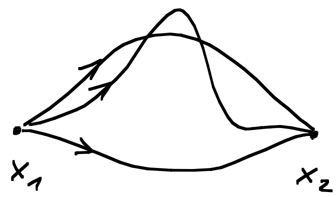
# 6 Modern Covariant Quantization – The Path Integral

## 6.1 Polyakov Path Integral

Point particle: Amplitude = sum over histories =




$$\langle x_2, t_2 | x_1, t_1 \rangle = \int \mathcal{D}x(t)\, e^{iS[x]}$$

$$x(t_{1/2}) = x_{1/2}$$

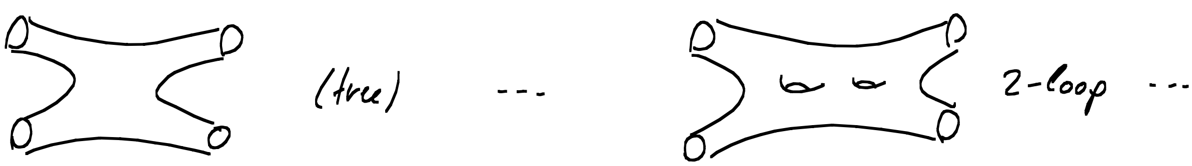
String:
analogously, we have


$$\langle \text{final} | \text{initial} \rangle = \sum_{\text{WS topologies}} \underbrace{\int \mathcal{D}X \mathcal{D}h \exp iS_p[X,h]}$$

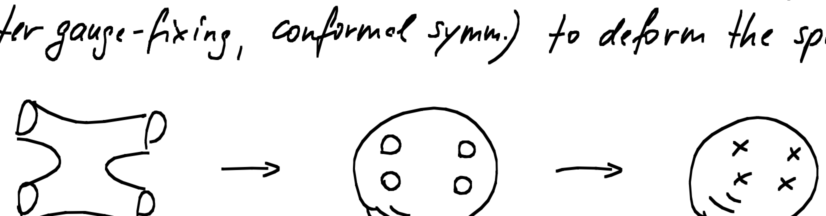
the loops of QFT are naturally included through this sum

integral over all WS embeddings & WS metrics, subject to b.c.s defined by initial & final state

- This propagation of a single string naturally generalizes to (e.g. 2→2) scattering:




- As we will see, our 2d QFT of $X^\mu$'s has enough symmetries (diff. + Weyl or, after gauge-fixing, conformal symm.) to deform the space e.g. as follows:



(sphere wit 4 distinguished points.)

- All information about the initial / final states can be encoded in specific local operators to be inserted at these special points:

$$\text{Amplitude} \sim \sum_{\substack{\text{Riemann}\\ \text{surfaces}}} \int DX\, Dh\, e^{iS_P[X,h]} \underbrace{V_1(X,h) \cdots V_4(X,h)}_{\text{vertex operators}}$$

- In analogy to QFT, the first step is to set up this functional integral without "operator insertions", i.e. to calculate the "partition function":

$$Z = \int DX\, Dh\, e^{iS}$$

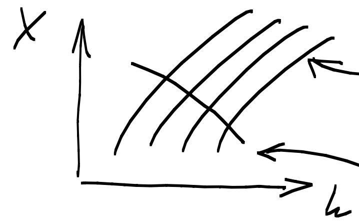
(for some fixed topology, e.g. $S^2$)

## 6.2 Gauge fixing by Fadeev-Popov procedure

- As in the quantization of gauge theories, $\int DX\, Dh$ is ill-defined (at least perturbatively) because:

  part of $X/h$-integration is just gauge change $\longrightarrow$ $S$ does not change $\longrightarrow$ $\int \cdots = \infty$

- Way out: fix gauge, i.e., integrate just over a "gauge slice":




- Recall that $h_{ab} \xrightarrow[\text{Diff \& Weyl}]{} h'_{ab} = e^{2\omega} h_{cd} \left(\frac{\partial \sigma^c}{\partial \sigma'^a}\right)\left(\frac{\partial \sigma^d}{\partial \sigma'^b}\right)$.

