Thermal ground state in
deconfining Yang-Mills thermodynamics

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- representation of partition function $Z$ (real scalar $\phi$) invented by Schwinger, Feynman 1950s, see e.g. [M. Le Bellac “Thermal Field Theory”]

$$Z = \text{Tr} e^{-\beta H} = \mathcal{N} \int_{\phi(x,0)=\phi_\alpha(x)}^{\phi(x,\beta)=\phi_\alpha(x)} \prod_{x,\tau'} d\phi(x, \tau') \times$$

$$\exp \left[ - \int_0^\beta d\tau'' \int d^3y \left( \frac{1}{2} \partial_{\tau''} \phi \partial_{\tau''} \phi + \frac{1}{2} \nabla \phi \cdot \nabla \phi + V(\phi) \right) \right]$$

$$\equiv \mathcal{N} \int_{\phi(x,0)=\phi_\alpha(x)}^{\phi(x,\beta)=\phi_\alpha(x)} \prod_{x,\tau'} d\phi(x, \tau') \exp \left[ - \int_0^\beta d\tau'' \int d^3y \mathcal{L}_E \right],$$

where $\beta \equiv 1/T$.

- in gauge theory: admissible changes of gauge respect periodicity of $A_\mu$
- in gauge-theory PT: additional gauge fixing required (Faddeev-Popov or better)
Euclidean finite-temperature field theory

- loop expansion of $N$-point functions in momentum space, propagator $\bar{D}$

$$
\bar{D}(p, \omega_n) = \frac{1}{\omega_n^2 + p^2 + m^2},
$$

where $\omega_n \equiv 2\pi n T \ (n \in \mathbb{Z})$ $n$th Matsubara frequency.

- re-expressing (but not changing the contour for $\tau''$ integration in Euclid. action) summation over $n$ and integration over $p$,

$$
\sum_n \int d^3 p, \text{ by Cauchy's integral theorem } \Rightarrow
$$

$$
- \frac{1}{\omega_n^2 + p^2 + m^2} \rightarrow \frac{i}{p^2 - m^2} + \delta(p^2 - m^2) \frac{2\pi}{e^{\beta|p_0|} - 1},
$$

where $\sum_n \int d^3 p \longrightarrow \int d^4 p$. 
Real-time interpretation of loop integrals

Remarks:

- A more elaborate $\tau''$ integration contour in the action was considered in [Umezawa, Matsumoto, and Tachiki (1982), Niemi and Semenoff (1984)]. This doubles real-time DOEs to avoid pinch singularities in PT.

- In Yang-Mills, where topological field configurations constructed for $0 \leq \tau'' \leq \beta$ (ground state!), such a change of contour for physics of propagating excitations is inconsistent.
Perturbative approach to pressure in Euclidean formulation

- In Linde 1980, the uselessness of PT after order $g^6$ is pointed out. (Scale-separation argument for $g \ll 1$: momenta of order $T$ (hard), $gT$ (soft), and $g^2T$ (ultrasoft); hard and soft OK; ultrasoft: weak screening of magnetic modes destroys perturbativity starting at $g^6$)
- SU(3) pressure in pure-YM PT

\begin{figure}
\centering
\includegraphics[width=\textwidth]{plot}
\caption{Graph showing the pressure-to-sound barrier ratio ($p/p_{SB}$) as a function of $T/\Lambda_{\text{MS}}$ for different orders of perturbation theory.}
\end{figure}

Trivial-holonomy calorons

- in singular gauge (winding number $|k| = 1$ is localized in a point) there is a **superposition principle** of instanton centers in prepotential $\Pi$ ['t Hooft (1976), Jackiw and Rebbi (1976)]:

$$
\bar{A}^+, a_{\mu}(x) = -\bar{\eta}^a_{\mu\nu} \partial_\nu \log \Pi, \\
\bar{A}^-, a_{\mu}(x) = -\eta^a_{\mu\nu} \partial_\nu \log \Pi.
$$

