7 Modern Covariant Quantization – BRST

7.1 The general structure of BRST transformations

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- let $Z = \int D\phi \, e^{-S_\phi[\phi]}$ (euclidean, i.e. $t \to -it$ with $S = \int (T+V)$; just for notational simplicity)

  $\uparrow$ [any set of fields, e.g. our $X^\mu(\xi)$ & $h_{ab}(\xi)$; the label "A" includes "μ", "ab" and even "ξ".]

- let S_ϕ have a gauge symm., to be fixed by a set of gauge conditions labelled by A: $\quad F^A(\phi) = 0$. $\quad$ [e.g. $h_{ab} - \eta_{ab} = 0$ in our case]

- Up to normalization, we find (following our previous logic):

$$Z = \int D\phi \, DB_A \, Db_A \, Dc^\alpha \, \exp\left\{ -S_\phi[\phi] - S_{gf}[B,\phi] - S_g[b,c,\phi] \right\}$$

where $S_{gf} = -iB_A F^A(\phi)$ (this produces the δ-fct. which fixes the gauge)

and $S_g = b_A c^\alpha \delta_\alpha F^A(\phi)$ (this calculates the determinant of $\delta_\alpha F^A(\phi)$, which is just the FP determinant)

$\uparrow$ These are the gauge group generators, satisfying $\quad [\delta_\alpha, \delta_\beta] = f_{\alpha\beta}{}^\gamma \delta_\gamma$

Note: Quite generally, the c-ghosts are labelled by the gauge group index α (in our case the vector index of ϵ^a) and the b-ghosts are labelled by the gauge-condition-index A (in our case the tensor indices of $h^{ab} = \eta^{ab}$). Naively, in our case we would expect one further c-ghost (for ω)

and one further b-ghost (for the trace-part of h^{ab}). However, we did not need them since we were able to carry the ω-integration in the definition of Δ_{FP}^{-1} out explicitly, making our β^{ab} traceless. This effectively reduced the matrix $\delta_\alpha F^A$ from "3x3" to "2x2".

- Fact: $S = S_\phi + S_{gf} + S_g$ is invariant under a "generalized gauge trf." with gauge parameter c^α (and some infinites. parameter ϵ):

$$\left.\begin{aligned} \delta_{BRST}\,\phi &= -i\epsilon c^\alpha \delta_\alpha \phi \\ \delta_{BRST}\,B_A &= 0 \\ \delta_{BRST}\,b_A &= \epsilon B_A \\ \delta_{BRST}\,c^\alpha &= \frac{i}{2}\epsilon c^\beta c^\gamma f_{\beta\gamma}{}^\alpha \end{aligned}\right\} \text{BRST-trf.}$$

(the corresponding Noether charge Q is also called BRST operator and will be crucial below)

7.2 The general structure of BRST quantization

- Fact: $\delta_{BRST}(b_A F^A) = i\epsilon(S_{gf} + S_g)$ (easy to check)
- Consider an infinitesimal variation of the gauge condition:

$$F^A \to F'^A = F^A + \delta_g F^A$$

g for "gauge", $\underline{\underline{\text{not}}}$ "ghost".

- An amplitude of typ $\int\limits_{x(t_{1/2}) = x_{1/2}} Dx\, e^{iS}$ must be gauge independent, i.e.:

$$\int\limits_{ini./fin.} D\phi DBDbDc\, \exp\{-(S_\phi + S_{gf} + S_g)\} - \left[\begin{matrix}\text{same with } S_{gf} + S_g \\ \to S_{gf} + S_g + \delta_g(S_{gf} + S_g)\end{matrix}\right] = 0$$

$$\Rightarrow \int\limits_{ini./fin.} D\phi .. Dc\; \delta_g(S_{gf} + S_g)\, e^{-S} = 0$$

$$\Rightarrow \langle final |\, \delta_g(S_{gf} + S_g)\, | initial \rangle = 0$$

Use $\delta_g (S_{gf} + S_g) = \delta_g \delta_{BRST} (b_A F^A) = \delta_{BRST} (b_A \delta_g F^A) \sim \{Q, b_A \delta_g F^A\}$

$\Rightarrow \langle \text{final} | \{Q, b_A \delta_g F^A\} | \text{initial} \rangle = 0$ for all $\delta_g F^A$.

