde{\alpha}^i_n}}{\sim} N_\perp - \tilde{N}_\perp$$

  - This light-cone level-matching condition ($N_\perp = \tilde{N}_\perp$) survives as a true constraint. [Otherwise, the constraints just lead to equations for α^-_n & $\tilde{\alpha}^-_n$ in terms of the α^i_n & $\tilde{\alpha}^i_n$, as in the open case.]

  - The Fock space $\equiv$ Hilbert space is now obtained in complete analogy to the open case: Act with α^i_n & $\tilde{\alpha}^i_n$ $(n<0)$ on $|0,p\rangle$ respecting $N_\perp = \tilde{N}_\perp$.

Excursion: Young Tableaux

Before summarizing, we introduce a useful notation:

- $SU(N)$ tensor representations, $t_{i_1 \dots i_n} \to U_{i_1}{}^{j_1} \dots U_{i_n}{}^{j_n} t_{j_1 \dots j_n}$

are not irreducible. Their irreducible components can be found by systematic symmetrization / antisymmetrization of the tensor indices. This is nicely visualized by Young tableaux, e.g.

$$t_i \longrightarrow \ydiagram{1}$$

$$t_{ij} \longrightarrow \ydiagram{2} \;\hat{=}\; t_{ij} + t_{ji} \qquad \text{(symm. part)}$$

$$\ydiagram{1,1} \;\hat{=}\; t_{ij} - t_{ji} \qquad \text{(antisymm. part)}$$

$$t_{ijk} \longrightarrow \ydiagram{3} \;\hat{=}\; \underbrace{t_{ijk} + t_{jik} + \cdots}_{\text{all 6 permutations}}$$

$$\ydiagram{1,1,1} \;\hat{=}\; \underbrace{t_{ijk} - t_{jik} + t_{jki} - \cdots}_{\text{all 6 perms. with appropr. signs}}$$

$$\ydiagram{2,1} \;\hat{=}\; \text{assign indices to boxes: } \begin{ytableau} i & j \\ k \end{ytableau}$$

"mixed" repr.:

$$\underbrace{\underbrace{t_{ijk} + t_{jik}}_{\substack{\text{symmetrize}\\ \text{rows}}} - t_{kji} - t_{jki}}_{\text{then antisymmetrize columns}}$$

the generalization to, e.g.,

$$\boxed{\text{the } \ydiagram{4,3,1} \text{ subrepr. of } t_{i_1 \dots i_8} \;\hat{=}\; \underbrace{\Box \otimes \Box \otimes \cdots \otimes \Box}_{\text{8 factors}}}$$

should be obvious from our examples.
(<u>Note</u>: the rows never become longer from top to bottom and the

columns never become longer from left to right.)

- This is a very powerful tool multiplying reps., counting their dims., getting <u>all</u> irred. reprs. of $SU(N)$.

- The math. basis of this is the classification of reps. of the permutation or symmetric groups S_n by Young tableaux. This then applies to $SU(N)$-tensors since the S_n action commutes with the $SU(N)$ action. This clearly also holds for other matrix groups, in particular $SO(N)$, where Young tableaux are hence also useful.

- For $SO(N)$ there are (at least) two caveats: One can also use the "trace" (contracting two indices) when reducing a higher-rank tensor. This has to be taken care of separately. (We will assume tracelessness when writing $SO(N)$ Young tableaux.) Also, in contrast to $SU(N)$, one does not get all reps. from powers of the fund. of $SO(N)$. One should really start from the spinors, and again Young tableaux can be used.

- Our use of Young tableaux for $SO(N)$ will be "naive", i.e. just as explained for $SU(N)$ and assuming tracelessnes.

(See many group-th. books, e.g. Georgi: Lie Algebras in Part. Phys.
Hamermesh: Group Theory & its Applic.s

- Now we summarize using two nice tables from BLT:

<u>Open</u>

Level	m^2	$SO(24)$ reps.	little group	its reps.
0	-2	$\|0,p\rangle$ $\bullet$ (1)	[$SO(1,24)$?]	—

1	0	$\alpha^i_{-1}\|0,p\rangle$ $\square$ (24)	$SO(24)$	$\square$ (24)
2	2	$\alpha^i_{-2}\|0,p\rangle$ $\alpha^i_{-1}\alpha^j_{-1}\|0,p\rangle$ $\square$ $\square\square + \bullet$ (24) (299) + (1)	$SO(25)$	$\square\square$ (324)
3	4	$\alpha^i_{-3}\|0,p\rangle$ $\alpha^i_{-2}\alpha^j_{-1}\|0,p\rangle$ $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\|0,p\rangle$ $\square$ $\begin{smallmatrix}\square\\\square\end{smallmatrix} + \square\square + \bullet$ $\square\square\square + \square$ (24) (276) + (299) + (1) (2576) + (24)	$SO(25)$	$\square\square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix}$ (2900) + (300)
		...		

Closed

0	-8	$\|0,p\rangle$ $\bullet$ (1)	—	—
1	0	$\alpha^i_{-1}\tilde{\alpha}^j_{-1}\|0,p\rangle$ $\square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix} + \bullet$ (299) + (276) + (1)	$SO(24)$	same as left
		...		...

$SO(D-2) \longrightarrow$ The fact that this always fits nicely into $SO(D-1)$ reps. at all higher levels is non-trivial!

5.3 Lorentz symmetry

- Noether theorem applied to $SO(1,D-1)$

$$\Rightarrow J^{\mu\nu} = T\int_0^\pi d\sigma\,(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu)$$

— generators of $SO(1,D-1)$ in class. Hamilton formul. or in QM

$$J^{\mu\nu} = \underbrace{x^\mu p^\nu - x^\nu p^\mu}_{\equiv \; \ell^{\mu\nu}} \underbrace{- i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n \right)}_{+ \; E^{\mu\nu}}$$

angul. mom. from c.o.m. motion — "inner" angular momentum from oscillations

- Our gauge choice distinguishes X^1 w.r.t. X^i $(i = 2, \dots, D-1)$ [since $X^+ = \frac{1}{\sqrt{2}}(X^0 + X^1)$ gets a special treatment].
- Is Lorentz-symm. still intact after quantizing in this gauge?
- J^{i-} mixes X^i & X^+, so a violation could show up in this generator
- Recall: $[J^{\mu\nu}, J^{\rho\sigma}] = i \eta^{\mu\rho} J^{\nu\sigma} - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma) + \begin{pmatrix} \mu \leftrightarrow \nu \\ \rho \leftrightarrow \sigma \end{pmatrix}$

$$\Rightarrow [J^{i-}, J^{j-}] = 0$$ should hold as part of the $SO(1, D-1)$ algebra.

- In contrast to this expectation, an explicit calculation using algebra of α, p, x^i uncovers an <u>anomaly</u>:

$$[J^{i-}, J^{j-}] = -\frac{1}{(p_+)^2} \sum_{m=1}^{\infty} \Delta_m \left(\alpha^i_{-m} \alpha^j_m - \alpha^j_{-m} \alpha^i_m \right)$$

$$\text{with} \quad \Delta_m = m \, \frac{26-D}{12} + \frac{1}{m} \left(\frac{D-26}{12} + 2(1-a) \right)$$

$\Rightarrow$ As already expected from the $D-2$ d.o.f. of the vector at level 1, we <u>really need</u> $D = 26$ & $a = 1$. [$\to$ problems]

<u>Final comment:</u>

- It is possible to map the light-cone Hilbert space on the phys. Fock space of covar. quantization / gauge-inv.. This proves the absence of ghosts in the latter ("No ghost theorem"). See GSW for details.