# Polarisation of the $Z$ boson in $pp \to ZH$.

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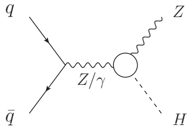
1. The process $pp \to ZH$.
   - It provides direct access to $HZZ$ and $HZ\gamma$ couplings.
   - It receives more attention after Butterworth, Davison, Rubin and Salam (2008).
2. Polarisation of the $Z$ boson.
   - The $Z$ boson is a spin 1 particle, thus can be in a polarised state.
   - Polarisation of the $Z$ boson in a mixed state is uniquely described by a single $3 \times 3$ Hermitian density matrix $\rho$, which is constructed of the production amplitudes. D.o.f of $\rho$ is 9 (without $tr(\rho) = 1$).
   - DM $\rho$ (i.e. polarisation) can be determined by studying the decay $Z \to f\bar{f}$:



## Introduction.

- DM $\rho$ (i.e. polarisation) can be determined by studying the decay $Z \to f\bar{f}$:

$$
\begin{aligned}
Z &: \ \left(m_Z, 0, 0, 0\right) \\
f &: \ \frac{m_Z}{2}\left(1,\ \sin\widehat{\theta}\cos\widehat{\phi},\ \sin\widehat{\theta}\sin\widehat{\phi},\ \cos\widehat{\theta}\right) \\
\bar{f} &: \ \frac{m_Z}{2}\left(1,\ -\sin\widehat{\theta}\cos\widehat{\phi},\ -\sin\widehat{\theta}\sin\widehat{\phi},\ -\cos\widehat{\theta}\right).
\end{aligned}
$$

The differential cross section has 9 independent angular distributions:

$$
\begin{aligned}
\frac{d\sigma(pp \to Z(\to f\bar{f})H)}{d\Omega\ d\cos\widehat{\theta}\ d\widehat{\phi}} =& F_1\left(1+\cos^2\widehat{\theta}\right) + F_2\left(1-3\cos^2\widehat{\theta}\right) + F_3\cos\widehat{\theta} \\
&+ F_4\sin\widehat{\theta}\cos\widehat{\phi} + F_5\sin 2\widehat{\theta}\cos\widehat{\phi} + F_6\sin^2\widehat{\theta}\cos 2\widehat{\phi} \\
&+ F_7\sin\widehat{\theta}\sin\widehat{\phi} + F_8\sin 2\widehat{\theta}\sin\widehat{\phi} + F_9\sin^2\widehat{\theta}\sin 2\widehat{\phi}.
\end{aligned}
$$

- "D.o.f of DM $\rho$" = "the number of the functions $F_i$" = 9.
- The 9 functions $F_i$ can be written in terms of the elements of DM $\rho$.
- "Measurement of all the 9 $F_i$" = "Determination of $\rho$ (i.e. polarisation)"

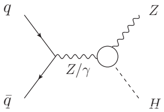
The purpose of this work is to present one approach to make use of the polarisation information in order to study $HZV(V = Z, \gamma)$ couplings.

## Outline

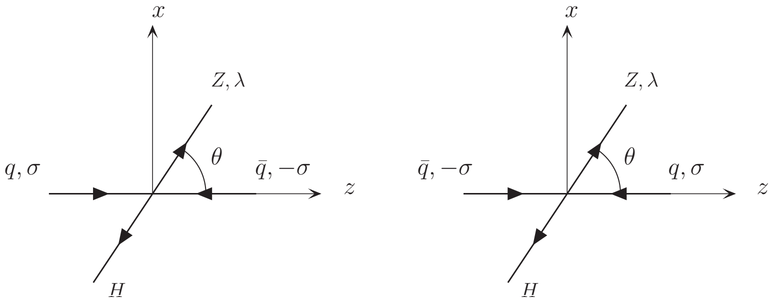
- Introduction.
- Production amplitudes.
- Decay angular distributions of the polarised $Z$ boson.
- Symmetry properties.
- Numerical studies.

## Production amplitudes: 2 c.m. frames.

We assume the standard interaction for $qqV$ and non-standard interactions for $HZZ$ and $HZ\gamma$:



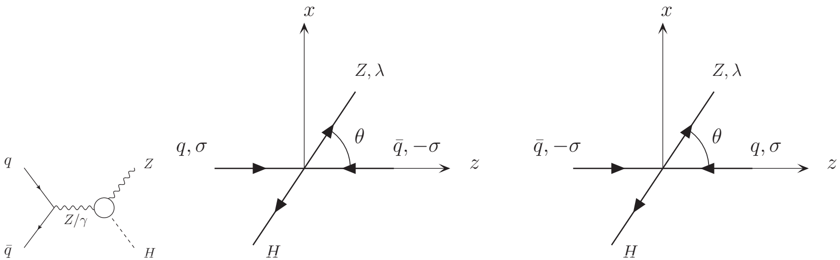

- $q\bar{q}$ c.m. frame (left), where $q$ moves along the positive direction of the $z$-axis.
- $\bar{q}q$ c.m. frame (right), where $\bar{q}$ moves along the positive direction of the $z$-axis:




$\lambda$: helicity of the $Z$ boson, $\sigma$: helicity of $q$, $-\sigma$: helicity of $\bar{q}$.

## Production amplitudes: factorisation of the $\theta$ dependent part.

Feynman diagram, $q\bar{q}$ c.m. frame (left) and $\bar{q}q$ c.m. frame (right):




Helicity amplitudes in $q\bar{q}$ c.m. frame are

$$\mathcal{M}^{\lambda=\pm}_{\sigma=\pm}(q\bar{q}) = \sigma\frac{1+\sigma\lambda\cos\theta}{\sqrt{2}}\hat{M}^{\lambda=\pm}_{\sigma}, \qquad \mathcal{M}^{\lambda=0}_{\sigma=\pm}(q\bar{q}) = \sin\theta\hat{M}^{\lambda=0}_{\sigma}.$$

Helicity amplitudes in $\bar{q}q$ c.m. frame are

$$\mathcal{M}^{\lambda=\pm}_{\sigma=\pm}(\bar{q}q) = -\sigma\frac{1-\sigma\lambda\cos\theta}{\sqrt{2}}\hat{M}^{\lambda=\pm}_{\sigma}, \qquad \mathcal{M}^{\lambda=0}_{\sigma=\pm}(\bar{q}q) = \sin\theta\hat{M}^{\lambda=0}_{\sigma}.$$

- Polar angle $\theta$ dependence is completely factorised.
- $\hat{M}^{\lambda}_{\sigma}$ depend on an explicit form of the $HZV(V=Z,\gamma)$ interactions.
- Notice the difference between the amplitudes in $q\bar{q}$ c.m. frame and those in $\bar{q}q$ c.m. frame: the sign in front of $\sigma$ in the $\theta$ dependent part.

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## Decay angular distributions of polarised $Z$: the full-amplitude squared.