To realize this, demand: $\boxed{Q\,|\text{phys}\rangle = 0}$ (& Q hermitian)

• Fact: $Q^2 = 0$ (Check e.g. $\delta^{\epsilon'}_{BRST} \delta^{\epsilon}_{BRST} b_A = \delta^{\epsilon'}_{BRST} (\epsilon B_A) = 0$ ✓ & analogously on all other fields)

$\Rightarrow$ States of the form $Q|\chi\rangle$ (any $|\chi\rangle$) are always physical and orthogonal to all physical states. They are "null".

$$(\langle \text{phys} | Q | \chi \rangle = \langle \chi | Q | \text{phys} \rangle^* = 0)$$

The true, physical Hilbert space takes a very natural mathematical form:

$$\underline{\underline{\mathcal{H}_{BRST} = \frac{\ker Q}{\operatorname{Im} Q}}} = \frac{\mathcal{H}_{closed}}{\mathcal{H}_{exact}} \quad \begin{matrix} \leftarrow \text{all } |\psi\rangle \text{ with } Q|\psi\rangle = 0 \\ \leftarrow \text{all states of form } Q|\psi\rangle \end{matrix}$$

(cf. closed vs. exact diff. forms)

$\underline{\underline{\mathcal{H}_{BRST} \text{ is the cohomology of } Q}}$

(cf. H^p as the cohomology of d acting on Ω^p)

7.3 BRST Quantization of bosonic string

• Recall: $L_m = L^X_m + L^g_m - a\delta_m$

where $L^X_m = :\frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \alpha_n: \quad \left(L^X_0 = \frac{p^2}{2} + \sum_{n=1}^{\infty} \alpha_{-n} \alpha_n \right)$

$L^g_m = :\sum_{n=-\infty}^{\infty} (m-n)\, b_{m+n} c_{-n} : \quad \left(L^g_0 = \sum_{n=1}^{\infty} n \left(b_{-n} c_n + c_{-n} b_n \right) \right)$

["a" can now be explicitly understood as resulting from normal-ordering the original expression. The ghost contribution compensates the effect of 2 of the D bosons "X". This justifies a posteriori our earlier Casimir calculation of "a". → Problems.]

- Noether theorem applied to BRST-symmetry:

$$\Rightarrow \quad Q = \sum_{-\infty}^{\infty} :\left(L^X_{-m} + \frac{1}{2} L^g_{-m} - a\delta_m\right) c_m :$$

$$= \sum_{-\infty}^{\infty} \left(L^X_{-m} - a\delta_m\right) c_m \; + \; :\sum_{m,n=-\infty}^{\infty} \frac{m-n}{2}\, b_{m+n} c_{-n} c_{-m} :$$

(In this second form, the factor "1/2" appears more naturally; see also the simple application further down.)

- One straightforwardly checks:

$$Q^2 = \frac{1}{2}\{Q,Q\} = \frac{1}{2}\sum_{m,n=-\infty}^{\infty} \Big(\underbrace{[L_m, L_n] - (m-n)L_{m+n}}_{\sim \text{ Virasoro anomaly}}\Big) c_{-m} c_{-n}$$

$\Rightarrow$ $\boxed{\text{Since the BRST formalism needs } Q^2 = 0, \text{ we are forced to take } D = 26 \;\&\; a = 1.}$

- For a fermionic oscillator algebra, the choice of vacuum is not as obvious as for a boson. Consider e.g. the b_0-c_0-subalgebra:

$$c_0^2 = b_0^2 = 0 \;; \quad \{c_0, b_0\} = 1 .$$

Define $|\downarrow\rangle$ by $b_0|\downarrow\rangle = 0$ and give the stat $c_0|\downarrow\rangle$ the name $|\uparrow\rangle = c_0|\downarrow\rangle$. Obviously, $c_0|\uparrow\rangle = 0$. Furthermore, $b_0|\uparrow\rangle = b_0 c_0|\downarrow\rangle = |\downarrow\rangle - c_0 b_0|\downarrow\rangle = |\downarrow\rangle$. That's it!

- Completely analogous 2-state representations exist for c_1, b_{-1} ; b_1, c_{-1} ; c_2, b_{-2} etc. The full Fock space is just the tensor product.

- Clearly, nothing distinguishes the pairs of states from the point of view of the algebra (as opposed to a bosonic oscillator!) and either could be the vacuum.
- It is the Hamilton (in our case L_0^g) that fixes the vacuum (in our case the states annihilated by b_n, c_n with $n > 0$).

- However, b_0, c_0 don't appear in L_0 and we appear to have full freedom to build a theory <u>on the vacuum $|\downarrow\rangle$ or on $|\uparrow\rangle$.</u> This corresponds to a doubling of the complete spectrum and we must resolve this ambiguity in one way or the other.

- The choice is dictated by what we have learned before (in OCQ & light-cone quantization):

  - Consider a state $|\chi\rangle$ built by acting with $\alpha_{-m}'s$ on $|\downarrow\rangle$ (i.e. no ghosts are excited).
  - Then $Q|\chi\rangle \overset{!}{=} 0 \Rightarrow \left[(L_0^X - 1)c_0 + \sum_{m>0} c_{-m} L_m^X\right]|\chi\rangle \overset{!}{=} 0$
  - Hence $(L_0^X - 1)|\chi\rangle \overset{!}{=} 0$ & $L_m^X|\chi\rangle \overset{!}{=} 0$. This is what we expect!
  - By contrast, if we had built $|\chi\rangle$ on $|\uparrow\rangle$, we would have $c_0|\chi\rangle = 0$ and no L_0^X-constraint.