- can be used to satisfy at $|k| = 1$ periodic b.c. in strip $(0 \leq \tau \leq \beta) \times \mathbb{R}^3$ [Harrington and Shepard (1978)]:

$$
\Pi(\tau, \mathbf{x}; \rho, \beta, x_0) = 1 + \sum_{l=-\infty}^{l=\infty} \frac{\rho^2}{(x - x_l)^2} \\
= 1 + \frac{\pi \rho^2}{\beta r} \frac{\sinh \left( \frac{2\pi r}{\beta} \right)}{\cosh \left( \frac{2\pi r}{\beta} \right) - \cos \left( \frac{2\pi \tau}{\beta} \right)},
$$

where $r \equiv |\mathbf{x}|$. 
Trivial-holonomy calorons, cntd.

- holonomy of $\overline{A}_{\mu}^{\pm, a}(x)$ at $r \to \infty$ trivial:

$$\prod_{r \to \infty} 1 + \frac{\pi \rho^2}{\beta r} \Rightarrow \lim_{r \to \infty} \overline{A}_{4}^{\pm} \propto \lim_{r \to \infty} \frac{1}{r^2} = 0 \Rightarrow$$

$$\mathcal{P} \exp \left[ i \int_{0}^{\beta} d\tau \overline{A}_{4}^{\pm} \right] = 1_2.$$

- Gaussian quantum weight [Gross, Pisarski, and Yaffe (1981)]:

$$S_{\text{eff}} = \frac{8\pi^2}{g^2} + \frac{4}{3} \sigma^2 + 16 A(\sigma) \quad (\sigma \equiv \frac{\pi \rho}{\beta}),$$

$$A(\sigma) \to -\frac{1}{6} \log \sigma \quad (\sigma \to \infty) \quad A(\sigma) \to -\frac{\sigma^2}{36} \quad (\sigma \to 0).$$

Conclusion of **semiclassical approx.**:
Trivial-holonomy-caloron weight exponentially suppressed at high $T$. 
Nontrivial holonomy: Static magnetic dipoles


- explicitly carried out in [Lee and Lu 1998, Kraan and Van Baal 1998]: \( A_4(\tau, r \to \infty) = -iut^3(0 \leq u \leq \frac{2\pi}{\beta}) \).

Exact cancellation between \( A_4 \)-mediated repulsion and \( A_i \)-mediated attraction; caloron radius \( \rho \) and thus monopole-core separation \( D = \frac{\pi}{\beta} \rho^2 \) increase from left to right (\( T \) and holonomy fixed).
Nontrivial holonomy, cntd.

computation of functional determinant about nontrivial holonomy carried out in [Gross, Pisarski, and Yaffe (1981), Diakonov et al. 2004] in (relevant) limit $\frac{D}{\beta} = \pi \left( \frac{\rho}{\beta} \right)^2 \gg 1$

conclusions:

- **total suppression** for nontrivial static holonomy in limit $V \to \infty$
- **attraction** of monop. and antimonop. for small holonomy $(0 \leq u \leq \frac{\pi}{\beta}(1 - \frac{1}{\sqrt{3}}); \frac{\pi}{\beta}(1 + \frac{1}{\sqrt{3}}) \leq u \leq 2 \frac{\pi}{\beta})$
- **repulsion** of monop. and antimonop. for large holonomy $(\frac{\pi}{\beta}(1 - \frac{1}{\sqrt{3}}) \leq u \leq \frac{\pi}{\beta}(1 + \frac{1}{\sqrt{3}}))$
- **Instability** of classical configuration under quantum noise $\Rightarrow$ Nontrivial holonomy does not enter a priori estimate of thermal ground state!
Inert field $\phi$

Observations and principles constraining construction of $\phi$:

$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \Rightarrow$ vanishing energy-momentum:

$$\Theta_{\mu\nu} = -2 \text{tr} \left\{ \delta_{\mu\nu} \left( \mp E \cdot B \pm \frac{1}{4} (2E \cdot B + 2B \cdot E) \right) \right. $$

$$\left. \mp (\delta_{\mu 4} \delta_{\nu i} + \delta_{\mu i} \delta_{\nu 4}) (E \times E)_i \right.$$  

$$\pm \delta_{\mu i} \delta_{\nu(j \neq i)} (E_i B_j - E_i B_j) \pm \delta_{\mu(j \neq i)} \delta_{\nu i} (E_j B_i - E_j B_i) \right\} \equiv 0.$$  

$\Rightarrow$ spatial isotropy and homogeneity of effective local field not associated with propagation of energy-momentum by fundamental gauge fields $\Rightarrow$ inert scalar $\phi$

$\Rightarrow$ modulo admissible gauge transformations $\phi$ does not depend on time

$\Rightarrow$ relevance of $\phi$ (BPS) by gauge-invariant coupling to coarse-grained $k = 0$ sector (perturbative renormalizability) $\Rightarrow$ $\phi$ adjoint scalar
Inert field $\phi$

Observations and principles constraining construction of $\phi$, cntd:

- $F_{\mu\nu} \equiv \pm \tilde{F}_{\mu\nu} \Rightarrow$ any local “power” of $F_{\mu\nu}$ with an insertion of $t^a$ vanishes

- **only trivial holonomy** in $F_{\mu\nu} \equiv \pm \tilde{F}_{\mu\nu}$ allowed

- $|\phi|$ is spacetime homogeneous $\Rightarrow$ information on $\phi$’s EOM is encoded in phase $\hat{\phi} \equiv \frac{\phi}{|\phi|}$

- definition of possible phases $\{\hat{\phi}\}$: due to BPS of $A^\pm_\mu$ no explicit $T$ dependence, flat measure for admissible integration over moduli (excluding temporal shifts and global gauge rotations), Wilson lines between spatial points along straight lines
Inert field $\phi$

**Unique** definition of $\{\hat{\phi}\}$ [Herbst and Hofmann 2004]:

$$\{\hat{\phi}^a\} \equiv \sum_{\pm} \text{tr} \int d^3 x \int d\rho \ t^a \ F_{\mu\nu}(\tau, 0) \ \{(\tau, 0), (\tau, x)\}$$

$$\times F_{\mu\nu}(\tau, x) \ \{(\tau, x), (\tau, 0)\} \ ,$$

where

$$\{(\tau, 0), (\tau, x)\} \equiv \mathcal{P} \exp \left[ i \int_{(\tau, 0)}^{(\tau, x)} dz_\mu A_\mu(z) \right] \ ,$$

$$\{(\tau, x), (\tau, 0)\} \equiv \{(\tau, 0), (\tau, x)\}^\dagger \ ,$$

and sum is over **Harrington-Shepard** (trivial-holonomy) caloron and anticaloron of scale $\rho$.

Higher $n$-point functions, higher topol. charge $k$? **No.**

(Would introduce mass dimension $d = 3 - n - m$ of object, $m > 1$ number of dimension-length caloron moduli at $k > 1$, but $d$ needs to vanish.)
Inert field $\phi$

Some observations, conventions:

- $\hat{\phi}$ indeed transforms as an adjoint scalar:
  \[
  \hat{\phi}^a(\tau) \rightarrow R^{ab}(\tau)\hat{\phi}^b(\tau),
  \]
  where $R^{ab}$ is $\tau$ dependent matrix of adjoint rep.

- $R^{ab}(\tau)t^b = \Omega^\dagger(\tau,0)t^a\Omega(\tau,0)$.

- What about shift of spatial center $0 \rightarrow z_\pm$?