The full-process is $q(\sigma) + \bar{q}(-\sigma) \to Z(\lambda) + H; \;\; Z(\lambda) \to f(\tau) + \bar{f}(-\tau)$.
The full-amplitudes are

$$\mathcal{T}_\sigma^\tau(q\bar{q}) = P_Z \sum_{\lambda=\pm,0} \mathcal{M}_\sigma^\lambda(q\bar{q}) \; D_\lambda^\tau, \qquad \mathcal{T}_\sigma^\tau(\bar{q}q) = P_Z \sum_{\lambda=\pm,0} \mathcal{M}_\sigma^\lambda(\bar{q}q) \; D_\lambda^\tau,$$

where decay amplitude $D_\lambda^\tau$ is common

$$D_\lambda^\tau = g_{Zf\bar{f}}^\tau m_Z \; d_\lambda^\tau, \qquad P_Z = (Q^2 - m_Z^2 + i m_Z \Gamma_Z)^{-1}.$$

The full-amplitude squared is

$$\begin{aligned}
\sum_{\sigma=\pm} |\mathcal{T}_\sigma^\tau(q\bar{q})|^2 &= \left|P_Z m_Z g_{Zf\bar{f}}^\tau\right|^2 \sum_\sigma \sum_{\lambda',\lambda} \left(d_{\lambda'}^\tau\right)^* \left\{\mathcal{M}_\sigma^{\lambda'}(q\bar{q})\right\}^* \mathcal{M}_\sigma^\lambda(q\bar{q}) \; d_\lambda^\tau \\
&= \left|P_Z m_Z g_{Zf\bar{f}}^\tau\right|^2 \sum_\sigma \sum_{\lambda',\lambda} \left(d_{\lambda'}^\tau\right)^* \; \rho_\sigma^{\lambda'\lambda}(q\bar{q}) \; d_\lambda^\tau \\
&= \left|P_Z m_Z g_{Zf\bar{f}}^\tau\right|^2 \; d^{\tau\dagger} \; \rho(q\bar{q}) \; d^\tau,
\end{aligned}$$

where DM $\rho(q\bar{q})$ is defined as

$$\rho^{\lambda'\lambda}(q\bar{q}) \equiv \sum_\sigma \rho_\sigma^{\lambda'\lambda}(q\bar{q}) \equiv \sum_\sigma \left\{\mathcal{M}_\sigma^{\lambda'}(q\bar{q})\right\}^* \mathcal{M}_\sigma^\lambda(q\bar{q}) \;\; : \;\; \text{DM elements.}$$

We do the same for the other full-amplitude:

$$\sum |\mathcal{T}_\sigma^\tau(\bar{q}q)|^2 = \left|P_Z m_Z g_{Zf\bar{f}}^\tau\right|^2 d^{\tau\dagger} \rho(\bar{q}q) \; d^\tau, \quad \rho^{\lambda'\lambda}(\bar{q}q) \equiv \sum \left\{\mathcal{M}_\sigma^{\lambda'}(\bar{q}q)\right\}^* \mathcal{M}_\sigma^\lambda(\bar{q}q).$$

## Decay angular distributions of polarised $Z$: the complete differential cross section.

The complete differential cross section is

$$\frac{d\sigma}{d\hat{s}\ dy\ d\cos\theta\ d\cos\widehat{\theta}\ d\widehat{\phi}} \propto q(x_1)\bar{q}(x_2)\ d^{\tau\dagger}\rho(q\bar{q})\ d^{\tau} + \bar{q}(x_1)q(x_2)\ d^{\tau\dagger}\rho(\bar{q}q)\ d^{\tau}.$$

$q(x_i)$ : PDF of a quark with energy fraction $x_i$.

$\hat{s} = sx_1x_2$ : $\mathrm{q\bar{q}}$ and $\mathrm{\bar{q}q}$ c.m. energy squared; $(m_Z + m_H)^2 < \hat{s} < s$.

$y = \frac{1}{2}\ln\frac{x_1}{x_2}$ : rapidity of the $\mathrm{q\bar{q}}$ and $\mathrm{\bar{q}q}$ c.m. frames; $-\ln\sqrt{\frac{s}{\hat{s}}} < y < \ln\sqrt{\frac{s}{\hat{s}}}$.

$x_1 = \sqrt{\hat{s}/s}e^{y}, \qquad x_2 = \sqrt{\hat{s}/s}e^{-y}$

$\theta$ : polar angle of the Z boson from the z − axis.

$\widehat{\theta}, \widehat{\phi}$ : Z decay angles.

- Density matrices $\rho$ are given for each phase space point (each value of $\hat{s}, y, \cos\theta$ in our case).
- We perform $\hat{s}$, $y$ and $\cos\theta$ integration.
- As a result, we obtain "the integrated DM $\rho$" $\rightarrow$ good from statistics's point of view.

## Decay angular distributions of polarised $Z$: First, $\cos\theta$ integration.

Let us remind that

- $\theta$ dependence is factorised as $\mathcal{M}_\sigma^{\lambda=\pm}(q\bar{q}) = \sigma\frac{1+\sigma\lambda\cos\theta}{\sqrt{2}}\hat{M}_\sigma^{\lambda=\pm}$ for e.g..
- Thus, $\rho_\sigma^{+0}(q\bar{q}) = \left\{\mathcal{M}_\sigma^{+}(q\bar{q})\right\}^*\mathcal{M}_\sigma^{0}(q\bar{q}) = \sigma\frac{1+\sigma\cos\theta}{\sqrt{2}}\sin\theta\left\{\hat{M}_\sigma^{+}\right\}^*\hat{M}_\sigma^{0}$

$$\frac{d\sigma}{d\hat{s}\ dy\ d\cos\widehat{\theta}\ d\widehat{\phi}} \propto \int_{-1}^{1} d\cos\theta\ q(x_1)\bar{q}(x_2)\ d^{\tau\dagger}\rho(q\bar{q})\ d^{\tau} + \bar{q}(x_1)q(x_2)\ d^{\tau\dagger}\rho(\bar{q}q)\ d^{\tau}$$
$$= q(x_1)\bar{q}(x_2)\ d^{\tau\dagger}\langle\rho(q\bar{q})\rangle\ d^{\tau} + \bar{q}(x_1)q(x_2)\ d^{\tau\dagger}\langle\rho(\bar{q}q)\rangle\ d^{\tau},$$

where

$$\langle\rho(q\bar{q})\rangle \equiv \int_{-1}^{1} d\cos\theta\rho(q\bar{q}) = \sum_\sigma \begin{pmatrix} \frac{4}{3}\left|\hat{M}_\sigma^{+}\right|^2 & \frac{2}{3}\left(\hat{M}_\sigma^{+}\right)^*\hat{M}_\sigma^{-} & \frac{\sigma\pi}{2\sqrt{2}}\left(\hat{M}_\sigma^{+}\right)^*\hat{M}_\sigma^{0} \\ \frac{2}{3}\hat{M}_\sigma^{+}\left(\hat{M}_\sigma^{-}\right)^* & \frac{4}{3}\left|\hat{M}_\sigma^{-}\right|^2 & \frac{\sigma\pi}{2\sqrt{2}}\left(\hat{M}_\sigma^{-}\right)^*\hat{M}_\sigma^{0} \\ \frac{\sigma\pi}{2\sqrt{2}}\hat{M}_\sigma^{+}\left(\hat{M}_\sigma^{0}\right)^* & \frac{\sigma\pi}{2\sqrt{2}}\hat{M}_\sigma^{-}\left(\hat{M}_\sigma^{0}\right)^* & \frac{4}{3}\left|\hat{M}_\sigma^{0}\right|^2 \end{pmatrix},$$

$$\langle\rho(\bar{q}q)\rangle \equiv \int_{-1}^{1} d\cos\theta\rho(\bar{q}q) = \sum_\sigma \begin{pmatrix} \frac{4}{3}\left|\hat{M}_\sigma^{+}\right|^2 & \frac{2}{3}\left(\hat{M}_\sigma^{+}\right)^*\hat{M}_\sigma^{-} & -\frac{\sigma\pi}{2\sqrt{2}}\left(\hat{M}_\sigma^{+}\right)^*\hat{M}_\sigma^{0} \\ \frac{2}{3}\hat{M}_\sigma^{+}\left(\hat{M}_\sigma^{-}\right)^* & \frac{4}{3}\left|\hat{M}_\sigma^{-}\right|^2 & -\frac{\sigma\pi}{2\sqrt{2}}\left(\hat{M}_\sigma^{-}\right)^*\hat{M}_\sigma^{0} \\ -\frac{\sigma\pi}{2\sqrt{2}}\hat{M}_\sigma^{+}\left(\hat{M}_\sigma^{0}\right)^* & -\frac{\sigma\pi}{2\sqrt{2}}\hat{M}_\sigma^{-}\left(\hat{M}_\sigma^{0}\right)^* & \frac{4}{3}\left|\hat{M}_\sigma^{0}\right|^2 \end{pmatrix}$$

- Notice the difference between $\langle\rho(q\bar{q})\rangle$ and $\langle\rho(\bar{q}q)\rangle$: the sign in front of $\sigma$.
- $\langle\rho(q\bar{q})\rangle + \langle\rho(\bar{q}q)\rangle \rightarrow$ "the $\sigma$ terms" vanish.