$\Rightarrow$ We define our Hilbert space as the "Q-cohomology" on the Fock space built on $|p,\downarrow\rangle$ (all p's) without using c_0. (i.e. with the extra constraint $b_0|\psi\rangle = 0$)

- The fact that this space does indeed fullfill the Hilbert space axioms is known as the "no ghost theorem" and can be proven more easily here than in OCQ ($\to$ Polchinski, I).

- We only try to provide some intuition using the simplest examples:

Before starting level by level:

It is easy to check that $\{Q, b_0\} = L_0$. Using $b_0|\psi\rangle = 0$, we have

$$Q|\psi\rangle = 0 \Rightarrow L_0|\psi\rangle = 0 \Rightarrow m^2 = 2(\underset{\substack{\uparrow \\ N^X + N^g \\ \text{(total level)}}}{N} - 1).$$

Hence the mass shell condition is guaranteed at all levels, as in OCQ.

(0) Level zero: $|0,p,\downarrow\rangle$

$$0 \stackrel{!}{=} Q|0,p,\downarrow\rangle = \left[\sum \left(L^X_{-m} - a\delta_m\right)c_m + \sum \frac{m-n}{2} : b_{m+n} c_{-m} c_{-n} :\right]|0,p,\downarrow\rangle$$

$$\xrightarrow[\text{wit annihilators}]{\text{drop all terms}} \quad 0 \stackrel{!}{=} \left[\left(\frac{p^2}{2} - a\right)c_0\right]|0,p,\downarrow\rangle$$

$$\Rightarrow p^2 = 2 \qquad \text{(tachyon)}$$

[Since Q does'nt change the level and p^2, this also implies that ther are no Q-exact phys. states at level zero.]

(1) Level one $\quad |\psi\rangle = (\varepsilon \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|0,p,\downarrow\rangle$

- Consider $Q|\psi\rangle$ term by term:

$$(L^X_0 - 1)c_0\,(\varepsilon \cdot \alpha_{-1})|\cdots\rangle = c_0\left(\frac{p^2}{2} + 1 - 1\right)(\varepsilon \cdot \alpha_{-1})|\cdots\rangle = 0 \quad \text{(since } p^2 = 0\text{)}$$

$$(L^X_0 - 1)\,c_0\,(\beta b_{-1})|\cdots\rangle = c_0\left(\frac{p^2}{2} - 1\right)\beta b_{-1}|\cdots\rangle$$

$$(L^X_0 - 1)\,c_0\,(\gamma c_{-1})|\cdots\rangle = c_0\left(\frac{p^2}{2} - 1\right)\gamma c_{-1}|\cdots\rangle$$

$$\sum \frac{m-n}{2} :b_{m+n} c_{-m} c_{-n} : (\beta b_{-1})|..\rangle = c_0 \beta b_{-1} |..\rangle$$
$$\sum \frac{m-n}{2} :b_{m+n} c_{-m} c_{-n} : (\gamma c_{-1})|..\rangle = c_0 \gamma c_{-1} |..\rangle$$

Cancels previous 2 lines, using also $p^2 = 0$

The only other non-trivial contributions come from $\sum_{m \neq 0} L_m^X c_{-m}$ and give:

$$(c_{-1}\, p\cdot\alpha_1 + c_1\, p\cdot\alpha_{-1})(\varepsilon\cdot\alpha_{-1} + \beta b_{-1} + \gamma c_{-1})|..\rangle$$
$$= [c_{-1}\, p\cdot\varepsilon + \beta (p\cdot\alpha_{-1})]|..\rangle = 0$$

if $\underline{\varepsilon \text{ transverse}}$ & $\underline{\beta = 0}$.

<u>Thus:</u>
- b-excitations are forbidden
- c-excitations are gauge freedom
- X-excitations with $\varepsilon \sim p$ are residual gauge freedom (as before)

This is also in the image of Q and is removed by $\cdots/_{\mathrm{Im}\,Q}$.

- transverse X-excitations are physical

- <u>Higher levels</u>: analogously, but won't go there...

<u>Advanced comment:</u> The argument given so far to eliminate the $|\uparrow\rangle$-states is not completely satisfactory. For the point-particle, Polch. I (Sect. 4.2) contains an argument that $|\downarrow\rangle$-states must be eliminated since their amplitudes would be $\sim \delta(k^2+m^2)$, which is known not to arise in QFT on general grounds. A similar argument for ST (as a UV-completion of QFT) can be made as follows: BLT show in the paragraph before eq. (5.53), that $|\uparrow\rangle$-states

are Q-exact except if they are on-shell. Polch. I shows in Sect. 9 that Q-exact states decouple from amplitudes. This takes us to a point where the point-particle QFT-argument given above can be applied to ST.