- Let $\zeta \equiv (\sigma' \!\!-\! \sigma, \omega)$ for notational simplicity, and write

$$h_{ab} \xrightarrow[\zeta]{} h'_{ab} \equiv h^{\zeta}_{ab}.$$

- Since $h$ has as many degrees of freedom as $\zeta$ (cf. our earlier discussion of gauge fixing), we can write: $\int Dh\, F[h] = \int D\zeta\, \det\left(\frac{\delta \hat{h}^{\zeta}}{\delta \zeta}\right) F[\hat{h}]$, with some fiducial or reference metric $\hat{h}$.

(<u>Note</u>: This is non-generic. In usual gauge theories, not all of the gauge field will be unphysical and one will be left with both an integral over the gauge field and the gauge parameter. Cf. problems.)

- Thus:
$$Z = \int D\xi\, DX \det\left(\frac{\delta \hat{h}^\xi}{\delta \xi}\right) e^{iS[X,\hat{h}^\xi]}$$
↓ renaming of integration variable (trivial)
$$= \int D\xi\, DX^\xi \det\left(\frac{\delta \hat{h}^\xi}{\delta \xi}\right) e^{iS[X^\xi,\hat{h}^\xi]}$$
↓ invariance of $S$ (obvious) & invariance of measure (highly non-trivial!)
$$= \int D\xi\, DX \underbrace{\det\left(\frac{\delta \hat{h}^\xi}{\delta \xi}\right)}_{\equiv \Delta_{FP}[\hat{h}^\xi]} e^{iS[X,\hat{h}]}$$
$\equiv \Delta_{FP}[\hat{h}^\xi]$ Fadeev-Popov-determinant

- $\Delta_{FP}$ can be further rewritten:
$$1 = \int Dh^{\xi'}\, \delta[h^{\xi'} - h^\xi] = \int D\xi' \Delta_{FP}[h^\xi]\, \delta[h^{\xi'} - h^\xi]$$
$$\Rightarrow \Delta_{FP}^{-1}[h^\xi] = \int D\xi'\, \delta[h^{\xi'} - h^\xi]$$
↓ at least for small $\xi$ this is obvious since $h^\xi = h + [\text{linear in } \xi]$
$$= \int D\xi'\, \delta[h^{\xi'\xi^{-1}} - h]$$
↓ by gauge-invariance of the measure $D\xi$.
$$= \int D(\xi'\xi^{-1})\, \delta[h^{\xi'\xi^{-1}} - h]$$
$$= \int D\xi''\, \delta[h^{\xi''} - h] = \Delta_{FP}^{-1}[h]$$

if this is true for small $\xi$, it's true for all $\xi$ by composition.

$\Rightarrow \Delta_{FP}$ is constant along gauge orbits.

- Thus: $Z = \int D\xi\, DX\, \Delta_{FP}[\hat{h}]\, e^{iS[X,\hat{h}]}$ or, absorbing the <u>constant</u> $\int D\xi$ in the normalization of $DX$:

$$\boxed{Z = \int DX \, \Delta_{FP}[\hat{h}] \, e^{iS[X,\hat{h}]}} \quad \text{with} \quad \Delta_{FP}[h] = \det \frac{\delta h^{\xi}}{\delta \xi}\Big|_{\xi=0}$$

(Since our previous manipulations were largely formal, one may say that this defines the Polyakov path integral.)

Comment: The measure $DX$ will be defined if we can give a norm on the vector space of fcts. $X^{\mu}$. A natural attempt would be

$$\|X\|^2 = \int d^2\sigma \sqrt{-h} \, X^{\mu} X^{\nu} \eta_{\mu\nu} \,.$$

↑ This factor ensures diff. invariance, but spoils Weyl invariance. This corresponds to an anomaly for Weyl tfs., which will only vanish in special situations. We have reason to hope for this in $D=26$ from OCQ & LCQ.