## Decay angular distributions of polarised $Z$: Second, $y$ integration.

$$\frac{d\sigma}{d\hat{s}\ d\cos\widehat{\theta}\ d\widehat{\phi}} \propto \int_{-y_{\rm cut}}^{y_{\rm cut}} dy\ q(x_1)\bar{q}(x_2)\ d^{\tau\dagger}\langle\rho(q\bar{q})\rangle\ d^{\tau} + \bar{q}(x_1)q(x_2)\ d^{\tau\dagger}\langle\rho(\bar{q}q)\rangle\ d^{\tau}$$

$$= \underbrace{\int_{0}^{y_{\rm cut}}}_{x_1>x_2} dy\ \underbrace{q(x_1)\bar{q}(x_2)}_{A}\ d^{\tau\dagger}\langle\rho(q\bar{q})\rangle\ d^{\tau} + \underbrace{\bar{q}(x_1)q(x_2)}_{B}\ d^{\tau\dagger}\langle\rho(\bar{q}q)\rangle\ d^{\tau}$$

$$+ \underbrace{\int_{-y_{\rm cut}}^{0}}_{x_2>x_1} dy\ \underbrace{q(x_1)\bar{q}(x_2)}_{B}\ d^{\tau\dagger}\langle\rho(q\bar{q})\rangle\ d^{\tau} + \underbrace{\bar{q}(x_1)q(x_2)}_{A}\ d^{\tau\dagger}\langle\rho(\bar{q}q)\rangle\ d^{\tau}$$

$$= \int_{0}^{y_{\rm cut}} dy\ \{q(x_1)\bar{q}(x_2)+\bar{q}(x_1)q(x_2)\}\ d^{\tau\dagger}\{\langle\rho(q\bar{q})\rangle + \langle\rho(\bar{q}q)\rangle\}\ d^{\tau}$$

$$= \int_{0}^{y_{\rm cut}} dy\ 2\{q(x_1)\bar{q}(x_2)+\bar{q}(x_1)q(x_2)\}$$

$$\times\ d^{\tau\dagger}\sum_{\sigma}\begin{pmatrix} \frac{4}{3}\left|\hat{M}_\sigma^{+}\right|^2 & \frac{2}{3}\left(\hat{M}_\sigma^{+}\right)^{*}\hat{M}_\sigma^{-} & 0 \\ \frac{2}{3}\hat{M}_\sigma^{+}\left(\hat{M}_\sigma^{-}\right)^{*} & \frac{4}{3}\left|\hat{M}_\sigma^{-}\right|^2 & 0 \\ 0 & 0 & \frac{4}{3}\left|\hat{M}_\sigma^{0}\right|^2 \end{pmatrix} .d^{\tau}$$

- Recall that $y = \frac{1}{2}\ln\frac{x_1}{x_2}$, $y_{\rm cut} = \ln\sqrt{s/\hat{s}}$, $x_1 = \sqrt{\hat{s}/s}\ e^{y}$, $x_2 = \sqrt{\hat{s}/s}\ e^{-y}$.
- the 4 off-diagonal elements of the DM vanish after the $\cos\theta$ and $y$ integration, due to the sign difference in $\langle\rho(q\bar{q})\rangle$ and $\langle\rho(\bar{q}q)\rangle$.

## Decay angular distributions of polarised $Z$: $\cos\theta$ integration but in a different way.

We subtract the contribution of $\cos\theta < 0$ from that of $\cos\theta > 0$:

$$\frac{d\sigma}{d\hat{s}dyd\cos\widehat{\theta}d\widehat{\phi}} \propto \left(\int_0^1 - \int_{-1}^0\right) d\cos\theta \ q(x_1)\bar{q}(x_2)\ d^{\tau\dagger}\rho(q\bar{q})\ d^{\tau} + \bar{q}(x_1)q(x_2)\ d^{\tau\dagger}\rho(\bar{q}q)\ d^{\tau}$$
$$= q(x_1)\bar{q}(x_2)\ d^{\tau\dagger}\overline{\langle\rho(q\bar{q})\rangle}\ d^{\tau} + \bar{q}(x_1)q(x_2)\ d^{\tau\dagger}\overline{\langle\rho(\bar{q}q)\rangle}\ d^{\tau},$$

where

$$\overline{\langle\rho(q\bar{q})\rangle} \equiv \left(\int_0^1 - \int_{-1}^0\right) d\cos\theta\ \rho(q\bar{q})$$
$$= \sum_{\sigma} \begin{pmatrix} \sigma\left|\hat{M}_{\sigma}^{+}\right|^2 & 0 & \frac{2}{3\sqrt{2}}\left(\hat{M}_{\sigma}^{+}\right)^*\hat{M}_{\sigma}^{0} \\ 0 & -\sigma\left|\hat{M}_{\sigma}^{-}\right|^2 & -\frac{2}{3\sqrt{2}}\left(\hat{M}_{\sigma}^{-}\right)^*\hat{M}_{\sigma}^{0} \\ \frac{2}{3\sqrt{2}}\hat{M}_{\sigma}^{+}\left(\hat{M}_{\sigma}^{0}\right)^* & -\frac{2}{3\sqrt{2}}\hat{M}_{\sigma}^{-}\left(\hat{M}_{\sigma}^{0}\right)^* & 0 \end{pmatrix},$$

$$\overline{\langle\rho(\bar{q}q)\rangle} \equiv \left(\int_0^1 - \int_{-1}^0\right) d\cos\theta\ \rho(\bar{q}q)$$
$$= \sum_{\sigma} \begin{pmatrix} -\sigma\left|\hat{M}_{\sigma}^{+}\right|^2 & 0 & \frac{2}{3\sqrt{2}}\left(\hat{M}_{\sigma}^{+}\right)^*\hat{M}_{\sigma}^{0} \\ 0 & \sigma\left|\hat{M}_{\sigma}^{-}\right|^2 & -\frac{2}{3\sqrt{2}}\left(\hat{M}_{\sigma}^{-}\right)^*\hat{M}_{\sigma}^{0} \\ \frac{2}{3\sqrt{2}}\hat{M}_{\sigma}^{+}\left(\hat{M}_{\sigma}^{0}\right)^* & -\frac{2}{3\sqrt{2}}\hat{M}_{\sigma}^{-}\left(\hat{M}_{\sigma}^{0}\right)^* & 0 \end{pmatrix},$$

- Now the diagonal elements of DM vanish after the $y$ integration.