(a) graphical representation of definition
(b) only possible generalization to $z_\pm \neq 0$

Shift of center amounts to spatially *global* gauge rotation induced by the group element $\Omega^\pm_z = \{(\tau,0), (\tau, z_\pm)\}$. 
Inert field $\phi$

Some observations, conventions, cntd:

- one has
  $$\int_{(\tau,0)}^{(\tau,x)} dz_\mu A_\mu(z)|_\pm = \pm \int_0^1 ds \, x_i A_i(\tau, sx)$$
  $$= \pm t_b x_b \partial_\tau \int_0^1 ds \, \log \Pi(\tau, sr, \rho) \Rightarrow$$
  integrand in the exponent of $\{(\tau, 0), (\tau, x)\}_\pm$ varies along a fixed direction in $\text{su}(2)$ (a hedge hog); **Path-ordering can be ignored.**

- temporal shift freedom in $A^\pm_\mu$: set $\tau_\pm = 0$ and re-instate later

- parity: $F_{\mu\nu}(\tau, x)_+ = F_{\mu\nu}(\tau, -x)_-$ and
  $$\{(\tau, 0), (\tau, x)\}_+ = (\{(\tau, x), (\tau, 0)\}_+)^\dagger = \{(\tau, 0), (\tau, -x)\}_-$$
  $$= (\{(\tau, -x), (\tau, 0)\}_-)^\dagger \Rightarrow$$
  $-$ contribution to the integrand in **definition** obtained by $x \to -x$ in $+$ contribution
Inert field $\phi$

Some observations, conventions, cntd:

after tedious computation [Herbst and Hofmann 2004]
+ contribution to integrand in definition reads:

$$- i \beta^{-2} \frac{32\pi^4}{3} x^a \frac{\pi^2 \rho^4 + \hat{\rho}^2 (2 + \cos(2\pi \hat{\tau}))}{r \left(2\pi^2 \hat{\rho}^2 + 1 - \cos(2\pi \hat{\tau})\right)^2} \times F[\hat{g}, \Pi],$$

where $\hat{\rho} \equiv \frac{\rho}{\beta}$, $\hat{r} \equiv \frac{r}{\beta}$, $\hat{\tau} \equiv \frac{\tau}{\beta}$, and functional $F$ is

$$F[\hat{g}, \Pi] = 2 \cos(2\hat{g}) \left(2 \frac{[\partial_\tau \Pi][\partial_r \Pi]}{\Pi^2} - \frac{\partial_\tau \partial_r \Pi}{\Pi}\right) + \sin(2\hat{g}) \left(2 \frac{[\partial_r \Pi]^2}{\Pi^2} - 2 \frac{[\partial_\tau \Pi]^2}{\Pi^2} + \frac{\partial_\tau^2 \Pi}{\Pi} - \frac{\partial_r^2 \Pi}{\Pi}\right),$$

and

$$\{(\tau, 0), (\tau, x)\}_\pm \equiv \cos \hat{g} \pm 2i t_b \frac{\chi^b}{r} \sin \hat{g}.$$  

One shows that $\hat{g}$ saturates exponentially fast for $\hat{r} > 1$. 
**Inert field $\phi$**

**discussion:**

- Angular integration would yield **zero** if radial integration was regular.

- **But:** radial integration diverges logarithmically due to term $\frac{\partial^2 \Pi}{\Pi}$; this term arises from the **magnetic-magnetic** correlation (recall: no convergence in PT due to weakly screened magnetic sector!)

- **Zero** $\times$ infinity yields undetermined, multiplicative, and real constants $\Xi_{\pm}$

- Without restriction of generality (global choice of gauge), angular integration regularized by defect azimuthal angle in 1-2 plane of $\text{su}(2)$ for both $+$ and $-$ contributions $\Rightarrow$

  **Members of $\{\hat{\phi}\}$ all move in hyperplane of $\text{su}(2)$!**

- Re-instate $\tau \rightarrow \tau + \tau_{\pm}$ $\Rightarrow$
Inert field $\phi$

discussion, cntd:

result:

$$\{ \hat{\phi}^a \} = \{ \Xi_+ (\delta^a_1 \cos \alpha_+ + \delta^a_2 \sin \alpha_+) \mathcal{A} (2\pi (\hat{\tau} + \hat{\tau}_+)) + \Xi_- (\delta^a_1 \cos \alpha_- + \delta^a_2 \sin \alpha_-) \mathcal{A} (2\pi (\hat{\tau} + \hat{\tau}_-)) \}, \quad \text{where}$$