## 6.3 Evaluating the FP-determinant

- We need the determinant of the map $\xi \to \delta h = h^{\xi} - h$ for small $\xi$.
- It is convenient to decompose $h$ (and hence $\delta h$) in traceless part and trace:

$$\delta h_{ab} = -(D_a \epsilon_b + D_b \epsilon_a) + 2\omega h_{ab}$$

$$= -(P\epsilon)_{ab} + (2\omega - D_c \epsilon^c) h_{ab} \,, \quad \text{where}$$

$$(P\epsilon)_{ab} \equiv D_a \epsilon_b + D_b \epsilon_a - (D_c \epsilon^c) h_{ab}$$

↑ $P$ is a diff. operator acting on a 2-vector and producing a rank-2 symm. traceless tensor.

- In this language, we have (see above):

$$\Delta_{FP}^{-1}[h] = \int D\epsilon\, D\omega\, \delta[\delta h] = \int D\epsilon\, D\omega\, \delta[(2\omega - D_a\epsilon^a)h - P\epsilon]$$

$$= \int D\epsilon\, D\omega\, D\beta \exp\left[i\int d^2\sigma \sqrt{-h}\, \beta^{ab}\{(2\omega - D_c\epsilon^c)h_{ab} - (P\epsilon)_{ab}\}\right]$$

($D\beta$: inserted by analogy to $\delta^3(\bar{x}) \sim \int d^3\bar{k}\, e^{i\bar{k}\bar{x}}$; $\beta^{ab}$: by def. symmetric)

$$= \int D\epsilon\, D\beta\, \delta[\beta^{ab}h_{ab}]\, e^{-i\int d^2\sigma\sqrt{-h}\,\beta^{ab}(P\epsilon)_{ab}}$$

$$= \int D\epsilon\, D\beta\, e^{-i\int d^2\sigma\sqrt{-h}\,\beta^{ab}(P\epsilon)_{ab}}\ ,\quad \text{where } \beta^{ab} \text{ is \underline{traceless} \& symmetric.}$$

- Recalling that $\int d^n x\, d^n y\, e^{-x^T M y} \sim (\det M)^{-1}$,
  we see:
  $$\Delta_{FP}^{-1}[h] \sim (\det P)^{-1}.$$
- We need a path-integral expression not for $\Delta^{-1}$ but for $\Delta$. This can by found by using Grassmann variables:

> <u>Recall:</u> Grassmann $\theta_i$ $(i = 1 \cdots n)$ are defined by
> $$\theta_i\theta_j = -\theta_j\theta_i \quad (\text{implying } \theta_i^2 = 0\ \forall i).$$
> Integration is defined by $\int d\theta_i = 0$ ; $\int d\theta_i\, \theta_j = \delta_{ij}$.
> $$\Rightarrow \int d^n\theta\, d^n\psi\, e^{\theta^T M \psi} \sim \det M \quad (\text{not } \det M^{-1}!)$$
> ($\theta$, $\psi$: both Grassmann)
>
> [Prove this using $e^x = \sum_n \frac{x^n}{n!}$ & $\int d^n\theta\, \theta_{i_1}\theta_{i_2}\cdots\theta_{i_n} = \epsilon_{i_1 i_2 \cdots i_n}$ !]

$$\Delta_{FP}[h] = \det P = \int Db\, Dc \exp\left[\int d^2\sigma\sqrt{-h}\, b^{ab}(Pc)_{ab}\right]$$

(the prefactor is arbitrary)

- $b$ – fermionic traceless symm. tensor
  $c$ – fermionic vector } FP-ghosts

- Using tracelessness of $b$ and int. by parts we find:

$$\int \sqrt{-h}\, b^{ab}(Pc)_{ab} \sim \int b^{ab}\left(D_a c_b - D_b c_a - h_{ab}(D_c c^c)\right) \sim \int b^{ab} D_a c_b \sim \int c_a D_b b^{ab}$$

and

$$\boxed{Z = \int DX\, Db\, Dc\, e^{iS_X + iS_g} \;;\quad S_g = -\frac{i}{2\pi}\int d^2\sigma \sqrt{-h}\, c_a D_b b^{ab}}$$

## 6.4 Quantizing the FP ghosts:

- We now quantize the full action, $S_X + S_g$. We choose $\hat{h}_{ab} = \delta_{ab}$ so that nothing changes concerning $S_X$. For $S_g$ we have:

$$S_g = \frac{i}{\pi}\int d^2\sigma \left(c^+ \partial_- b_{++} + c^- \partial_+ b_{--}\right)$$ ; quantize like usual fermions, i.e. with anticommutators.