## Decay angular distributions of polarised $Z$: Second, $y$ integration.

$$\frac{d\sigma}{d\cos\widehat{\theta}\ d\widehat{\phi}}\Big|_{\mathcal{A}} \propto \int_{(m_H+m_Z)^2}^{s} d\hat{s} \int_0^{y_{\rm cut}} dy\ 2\{q(x_1)\bar{q}(x_2)+\bar{q}(x_1)q(x_2)\}$$

$$\times\ d^{\tau\dagger} \sum_{\sigma} \begin{pmatrix} \frac{4}{3}|\hat{M}_\sigma^+|^2 & \frac{2}{3}(\hat{M}_\sigma^+)^*\hat{M}_\sigma^- & 0 \\ \frac{2}{3}\hat{M}_\sigma^+(\hat{M}_\sigma^-)^* & \frac{4}{3}|\hat{M}_\sigma^-|^2 & 0 \\ 0 & 0 & \frac{4}{3}|\hat{M}_\sigma^0|^2 \end{pmatrix} d^{\tau},$$

$$\frac{d\sigma}{d\cos\widehat{\theta}\ d\widehat{\phi}}\Big|_{\mathcal{C}} \propto \int_{(m_H+m_Z)^2}^{s} d\hat{s} \int_0^{y_{\rm cut}} dy\ 2\{q(x_1)\bar{q}(x_2)+\bar{q}(x_1)q(x_2)\}$$

$$\times\ d^{\tau\dagger} \sum_{\sigma} \begin{pmatrix} 0 & 0 & \frac{2}{3\sqrt{2}}(\hat{M}_\sigma^+)^*\hat{M}_\sigma^0 \\ 0 & 0 & -\frac{2}{3\sqrt{2}}(\hat{M}_\sigma^-)^*\hat{M}_\sigma^0 \\ \frac{2}{3\sqrt{2}}\hat{M}_\sigma^+(\hat{M}_\sigma^0)^* & -\frac{2}{3\sqrt{2}}\hat{M}_\sigma^-(\hat{M}_\sigma^0)^* & 0 \end{pmatrix} d^{\tau},$$

- $\mathcal{A}$: $\int_{-1}^{1} d\cos\theta$, $\mathcal{C}$: $(\int_0^1 - \int_{-1}^{0})d\cos\theta$
- These 2 results compensate each other (enough and sufficient!) $\rightarrow$ The DM can be uniquely determined.
- This means that we can use the full information of polarisation (^^).
- The DM can be determined by measuring the $Z$ decay angular distributions (shown below).

Decay angular distributions of polarised $Z$: 9 non-zero angular coefficients $F$.

$$\frac{d\sigma}{d\cos\widehat{\theta}d\widehat{\phi}}\Big|_{i(=\mathcal{A},\mathcal{C})} = F_{i1}\left(1+\cos^2\widehat{\theta}\right) + F_{i2}\left(1-3\cos^2\widehat{\theta}\right) + F_{i3}\cos\widehat{\theta} + F_{i4}\sin\widehat{\theta}\cos\widehat{\phi}$$

$$+ F_{i5}\sin 2\widehat{\theta}\cos\widehat{\phi} + F_{i6}\sin^2\widehat{\theta}\cos 2\widehat{\phi} + F_{i7}\sin\widehat{\theta}\sin\widehat{\phi} + F_{i8}\sin 2\widehat{\theta}\sin\widehat{\phi} + F_{i9}\sin^2\widehat{\theta}\sin 2\widehat{\phi},$$

where

$$F_{\mathcal{A}(\mathcal{C})a} \propto \int_{(m_H+m_Z)^2}^{s} d\hat{s} \int_0^{\ln\sqrt{s/\hat{s}}} dy\ 2\left[q(x_1)\bar{q}(x_2) + \bar{q}(x_1)q(x_2)\right] \sum_\sigma \frac{1}{3} f_{\mathcal{A}(\mathcal{C})a},$$

where

$$f_{\mathcal{A}1} = 2\left(\left|\hat{M}_\sigma^+\right|^2 + \left|\hat{M}_\sigma^-\right|^2 + \left|\hat{M}_\sigma^0\right|^2\right), \quad f_{\mathcal{C}1} = 0,$$

$$f_{\mathcal{A}2} = 2\left|\hat{M}_\sigma^0\right|^2, \quad f_{\mathcal{C}2} = 0,$$

$$f_{\mathcal{A}3} = 4\left(\left|\hat{M}_\sigma^+\right|^2 - \left|\hat{M}_\sigma^-\right|^2\right)\tau, \quad f_{\mathcal{C}3} = 0,$$

$$f_{\mathcal{A}4} = 0, \quad f_{\mathcal{C}4} = 2Re\left[\left(\hat{M}_\sigma^+\right)^*\hat{M}_\sigma^0 - \left(\hat{M}_\sigma^0\right)^*\hat{M}_\sigma^-\right]\tau,$$

$$f_{\mathcal{A}5} = 0, \quad f_{\mathcal{C}5} = Re\left[\left(\hat{M}_\sigma^+\right)^*\hat{M}_\sigma^0 + \left(\hat{M}_\sigma^0\right)^*\hat{M}_\sigma^-\right],$$

$$f_{\mathcal{A}6} = 2Re\left[\left(\hat{M}_\sigma^+\right)^*\hat{M}_\sigma^-\right], \quad f_{\mathcal{C}6} = 0,$$

$$f_{\mathcal{A}7} = 0, \quad f_{\mathcal{C}7} = 2Im\left[\left(\hat{M}_\sigma^+\right)^*\hat{M}_\sigma^0 - \left(\hat{M}_\sigma^0\right)^*\hat{M}_\sigma^-\right]\tau,$$

$$f_{\mathcal{A}8} = 0, \quad f_{\mathcal{C}8} = Im\left[\left(\hat{M}_\sigma^+\right)^*\hat{M}_\sigma^0 + \left(\hat{M}_\sigma^0\right)^*\hat{M}_\sigma^-\right],$$

$$f_{\mathcal{A}9} = 2Im\left[\left(\hat{M}_\sigma^+\right)^*\hat{M}_\sigma^-\right], \quad f_{\mathcal{C}9} = 0.$$

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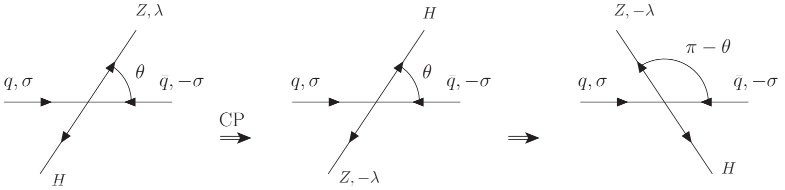
## Symmetry properties: consequences of CP and $\mathrm{CP\widetilde{T}}$ conservation in the amplitude.

CP transformation and a rotation around the $z$-axis by $\pi$:




CP conservation leads to

$$\mathcal{M}_\sigma^\lambda(q\bar{q})(\theta) = \mathcal{M}_\sigma^{-\lambda}(q\bar{q})(\pi - \theta).$$

In terms of $\hat{M}_\sigma^\lambda$, this relation is simply

$$\hat{M}_\sigma^\lambda = \hat{M}_\sigma^{-\lambda}.$$

In the similar manner, we find $\mathrm{CP\widetilde{T}}$ conservation, whose violation indicates the existence of re-scattering effects, leads to

$$\mathcal{M}_\sigma^\lambda(q\bar{q})(\theta) = \left\{\mathcal{M}_\sigma^{-\lambda}(q\bar{q})(\pi - \theta)\right\}^*, \quad \text{or} \quad \hat{M}_\sigma^\lambda = \left(\hat{M}_\sigma^{-\lambda}\right)^*.$$

Symmetry properties: consequences of CP and $\mathrm{CP\widetilde{T}}$ conservation in the 9 coefficients $F$.

$$\mathrm{CP\ conservation}: \hat{M}_\sigma^{\lambda} = \hat{M}_\sigma^{-\lambda}.$$
$$\mathrm{CP\widetilde{T}\ conservation}: \hat{M}_\sigma^{\lambda} = \left(\hat{M}_\sigma^{-\lambda}\right)^*.$$

$F_{\mathcal{A}9} \propto Im\left[\left(\hat{M}_\sigma^{+}\right)^* \hat{M}_\sigma^{-}\right]$:

$$\mathrm{CP\ conservation}: Im\left[\left(\hat{M}_\sigma^{+}\right)^* \hat{M}_\sigma^{-}\right] = Im\left[|\hat{M}_\sigma^{-}|^2\right] = 0,$$
$$\mathrm{CP\widetilde{T}\ conservation}: Im\left[\left(\hat{M}_\sigma^{+}\right)^* \hat{M}_\sigma^{-}\right] = Im\left[(\hat{M}_\sigma^{-})^2\right] \neq 0,$$

$\longrightarrow$ $F_{\mathcal{A}9}$ is zero if CP is conserved.