$\mathcal{A}$

$\tau$ dependence of function $\mathcal{A}(\frac{2\pi \tau}{\beta})$; saturation property (cutoff independence) for $\hat{\rho}$ integration.
\xi \text{ dependence of } \Xi_{\pm}

\rho_{\text{max}} \equiv \xi \beta:

\int d\rho \to \int_{0}^{\zeta} d\rho, \quad (\zeta > 0).

\begin{itemize}
  \item \(\Xi_{\pm} = 272 \zeta^3 \times \text{unknown, fixed real}, \quad (\zeta > 10)\)
  \item integral over \(\rho\) is strongly dominated by contributions just below upper limit
  \item semiclassical discussion of nontrivial-holonomy calorons in limit \(D_{\beta} = \pi \left(\frac{\rho}{\beta}\right)^2 \gg 1\) [Diakonov et al. 2004] is justified.
\end{itemize}
Kernel of a differential operator $D$ and potential for $\phi$

- set \{\hat{\phi}\} contains two real parameters for each “polarization”: $\Xi_\pm$ and $\tau_\pm$; \{\hat{\phi}\} is annihilated by \textbf{linear, second-order} differential operator $D = \partial_\tau^2 + \left(\frac{2\pi}{\beta}\right)^2 \Rightarrow$

\{\hat{\phi}\} coincides with \textbf{kernel} of $D$ and determines $D$ uniquely

- linearity $\Rightarrow$ also $D\phi = 0$

- \textbf{but:} $D$ depends on $\beta$ explicitly, not allowed (BPS, caloron action given by topolog. charge)

- therefore seek potential $V(|\phi|^2)$ such that (Euclidean) action principle applied to

$$ \mathcal{L}_\phi = \text{tr} \left( (\partial_\tau \phi)^2 + V(\phi^2) \right). $$

yields solutions annihilated by $D$, where $\mathcal{L}_\phi$ does not depend on $\beta$ explicitly; demand that energy density $\Theta_{44} = 0$ on those solutions
Potential for and modulus of $\phi$

- pick motion in 1-2 plane of su(2) (gauge invariance $\Rightarrow V$ central potential $\Rightarrow$ cons. angular momentum); ansatz:
  
  $$\phi = 2 |\phi| t_1 \exp(\pm \frac{4\pi i}{\beta} t_3 \tau).$$

  (circular motion in 1-2 plane, $|\phi|$ time independent!)

- apply E-L to $\mathcal{L}_\phi$ $\Rightarrow$
  
  $$\partial_\tau^2 \phi^a = \frac{\partial V(|\phi|^2)}{\partial |\phi|^2} \phi^a \text{ (in components)} \Leftrightarrow$$

  $$\partial_\tau^2 \phi = \frac{\partial V(\phi^2)}{\partial \phi^2} \phi \text{ (in matrix form)}.$$ 

- $\Theta_{44} = 0$ on ansatz $\phi$ $\Rightarrow$ $|\phi|^2 \left(\frac{2\pi}{\beta}\right)^2 - V(|\phi|^2) = 0$ but also:
  
  $$\partial_\tau^2 \phi + \left(\frac{2\pi}{\beta}\right)^2 \phi = 0 \Rightarrow$$

  $$\frac{\partial V(|\phi|^2)}{\partial |\phi|^2} = -\frac{V(|\phi|^2)}{|\phi|^2}.$$
Potential for and modulus of $\phi$, cntd

$\implies V(|\phi|^2) = \frac{\Lambda^6}{|\phi|^2}$

where $\Lambda$ integration constant of mass dim. unity.

$\implies |\phi| = \sqrt{\frac{\Lambda^3 \beta}{2\pi}}$ (power-like decay of field $\phi$ with increasing $T$)

The field $\phi$ describes coarse-grained effect of noninteracting trivial-holonomy calorons and anticalorons. It does not propagate, and its modulus $|\phi|$ sets the scale of off-shellness down to which quantum fluctuations, arising from the sector $k = 0$, must be considered “integrated out” in full effective theory.

