Exercises

E.1 Consider the Lagrange density $\mathcal{L}(x)$ for the free scalar field $\varphi(x)$ and the action functional

$$S[\varphi] = \int d^4x \ \mathcal{L}(x) .$$

Calculate the variation of $S[\varphi]$ under a variation of the field

$$\varphi(x) \to \varphi(x) + \delta\varphi(x)$$

where $\delta\varphi(x) \to 0$ at infinity. Show that $\delta S[\varphi] = 0$ is precisely fulfilled by the fields satisfying the Klein-Gordon equation.

✓

E.2 Show that the commutation relations for the creation and annihilation operators $a^{+}(\vec{k})$ and $a(\vec{k}')$ of the free scalar field operator $\phi(x)$ imply

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = 0,$$

$$[\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{y},t)] = 0,$$

$$[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y}).$$

✓

E.3 Functional derivative.

Consider functions $\vec{x} \to \psi(\vec{x})$, $\vec{x} \in R_3$, $\psi(\vec{x})$ real. Let

$F[\psi]$ be a functional of ψ.

Under a variation of ψ we will have

$$F[\psi + \delta\psi] = F[\psi] + \int d\vec{x}^3\, G[\psi, \vec{x}]\, \delta\psi(\vec{x}) + \mathcal{O}(\delta\psi^2).$$

$G[\psi, \vec{x}]$ is called the functional derivative of F:

$$G[\psi, \vec{x}] =: \frac{\delta F[\psi]}{\delta\psi(\vec{x})}.$$

Calculate the functional derivative of the following functionals

(a) $F[\psi] = \int d^3x \, \frac{1}{2} V(\vec{x}) \, \psi^2(\vec{x})$,

(b) $F[\psi] = \frac{1}{2} \int d^3x \left(\vec{\nabla} \psi(\vec{x}) \right)^2$,

(c) $F[\psi] = \psi(\vec{y})$

$= \int d^3x \, \delta^3(\vec{y} - \vec{x}) \, \psi(\vec{x})$.