$F_{\mathcal{C}7} \propto Im\left[\left(\hat{M}_\sigma^{+}\right)^* \hat{M}_\sigma^{0} - \left(\hat{M}_\sigma^{0}\right)^* \hat{M}_\sigma^{-}\right]$:

$$\mathrm{CP\ conservation}: Im\left[\left(\hat{M}_\sigma^{+}\right)^* \hat{M}_\sigma^{0} - \left(\hat{M}_\sigma^{0}\right)^* \hat{M}_\sigma^{-}\right] = Im\left\{2i Im\left[\left(\hat{M}_\sigma^{+}\right)^* \hat{M}_\sigma^{0}\right]\right\} \neq 0,$$
$$\mathrm{CP\widetilde{T}\ conservation}: Im\left[\left(\hat{M}_\sigma^{+}\right)^* \hat{M}_\sigma^{0} - \left(\hat{M}_\sigma^{0}\right)^* \hat{M}_\sigma^{-}\right] = Im\left[\left(\hat{M}_\sigma^{+}\right)^* \hat{M}_\sigma^{0} - \hat{M}_\sigma^{0}\left(\hat{M}_\sigma^{+}\right)^*\right] = 0,$$

$\longrightarrow$ $F_{\mathcal{C}7}$ is zero if re-scattering effects are absent.

## Symmetry properties: consequences of CP and $\mathrm{CP\widetilde{T}}$ conservation in the 9 coefficients $F$.

| Combinations of the density matrix elements | Symmetry properties CP | Symmetry properties $\mathrm{CP\widetilde{T}}$ | Observables | $f$ charge |
|---|---|---|---|---|
| $\lvert\hat{M}_\sigma^+\rvert^2 + \lvert\hat{M}_\sigma^-\rvert^2 + \lvert\hat{M}_\sigma^0\rvert^2$ | $+$ | $+$ | $F_{\mathcal{A}1}$ | - |
| $\lvert\hat{M}_\sigma^0\rvert^2$ | $+$ | $+$ | $F_{\mathcal{A}2}$ | - |
| $\lvert\hat{M}_\sigma^+\rvert^2 - \lvert\hat{M}_\sigma^-\rvert^2$ | $-$ | $-$ | $F_{\mathcal{A}3}$ | $\circ$ |
| $Re\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^0 + (\hat{M}_\sigma^0)^*\hat{M}_\sigma^-\right]$ | $+$ | $+$ | $F_{\mathcal{C}5}$ | - |
| $Re\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^0 - (\hat{M}_\sigma^0)^*\hat{M}_\sigma^-\right]$ | $-$ | $-$ | $F_{\mathcal{C}4}$ | $\circ$ |
| $Re\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^-\right]$ | $+$ | $+$ | $F_{\mathcal{A}6}$ | - |
| $Im\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^0 + (\hat{M}_\sigma^0)^*\hat{M}_\sigma^-\right]$ | $-$ | $+$ | $F_{\mathcal{C}8}$ | - |
| $Im\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^0 - (\hat{M}_\sigma^0)^*\hat{M}_\sigma^-\right]$ | $+$ | $-$ | $F_{\mathcal{C}7}$ | $\circ$ |
| $Im\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^-\right]$ | $-$ | $+$ | $F_{\mathcal{A}9}$ | - |

- The symbol " $-$ " means that the combination is zero if the symmetry (CP or $\mathrm{CP\widetilde{T}}$) is conserved.
- e.g. $F_{\mathcal{C}8}$ and $F_{\mathcal{A}9}$ are zero if CP is conserved. In other words, these can be non-zero if CP is violated.
- $F_{\mathcal{A}1}$ is directly related to the total cross section. The other 8 observables can provide us the different information on $HZV$ couplings than the total cross section.

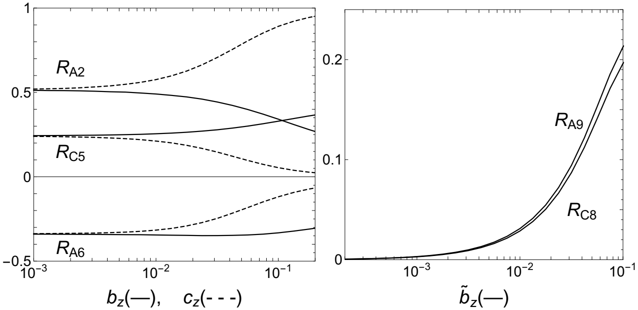
## Outline

- Introduction.
- Production amplitudes.
- Decay angular distributions of the polarised $Z$ boson.
- Symmetry properties.
- Numerical studies.

## Numerical studies: dependence on effective Lagrangian parameters.

We obtain a non-standard $HZZ$ coupling from the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = a_Z \frac{g_Z}{2} m_Z H Z_\mu Z^\mu + b_Z \frac{H}{v} Z_{\mu\nu} Z^{\mu\nu} + \frac{c_Z}{v} \left[ (\partial^\mu H) Z^\nu - (\partial^\nu H) Z^\mu \right] Z_{\mu\nu} + \widetilde{b}_Z \frac{H}{v} Z_{\mu\nu} \widetilde{Z}^{\mu\nu}.$$




- Define $R_i = F_i / F_{\mathcal{A}1}$, e.g. $R_{\mathcal{C}5} = F_{\mathcal{C}5}/F_{\mathcal{A}1}$. Recall that $F_{\mathcal{A}1} \propto$ "the total cross section".
- Left: $R_{\mathcal{A}2}$, $R_{\mathcal{C}5}$, $R_{\mathcal{A}6}$ are shown as deviation from the SM values caused by adding non-zero CP-even parameters $b_Z$ and $c_Z$.
- Right: $R_{\mathcal{C}8}$, $R_{\mathcal{A}9}$ are shown as deviation from the SM values caused by adding non-zero CP-odd parameter $\widetilde{b}_Z$.

## Summary.

- Polarisation of the $Z$ boson in $pp \to ZH$ is uniquely described by a $3 \times 3$ Hermitian density matrix $\rho$, which is constructed of the production amplitudes.
- DM $\rho$ (i.e. polarisation) can be determined by studying the decay $Z \to f\bar{f}$.
- We have presented one approach to make use of the polarisation information in order to study $HZV(V = Z, \gamma)$ couplings.
- We have obtained the 9 observables as the coefficients of the Z decay angular distributions after integration over $\cos\theta$, $y$ and $\hat{s}$. One of them is directly related to the total cross section.
- These 9 observables are enough and sufficient to determine all the 9 independent combinations of the DM elements. This means, "the use of these 9 observables" = "the use of the full information of polarisation".
- We have numerically shown that our new observables can be useful to determine effective Lagrangian parameters for the $HZZ$ coupling.

Thank you so much for your attention.

Back up.