Indeed: cutting off $\rho$ and $r$ integrations at $|\phi|^{-1}$, $\tau$ dependence of $\mathcal{A}(\frac{2\pi \tau}{\beta})$ is perfect sine

(Error at level smaller than $10^{-22}$ if knowledge about $T_c = \frac{\lambda_c \Lambda}{2\pi}$ with $\lambda_c = 13.87$ is used, later.)
In addition to E-L equation $\phi$ satisfies **first-order**, BPS equation:

$$\partial_\tau \phi = \pm 2i \Lambda^3 t_3 \phi^{-1} = \pm i \sqrt{\mathcal{V}}(\phi).$$

*Because $\phi$ satisfies both, second-order E-L and first-order BPS equation, usual shift ambiguity in ground-state energy density, as allowed by E-L equation, **absent in** $SU(2)$ Yang-Mills thermodynamics.*
Effective action for deconfining phase

Coupling the coarse-grained $k = 0$ sector to $\phi$, following constraints:

- perturbative renormalizability
  
  ['t Hooft, Veltman, Lee, and Zinn-Justin 1971-1973]
  
  $\Rightarrow$ form invariance of action for effective $k = 0$ gauge field $a_\mu$
  
  from integrating fundamental $k = 0$ fluctuations only, no higher dim. ops. for $a_\mu$ only

- no energy-momentum transfer to $\phi$ $\Rightarrow$ absence of higher dim. ops. involving $a_\mu$ and $\phi$

- gauge invariance $\Rightarrow \partial_\mu \phi \rightarrow D_\mu \phi \equiv \partial_\mu \phi - ie[a_\mu, \phi]$ (effective coupling); no momentum transfer to $\phi$ if (unitary gauge $\phi = 2|\phi| t_3$) massive 1,2 modes propagate on-shell only

\[(a)\]

\[(b)\]
Effective action and ground-state estimate

unique effective action density:

\[ \mathcal{L}_{\text{eff}}[a_\mu] = \text{tr} \left( \frac{1}{2} G_{\mu\nu} G_{\mu\nu} + (D_\mu \phi)^2 + \frac{\Lambda^6}{\phi^2} \right), \]

where \( G_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - ie[a_\mu, a_\nu] \equiv G_{\mu\nu}^a t_a \)

ground-state estimate:

- E-L EOM from \( \mathcal{L}_{\text{eff}}[a_\mu] \)

\[ D_\mu G_{\mu\nu} = ie[\phi, D_\nu \phi]. \]

- solved by zero-curvature (pure-gauge) config. \( a_{\mu}^{\text{gs}} \):

\[ a_{\mu}^{\text{gs}} = \mp \delta_{\mu 4} \frac{2\pi}{e\beta} t_3 \quad (D_\nu \phi \equiv G_{\mu\nu} \equiv 0) \quad \Rightarrow \]

\[ \rho^{\text{gs}} = -P^{\text{gs}} = 4\pi \Lambda^3 T. \]

Unresolvable interactions between \( k = 0 \) and \( |k| = 1 \) lifted \( \rho^{\text{gs}} \) from zero (BPS). EOS of a cosmological constant; pressure negative. (Short-lived, attracting magnetic (anti)monopoles by temporary shifts of (anti)caloron holonomies from trivial to small through absorption of hard plane-wave fluctuations.)
Winding to unitary gauge: $\mathbb{Z}_2$ degeneracy

- consider gauge rotation $\tilde{\Omega}(\tau) = \Omega_{gl} Z(\tau) \Omega(\tau)$ where $\Omega(\tau) \equiv \exp[\pm 2\pi i \frac{\tau}{\beta} t_3]$, $Z(\tau) = \left(2\Theta(\tau - \frac{\beta}{2}) - 1\right)1_2$, and $\Omega_{gl} = \exp[i \frac{\pi}{2} t_2]$