$$\Rightarrow \quad \{b_{++}(\tau,\sigma), c^+(\tau,\sigma')\} = 2\pi\,\delta(\sigma-\sigma')$$
$$\{b_{--}(\tau,\sigma), c^-(\tau,\sigma')\} = 2\pi\,\delta(\sigma-\sigma').$$
& zero otherwise

- EOMs: $\partial_- c^+ = \partial_- b_{++} = 0 \;;\; \partial_+ c^- = \partial_+ b_{--} = 0$

- Mode decomposition (closed string): $c^+ = \sqrt{2}\sum_{-\infty}^{\infty} c_n e^{-2in\sigma^+}$

(Note: No analogue of the $x^\mu/p^\mu$-part of the $X^\mu$-decomposition since $c^+$ & $c^-$ are separately periodic.)

$$c^- = \sqrt{2}\sum_{-\infty}^{\infty} \tilde{c}_n e^{-2in\sigma^-}$$
$$b_{++} = \;..\; b_n \;..$$
$$b_{--} = \;..\; \tilde{b}_n \;..$$

with $\{c_m, b_n\} = \delta_{m+n}$ & zero otherwise.
$\{\tilde{c}_m, \tilde{b}_n\} = \delta_{m+n}$

- Mode decomposition (open string): Implement b.c. by "modding out" $\mathbb{Z}_2$:

i.e. $X^\mu(\tau,\sigma)$ on $\sigma \in (-\pi,\pi)$; subject to requirement $X^\mu(\tau,\sigma) = (PX^\mu)(\tau,\sigma)$, i.e. $\mathbb{Z}_2$-invariance where $P$ is the "reflection operation" generating $\mathbb{Z}_2$. Obviously, $PX^\mu(\tau,\sigma) \equiv X^\mu(\tau,-\sigma)$. Crucially,

$$(Pc)^0(\tau,\sigma) = c^0(\tau,-\sigma)$$
$$(Pc)^1(\tau,\sigma) = -c^1(\tau,-\sigma) \quad \Rightarrow \quad (Pc)^\pm(\tau,\sigma) = c^\mp(\tau,-\sigma)$$

↑ $P$ is spatial reflection!

Thus, $\mathbb{Z}_2$-invariance means $c^+(\tau,\sigma) = c^-(\tau,-\sigma)$

($\&$ $b_{++}(\tau,\sigma) = b_{--}(\tau,-\sigma)$ by analogy).

$\Rightarrow c_n = \tilde{c}_n$ & $b_n = \tilde{b}_n$ for open string. The only commut. rels. are $\{c_m, b_n\} = \delta_{m+n}$ & otherwise zero.

- We now continue with just the <u>open string</u>:

$$S^g \longrightarrow T^g_{ab} \longrightarrow L^g_m \quad \text{(same logic for } S^X\text{)}$$

$$L^g_m = \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma\, e^{im\sigma}\, T^g_{++} = \sum_{n=-\infty}^{\infty} (m-n)\, b_{m+n}\, c_{-n}$$

$$\boxed{[L^g_m, L^g_n] = (m-n)L^g_{m+n} + A^g(m)\,\delta_{m+n} \; ; \; A_g(m) = \frac{1}{6}(m - 13m^3)}$$

- Finally: $S = S^X + S^g$ ; $L_m \equiv L^X_m + L^g_m - a\delta_m$

<u>Anomaly of $L_m$-algebra</u> (total anomaly)

$$\boxed{A(m) = \frac{D}{12}(m^3 - m) + \frac{1}{6}(m - 13m^3) + 2am}$$

By contrast to our discussion of old covar. quantization in Sects. 3/4, we now include the normal-ordering constant in the def. of $L$.

We see: $A(m) = 0 \iff \{ D = 26 \ \& \ a = 1 \}$

[This means that Weyl-invariance is non-anomalous and that, after gauge fixing, we will get a CFT; see also later.]