## Spin, polarisation and polarisation density matrices

Spin angular momentum (Spin) induces additional degrees of freedom for a state of a particle. For a given spin $s$, a state vector is a $(2s+1)$-dimensional complex vector:

$$|1\rangle, |2\rangle, \cdots, |2s+1\rangle : \text{eigenfunctions of } \hat{s}_z \text{ for instance.}$$

$$\sum_{i=1}^{2s+1} |i\rangle\langle i| = 1,$$

$$|\alpha\rangle = \sum_{i=1}^{2s+1} |i\rangle\langle i|\alpha\rangle = a_1|1\rangle + a_2|2\rangle + \cdots + a_{2s+1}|2s+1\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2s+1} \end{pmatrix},$$

where $a_i$ $(i = 1, 2, \cdots, 2s+1)$ are complex numbers, which transform under spatial rotations (SU(2)).

d.o.f. of $|\alpha\rangle$ is $(2s+1) \times 2 - \underbrace{1}_{\langle\alpha|\alpha\rangle=1} - \underbrace{1}_{\text{overall phase}} = 4s$

$\rightarrow$ vector $|\alpha\rangle$ has a particular direction characterised by $4s$ real parameters.

= polarisation.

## Spin, polarisation and polarisation density matrices

Let us consider a state of a spin 1/2 particle:

$$|\alpha\rangle = a_1|1\rangle + a_2|2\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \langle\alpha|\alpha\rangle = 1,$$

where eigenfunctions of $\hat{s}_z = \sigma_z/2$ are chosen as base vectors:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Expectation values of the spin generators are

$$\langle\alpha|\hat{s}_z|\alpha\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}(|a_1|^2 - |a_2|^2),$$

$$\langle\alpha|(\hat{s}_x, \hat{s}_y, \hat{s}_z)|\alpha\rangle = \Big(Re[a_1^* a_2], Im[a_1^* a_2], (|a_1|^2 - |a_2|^2)/2\Big).$$

The expectation values can be also derived by using a matrix:

$$\langle\alpha|\hat{s}_z|\alpha\rangle = \sum_{i=1,2}\langle\alpha|i\rangle\langle i|\hat{s}_z|\alpha\rangle = \sum_{i=1,2}\langle i|\hat{s}_z|\alpha\rangle\langle\alpha|i\rangle = tr(\hat{s}_z\rho_\alpha),$$

$$\rho_\alpha \equiv |\alpha\rangle\langle\alpha| = \begin{pmatrix} |a_1|^2 & a_1 a_2^* \\ a_1^* a_2 & |a_2|^2 \end{pmatrix}, \quad tr(\rho_\alpha) = 1, \; \rho_\alpha^\dagger = \rho_\alpha.$$

$\rightarrow$ polarisation density matrix

## Spin, polarisation and pol. density matrices

A spin 1/2 particle in a mixed state which consists of

$$
\begin{aligned}
&|\alpha\rangle \text{ with probability } p_\alpha,\\
&|\beta\rangle \text{ with probability } p_\beta,\\
&\vdots,\\
&p_\alpha + p_\beta + \cdots = 1.
\end{aligned}
$$

Expectation value of $\hat{s}_z$ for the particle in this mixed state is

$$
\begin{aligned}
\langle \hat{s}_z \rangle &= \langle \alpha|\hat{s}_z|\alpha\rangle \times p_\alpha + \langle \beta|\hat{s}_z|\beta\rangle \times p_\beta + \cdots \\
&= tr\left(\hat{s}_z \rho_\alpha\right) \times p_\alpha + tr\left(\hat{s}_z \rho_\beta\right) \times p_\beta + \cdots \\
&= tr\left[\hat{s}_z(\rho_\alpha p_\alpha + \rho_\beta p_\beta + \cdots)\right] \\
&\equiv tr(\hat{s}_z \rho).
\end{aligned}
$$

The density matrix $\rho$ satisfies (recall that $tr(\rho_\alpha) = 1,\ \rho_\alpha^\dagger = \rho_\alpha$)

$$
tr(\rho) = 1,\ \rho^\dagger = \rho.
$$

- Polarisation of a spin 1/2 particle in a mixed state is uniquely described by a single $2 \times 2$ density matrix $\rho$, whose d.o.f is 3.
- In general, for a given spin $s$, d.o.f of a density matrix $\rho$ is $(2s+1)^2 - 1$.

## Spin, polarisation and polarisation density matrices

A state vector of $Z$ boson as a result of scattering $q\bar{q} \to ZH$ can be written in terms of scattering amplitudes in helicity basis as

$$|Z_\alpha\rangle = \frac{1}{n_\alpha} \sum_{\lambda=\pm,0} \mathcal{M}_\alpha^\lambda |\lambda\rangle, \quad \langle Z_\alpha | Z_\alpha \rangle = 1.$$

$$|\lambda\rangle : \text{helicity eigenvectors of the } Z \text{ boson; } \sum_{\lambda=\pm,0} |\lambda\rangle\langle\lambda| = 1.$$

$$n_\alpha^2 = |\mathcal{M}_\alpha^+|^2 + |\mathcal{M}_\alpha^-|^2 + |\mathcal{M}_\alpha^0|^2 : \text{normalisation factor.}$$

$$\alpha : \text{specifying a helicity state of } q\bar{q}.$$

To confirm this, notice

$$\langle \lambda = + | Z_\alpha \rangle = \frac{1}{n_\alpha} \mathcal{M}_\alpha^+, \quad |\langle \lambda = + | Z_\alpha \rangle|^2 = \frac{1}{n_\alpha^2} |\mathcal{M}_\alpha^+|^2, \quad \sum_{\lambda=\pm,0} |\langle \lambda | Z_\alpha \rangle|^2 = 1.$$

Density matrix $\rho_\alpha$ of the produced $Z$ boson in helicity basis is

$$\rho_\alpha = |Z_\alpha\rangle\langle Z_\alpha| = \frac{1}{n_\alpha^2} \begin{pmatrix} |\mathcal{M}_\alpha^+|^2 & \mathcal{M}_\alpha^+ \mathcal{M}_\alpha^{-*} & \mathcal{M}_\alpha^+ \mathcal{M}_\alpha^{0*} \\ \mathcal{M}_\alpha^- \mathcal{M}_\alpha^{+*} & |\mathcal{M}_\alpha^-|^2 & \mathcal{M}_\alpha^- \mathcal{M}_\alpha^{0*} \\ \mathcal{M}_\alpha^0 \mathcal{M}_\alpha^{+*} & \mathcal{M}_\alpha^0 \mathcal{M}_\alpha^{-*} & |\mathcal{M}_\alpha^0|^2 \end{pmatrix}.$$

## Spin, polarisation and polarisation density matrices

If another helicity state of $q\bar{q}$ is allowed, the $Z$ boson is in a mixed state with respect to polarisation. Density matrix of the $Z$ boson in such a state is

$$\rho = |Z_\alpha\rangle\langle Z_\alpha| \times p_\alpha + |Z_\beta\rangle\langle Z_\beta| \times p_\beta \qquad (p_\alpha + p_\beta = 1)$$
$$= \frac{1}{n}\left[\begin{pmatrix} |\mathcal{M}_\alpha^+|^2 & \mathcal{M}_\alpha^+\mathcal{M}_\alpha^{-*} & \mathcal{M}_\alpha^+\mathcal{M}_\alpha^{0*} \\ \mathcal{M}_\alpha^-\mathcal{M}_\alpha^{+*} & |\mathcal{M}_\alpha^-|^2 & \mathcal{M}_\alpha^-\mathcal{M}_\alpha^{0*} \\ \mathcal{M}_\alpha^0\mathcal{M}_\alpha^{+*} & \mathcal{M}_\alpha^0\mathcal{M}_\alpha^{-*} & |\mathcal{M}_\alpha^0|^2 \end{pmatrix} + \begin{pmatrix} |\mathcal{M}_\beta^+|^2 & \mathcal{M}_\beta^+\mathcal{M}_\beta^{-*} & \mathcal{M}_\beta^+\mathcal{M}_\beta^{0*} \\ \mathcal{M}_\beta^-\mathcal{M}_\beta^{+*} & |\mathcal{M}_\beta^-|^2 & \mathcal{M}_\beta^-\mathcal{M}_\beta^{0*} \\ \mathcal{M}_\beta^0\mathcal{M}_\beta^{+*} & \mathcal{M}_\beta^0\mathcal{M}_\beta^{-*} & |\mathcal{M}_\beta^0|^2 \end{pmatrix}\right],$$

where we set $n = 1$ so that

$$tr(\rho) = |\mathcal{M}_\alpha^+|^2 + |\mathcal{M}_\alpha^-|^2 + |\mathcal{M}_\alpha^0|^2 + |\mathcal{M}_\beta^+|^2 + |\mathcal{M}_\beta^-|^2 + |\mathcal{M}_\beta^0|^2.$$

- $tr(\rho)$ gives just the $q\bar{q} \to ZH$ cross section. D.o.f of our DM $\rho$ is $8 + 1 = 9$.
- DM $\rho$ contains more information than the $q\bar{q} \to ZH$ cross section (8 additional information).
- "We use the full information of polarisation." = "We relate all the elements of $\rho$ with measurable observables."