- $\tilde{\Omega}(\tau)$ transforms $a_{\mu}^{gs}$ to $a_{\mu}^{gs} \equiv 0$ and $\phi$ to $\phi = 2 t^3 |\phi|$

- $\tilde{\Omega}(\tau)$ is admissible because respects periodicity of $\delta a_{\mu}$:

$$a_{\mu} \rightarrow \tilde{\Omega}(a_{\mu}^{gs} + \delta a_{\mu})\tilde{\Omega}^\dagger + \frac{i}{e} \tilde{\Omega} \partial_{\mu} \tilde{\Omega}^\dagger$$

$$= \Omega_{gl} \left(\Omega(a_{\mu}^{gs} + \delta a_{\mu})\Omega^\dagger + \frac{i}{e} \left(\Omega \partial_{\mu} \Omega^\dagger + Z \partial_{\mu} Z\right)\right)\Omega_{gl}^\dagger$$

$$= \Omega_{gl} \left(\Omega \delta a_{\mu} \Omega^\dagger + \frac{2i}{e} \delta(\tau - \frac{\beta}{2})Z\right)\Omega_{gl}^\dagger = \Omega_{gl} \Omega \delta a_{\mu} (\Omega_{gl} \Omega)^\dagger.$$

- $\tilde{\Omega}(\tau)$ transforms Polyakov loop from $-1_2$ to $1_2 \Rightarrow$ ground-state estimate is (electric) $\mathbb{Z}_2$ degenerate $\Rightarrow$ deconfining phase
Mass spectrum; outlook resummed radiative corrections

- computation in physical and completely fixed \textit{unitary}, Coulomb gauge \((\phi = 2t^3|\phi|, \partial_i a_3^i = 0)\)

- mass spectrum: \(m^2 \equiv m_1^2 = m_2^2 = 4e^2 \frac{\Lambda^3}{2\pi T}, m_3 = 0\).

- resummation of \textit{polarization tensor of massless mode} as

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{polarization_tensor_diagram.png}
\end{array}
\]

\Rightarrow \text{ small linear-in-}T \text{ correction to tree-level ground-state estimate [Falquez, Hofmann, Baumbach 2010]}

\[
\begin{align*}
\text{tree-level:} & \quad \frac{\rho_{gs}}{T^4} = 3117.09 \lambda^{-3}, \\
\text{one-loop resummed:} & \quad \frac{\Delta \rho_{gs}}{T^4} = 3.95 \lambda^{-3}.
\end{align*}
\]

- large hierarchy between loop orders (conjecture about \textit{termination at finite irreducible order}, see second talk), so one-loop correction \textit{practically exact}
$T$ dependence of $e$: selfconsistent thermal quasiparticles

$P$ and $\rho$ at one loop:

\[
P(\lambda) = -\Lambda^4 \left\{ \frac{2\lambda^4}{(2\pi)^6} \left[ 2\bar{P}(0) + 6\bar{P}(2a) \right] + 2\lambda \right\},
\]

\[
\rho(\lambda) = \Lambda^4 \left\{ \frac{2\lambda^4}{(2\pi)^6} \left[ 2\bar{\rho}(0) + 6\bar{\rho}(2a) \right] + 2\lambda \right\},
\]

where

\[
\bar{P}(y) \equiv \int_0^\infty dx \, x^2 \log \left[ 1 - \exp\left( -\sqrt{x^2 + y^2} \right) \right],
\]

\[
\bar{\rho}(y) \equiv \int_0^\infty dx \, x^2 \frac{\sqrt{x^2 + y^2}}{\exp(\sqrt{x^2 + y^2}) - 1},
\]

and $a \equiv \frac{m}{2T} = 2\pi e\lambda^{-3/2}$. For later use introduce function $D(2a)$ as

\[
\partial_{y^2} \bar{P}\bigg|_{y=2a} = -\frac{1}{4\pi^2} \int_0^\infty dx \, \frac{x^2}{\sqrt{x^2 + (2a)^2}} \frac{1}{e^{\sqrt{x^2 + (2a)^2}} - 1} \equiv -\frac{1}{4\pi^2} D(2a).
\]
Legendre transformation and evolution equation

- for $m(T)$ to respect Legendre trafo (fundamental partition function) between $P$ and $\rho \Leftrightarrow \partial_m P = 0$
- \Rightarrow first-order evolution equation

$$\partial_a \lambda = -\frac{24\lambda^4 a}{(2\pi)^6} \frac{D(2a)}{1 + \frac{24\lambda^3 a^2}{(2\pi)^6} D(2a)}.$$ 

or

$$1 = -\frac{24\lambda^3}{(2\pi)^6} \left( \lambda \frac{da}{d\lambda} + a \right) a D(2a).$$

- \Rightarrow dependence $a(\lambda)$ monotonic decreasing
  \Rightarrow for $\lambda \gg 1$ $a$ must fall below unity

- fixed points of evolution equation:

  repulsive at $a = 0$ ($\lambda \to \infty$)
  attractive at $a = \infty$ ($\lambda = \lambda_c$)
Solution to evolution equation

- \( a \ll 1 \) [Dolan, Jackiw 1974] \( \Rightarrow \) \( 1 = -\frac{\lambda^3}{(2\pi)^4} \left( \lambda \frac{da}{d\lambda} + a \right) \); solution \( (a(\lambda_i) = a_i \ll 1) \):

\[
a(\lambda) = 4\sqrt{2\pi^2} \lambda^{-3/2} \left( 1 - \frac{\lambda}{\lambda_i} \left[ 1 - \frac{a_i^2 \lambda_i^3}{32\pi^4} \right] \right)^{1/2}.
\]

\( \Rightarrow \) attractor \( a(\lambda) = 4\sqrt{2\pi^2} \lambda^{-3/2} \) as long as \( a \ll 1 \)

\( \Rightarrow \) \( e = \sqrt{8\pi} \) as long as \( a \ll 1 \) (amusingly: \( S = \frac{8\pi^2}{e^2} = 1 \))

(geometric interpretation of \( \hbar \) in terms of caloron winding number)

- full solution for \( e(\lambda) \Rightarrow \lambda_c = 13.87 \):
$T$ dependence of $P$ and $\rho$

- notice **negativity** of $P$ shortly above $\lambda_c$
- relative correction to one-loop quasiparticle $P$ and $\rho$ by radiative effects: $< 1\%$, see second talk
Summary:

- Brief motivation why nonperturbative approach to YMTD is necessary: mass generation, poor convergence of perturbation orders.
- Mini review on calorons: trivial vs. nontrivial holonomy for $|k| = 1$ plus semiclassical approximation.
- Construction of thermal ground-state estimate: inert field $\phi$; BPS and E-L; potential.
- Discussion of constraints on effective action: perturbation renormalizability plus inertness of $\phi \Rightarrow$ unique answer.
- Full ground-state estimate, deconfining nature, tree-level quasiparticles.
- Evolution of effective coupling.
- $T$ dependence of pressure and energy density.
Outlook

- running of fundamental coupling: trace anomaly
- radiative corrections: polarization tensor of massless mode
- radiative corrections: stable but unresolvable monopoles
- radiative corrections: two-loop and three-loop cases
- radiative corrections: loop expansion of pressure, conjecture on termination at finite irreducible order
- two other phases:
  - **preconfining** (thermal ground state: condensate of massless monopoles and antimonopoles)
  - **confining** (ground state of zero energy density: condensate of single, round-point like center-vortex loops)
Physics

Some physics implications:

(i) mechanism for ew SB (LHC: not much of a Higgs signal so far)
(ii) postulate: SU(2) \((10^{-4} \text{ eV})\) describes photon propagation

⇒ black-body spectral anomaly at \(T \sim 5 - 20 \text{ K}\) and low frequencies
(cold H1 clouds, large-angle anomalies in TT of CMB, UEGE)
⇒ Planck-scale axion plus such an SU(2) yield Dark Energy

Thank you.