## Helicity amplitudes and constraints from symmetries

S-matrix: $S = 1 + iT$

$$S^\dagger S = 1 \text{ (unitarity)},$$
$$-i(T - T^\dagger) = T^\dagger T,$$
$$-i\big(\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle\big) = \sum_n \langle f|T^\dagger|n\rangle\langle n|T|i\rangle,$$
$$-i(T_{fi} - T^*_{if}) = \sum_n T^*_{nf} T_{ni},$$
$$-i(\mathcal{M}_{fi} - \mathcal{M}^*_{if}) = (2\pi)^4 \sum_n \delta^4(P_i - P_n)\mathcal{M}^*_{nf}\mathcal{M}_{ni}.$$

*CPT* invariance leads to

$$\mathcal{M}_{fi} = \mathcal{M}_{\hat{i}\hat{f}},$$

where $\hat{i}(\hat{f})$ denotes the *CPT* conjugate state of $i(f)$. The above unitarity condition becomes

$$-i(\mathcal{M}_{fi} - \mathcal{M}^*_{\hat{f}\hat{i}}) = (2\pi)^4 \sum_n \delta^4(P_i - P_n)\mathcal{M}^*_{nf}\mathcal{M}_{ni}.$$

Unitarity *and CPT* invariance tell us that $\mathcal{M}_{fi} \neq \mathcal{M}^*_{\hat{f}\hat{i}}$ indicates the existence of rescattering effects (Hagiwara et al 1987).

## Decay angular distributions of polarised $Z$

Perform a translation $\widehat{\theta} \to \pi - \widehat{\theta}$ *and* $\widehat{\phi} \to \widehat{\phi} + \pi$:

$$
\begin{aligned}
\left.\frac{d\sigma}{d\hat{s}\; d\cos\widehat{\theta}\; d\widehat{\phi}}\right|_{i(=\mathcal{A},\mathcal{B})} \propto\; & F_{i1}\left(1+\cos^2\widehat{\theta}\right) + F_{i2}\left(1-3\cos^2\widehat{\theta}\right) - F_{i3}\cos\widehat{\theta} \\
& - F_{i4}\sin\widehat{\theta}\cos\widehat{\phi} + F_{i5}\sin 2\widehat{\theta}\cos\widehat{\phi} + F_{i6}\sin^2\widehat{\theta}\cos 2\widehat{\phi} \\
& - F_{i7}\sin\widehat{\theta}\sin\widehat{\phi} + F_{i8}\sin 2\widehat{\theta}\sin\widehat{\phi} + F_{i9}\sin^2\widehat{\theta}\sin 2\widehat{\phi},
\end{aligned}
$$

where we observe the change of the sign in front of the $F_{i3}$, $F_{i4}$ and $F_{i7}$ terms.

This means

- The $F_{i3}$, $F_{i4}$ and $F_{i7}$ terms are statistically zero, if we do not distinguish the fermion $f$ from the antifermion $\bar{f}$.
- The events with $Z \to jj$ cannot be used to measure the coefficients $F_{\mathcal{A}3}$, $F_{\mathcal{B}4}$ and $F_{\mathcal{B}7}$, in view of difficulty of flavor identification of the jets. (- -)

## Decay angular distributions of polarised $Z$

### Second, $y$ integration in a different way:

$$\left.\frac{d\sigma}{d\hat{s}d\cos\widehat{\theta}d\widehat{\phi}}\right|_{\mathcal{B}} \propto \left(\int_0^{y_{\rm cut}} - \int_{-y_{\rm cut}}^0\right) dy\, q(x_1)\bar{q}(x_2) d^{\tau\dagger}\langle\rho(q\bar{q})\rangle d^{\tau} + \bar{q}(x_1)q(x_2)\, d^{\tau\dagger}\langle\rho(\bar{q}q)\rangle\, d^{\tau}$$

$$= \underbrace{\int_0^{y_{\rm cut}}}_{x_1>x_2} dy\, \underbrace{q(x_1)\bar{q}(x_2)}_{A}\, d^{\tau\dagger}\langle\rho(q\bar{q})\rangle\, d^{\tau} + \underbrace{\bar{q}(x_1)q(x_2)}_{B}\, d^{\tau\dagger}\langle\rho(\bar{q}q)\rangle\, d^{\tau}$$

$$- \underbrace{\int_{-y_{\rm cut}}^0}_{x_2>x_1} dy\, \underbrace{q(x_1)\bar{q}(x_2)}_{B}\, d^{\tau\dagger}\langle\rho(q\bar{q})\rangle\, d^{\tau} + \underbrace{\bar{q}(x_1)q(x_2)}_{A}\, d^{\tau\dagger}\langle\rho(\bar{q}q)\rangle\, d^{\tau}$$

$$= \int_0^{y_{\rm cut}} dy\, \left\{q(x_1)\bar{q}(x_2) - \bar{q}(x_1)q(x_2)\right\}\, d^{\tau\dagger}\left\{\langle\rho(q\bar{q})\rangle - \langle\rho(\bar{q}q)\rangle\right\}\, d^{\tau}$$

$$= \int_0^{y_{\rm cut}} dy\, 2\left\{q(x_1)\bar{q}(x_2) - \bar{q}(x_1)q(x_2)\right\}$$

$$\times\, d^{\tau\dagger}\sum_{\sigma}\begin{pmatrix} 0 & 0 & c_3\sigma\left(\hat{M}_\sigma^+\right)^*\hat{M}_\sigma^0 \\ 0 & 0 & c_3\sigma\left(\hat{M}_\sigma^-\right)^*\hat{M}_\sigma^0 \\ c_3\sigma\hat{M}_\sigma^+\left(\hat{M}_\sigma^0\right)^* & c_3\sigma\hat{M}_\sigma^-\left(\hat{M}_\sigma^0\right)^* & 0 \end{pmatrix} d^{\tau}.$$

- The vanished elements of the DM in the previous approach are revived (^^)

# Decay angular distributions of polarised $Z$

Second, $y$ integration in the 2 different ways as before:

$$\left.\frac{d\sigma}{d\hat{s}d\cos\widehat{\theta}d\widehat{\phi}}\right|_{\mathcal{C}} \propto \int_{-\ln\sqrt{\frac{s}{\hat{s}}}}^{\ln\sqrt{\frac{s}{\hat{s}}}} dy\ q(x_1)\bar{q}(x_2)\ d^{\tau\dagger}\overline{\langle\rho(q\bar{q})\rangle}\ d^{\tau} + \bar{q}(x_1)q(x_2)\ d^{\tau\dagger}\overline{\langle\rho(\bar{q}q)\rangle}\ d^{\tau}$$

$$= \int_{0}^{\ln\sqrt{\frac{s}{\hat{s}}}} dy\ 2\{q(x_1)\bar{q}(x_2)+\bar{q}(x_1)q(x_2)\}$$

$$\times\, d^{\tau\dagger}\sum_{\sigma}\begin{pmatrix} 0 & 0 & c_4\left(\hat{M}_{\sigma}^{+}\right)^{*}\hat{M}_{\sigma}^{0} \\ 0 & 0 & -c_4\left(\hat{M}_{\sigma}^{-}\right)^{*}\hat{M}_{\sigma}^{0} \\ c_4\hat{M}_{\sigma}^{+}\left(\hat{M}_{\sigma}^{0}\right)^{*} & -c_4\hat{M}_{\sigma}^{-}\left(\hat{M}_{\sigma}^{0}\right)^{*} & 0 \end{pmatrix} d^{\tau},$$

$$\left.\frac{d\sigma}{d\hat{s}d\cos\widehat{\theta}d\widehat{\phi}}\right|_{\mathcal{D}} \propto \left(\int_{0}^{\ln\sqrt{\frac{s}{\hat{s}}}} - \int_{-\ln\sqrt{\frac{s}{\hat{s}}}}^{0}\right) dy\, q(x_1)\bar{q}(x_2)\ d^{\tau\dagger}\overline{\langle\rho(q\bar{q})\rangle} d^{\tau} + \bar{q}(x_1)q(x_2) d^{\tau\dagger}\overline{\langle\rho(\bar{q}q)\rangle}$$

$$= \int_{0}^{\ln\sqrt{\frac{s}{\hat{s}}}} dy\ 2\{q(x_1)\bar{q}(x_2)-\bar{q}(x_1)q(x_2)\}$$

$$\times\, d^{\tau\dagger}\sum_{\sigma}\begin{pmatrix} c_5\sigma\left|\hat{M}_{\sigma}^{+}\right|^2 & 0 & 0 \\ 0 & -c_5\sigma\left|\hat{M}_{\sigma}^{-}\right|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} d^{\tau}.$$

## Decay angular distributions of polarised $Z$

| Combinations of the density matrix elements | Symmetry properties $CP$ | $CP\widetilde{T}$ | Observables | $f$ charge |
|---|---|---|---|---|
| $c_1\lvert\hat{M}_\sigma^+\rvert^2 + c_1\lvert\hat{M}_\sigma^-\rvert^2 + c_2\lvert\hat{M}_\sigma^0\rvert^2$ | $+$ | $+$ | $F_{\mathcal{A}1}$ | - |
| $\lvert\hat{M}_\sigma^0\rvert^2$ | $+$ | $+$ | $F_{\mathcal{A}2}$ | - |
| $\lvert\hat{M}_\sigma^+\rvert^2 + \lvert\hat{M}_\sigma^-\rvert^2$ | $+$ | $+$ | $F_{\mathcal{D}3}$ | $\circ$ |
| $\lvert\hat{M}_\sigma^+\rvert^2 - \lvert\hat{M}_\sigma^-\rvert^2$ | $-$ | $-$ | $F_{\mathcal{A}3}$ | $\circ$ |
| | | | $F_{\mathcal{D}1}$ | - |
| $Re\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^0 + (\hat{M}_\sigma^0)^*\hat{M}_\sigma^-\right]$ | $+$ | $+$ | $F_{\mathcal{B}4}$ | $\circ$ |
| | | | $F_{\mathcal{C}5}$ | - |
| $Re\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^0 - (\hat{M}_\sigma^0)^*\hat{M}_\sigma^-\right]$ | $-$ | $-$ | $F_{\mathcal{B}5}$ | - |
| | | | $F_{\mathcal{C}4}$ | $\circ$ |
| $Re\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^-\right]$ | $+$ | $+$ | $F_{\mathcal{A}6}$ | - |
| $Im\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^0 + (\hat{M}_\sigma^0)^*\hat{M}_\sigma^-\right]$ | $-$ | $+$ | $F_{\mathcal{B}7}$ | $\circ$ |
| | | | $F_{\mathcal{C}8}$ | - |
| $Im\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^0 - (\hat{M}_\sigma^0)^*\hat{M}_\sigma^-\right]$ | $+$ | $-$ | $F_{\mathcal{B}8}$ | - |
| | | | $F_{\mathcal{C}7}$ | $\circ$ |
| $Im\left[(\hat{M}_\sigma^+)^*\hat{M}_\sigma^-\right]$ | $-$ | $+$ | $F_{\mathcal{A}9}$ | - |

- Among the 36 ($= 4 \times 9$) coefficients, only the 15 coefficients can be non-zero.
- Observation of the 9 coefficients does not require the charge identification of $f$.
- These 9 coefficients are enough and sufficient to determine all the 9 independent combinations of the DM elements.

With appropriate integration over $\cos\Theta$ and $\phi$, it is possible to isolate the angular distributions:

$$\int_{-1}^{1} d\cos\Theta \frac{d\sigma}{d\cos\Theta d\phi} = \frac{8}{3}\mathcal{C}_1 + \frac{\pi}{2}\mathcal{C}_4\cos\phi + \frac{4}{3}\mathcal{C}_6\cos 2\phi + \frac{\pi}{2}\mathcal{C}_7\sin\phi + \frac{4}{3}\mathcal{C}_9\sin 2\phi, \tag{1a}$$

$$\left(\int_{0}^{1} - \int_{-1}^{0}\right) d\cos\Theta \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_3 + \frac{4}{3}\mathcal{C}_5\cos\phi + \frac{4}{3}\mathcal{C}_8\sin\phi, \tag{1b}$$

and

$$\frac{1}{2\pi}\int_{0}^{2\pi} d\phi \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_1(1+\cos^2\Theta) + \mathcal{C}_2(1-3\cos^2\Theta) + \mathcal{C}_3\cos\Theta, \tag{2a}$$

$$\frac{1}{4}\left(\int_{0}^{\pi/2} - \int_{\pi/2}^{\pi} - \int_{\pi}^{3\pi/2} + \int_{3\pi/2}^{2\pi}\right) d\phi \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_4\sin\Theta + \mathcal{C}_5\sin 2\Theta, \tag{2b}$$

$$\frac{1}{4}\left(\int_{0}^{\pi/4} - \int_{\pi/4}^{\pi/2} - \int_{\pi/2}^{3\pi/4} + \int_{3\pi/4}^{\pi} + \int_{\pi}^{5\pi/4} - \int_{5\pi/4}^{3\pi/2} - \int_{3\pi/2}^{7\pi/4} + \int_{7\pi/4}^{2\pi}\right) d\phi \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_6\sin^2\Theta, \tag{2c}$$

$$\frac{1}{4}\left(\int_{0}^{\pi} - \int_{\pi}^{2\pi}\right) d\phi \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_7\sin\Theta + \mathcal{C}_8\sin 2\Theta, \tag{2d}$$

$$\frac{1}{4}\left(\int_{0}^{\pi/2} - \int_{\pi/2}^{\pi} + \int_{\pi}^{3\pi/2} - \int_{3\pi/2}^{2\pi}\right) d\phi \frac{d\sigma}{d\cos\Theta d\phi} = \mathcal{C}_9\sin^2\Theta. \tag{2e}$$

By combining the 2 approaches in eq. (1) and the 5 approaches in eq. (2), we obtain the 10 ($= 2 \times 5$) combinations. The 2 of them simply give zero (i.e. eqs. (1b) and (2c), and eqs. (1b) and (2e)). Each of the remaining 8 combinations gives one of $\mathcal{C}_i$ ($i = 1, 3, 4, 5, 6, 7, 8, 9$). For example, eqs. (1b) and (2b) gives $\mathcal{C}_5$. Only $\mathcal{C}_2$ is not determined in this method.