4 Quantum Chromodynamics, QCD

In this chapter we shall discuss the basics of QCD, of the theory which describes, as we believe, all strong interaction phenomena.

4.1 The Lagrange density of QCD

We have seen that there is ample evidence from experiments that hadrons are built out of constituents. We learnt about the quarks and their quantum numbers.

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Quarks carry charge, since they are "seen" in deep inelastic lepton-nucleon scattering and they come in three colours. We know the latter for instance from $\pi^0 \to \gamma\gamma$ decay and the ratio R in e^+e^- annihilation to hadrons. How should the theory for these quarks look like?

Quarks have spin $1/2$, thus we describe them by a Dirac spinor field. For quark flavour q we write

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$$q(x) = \begin{pmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{pmatrix} . \qquad (4.1)$$

Here $q = u, d, s, c, b, t$ and $1, 2, 3$ are the colour indices. That is, for each quark flavour we have three Dirac fields corresponding to the three colours. The Lagrange density describing quarks and their interactions should certainly include the kinetic term for these quarks.

$$\mathcal{L}^{(0)}_{\text{quarks}}(x) = \sum_q \bar{q}(x)\Big(\frac{i}{2}\gamma^\lambda \overrightarrow{\partial}_\lambda - \frac{i}{2}\gamma^\lambda \overleftarrow{\partial}_\lambda - m_q\Big) q(x), \tag{4.2}$$

where m_q is the mass of quark flavour q. With eq. (4.2) we can describe the motion of free quarks and surely this cannot be everything.

We shall motivate and introduce the interaction of quarks in a way very similar to what we did for QED.

We know that mesons are $q\bar{q}$ bound states, baryons qqq bound states. Thus a π^+ meson, for instance, is a state

$$\pi^+ \sim u_1 \bar{d}_1 + u_2 \bar{d}_2 + u_3 \bar{d}_3, \tag{4,3}$$

an Ω^- a state

$$\Omega^- \sim s_\alpha s_\beta s_\gamma \, \varepsilon_{\alpha\beta\gamma}, \tag{4.4}$$

where we write out the colour structure. A typical observable, the hadronic part of the electromagnetic current is given by

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$$J_\mu^{\text{hadronic}}(x) = e \sum_q Q_q \, \bar{q}(x) \gamma^\mu q(x) \qquad (4.5)$$

with

$$\begin{aligned} Q_u = Q_c = Q_t &= 2/3 , \\ Q_d = Q_s = Q_b &= -1/3 . \end{aligned} \qquad (4.6)$$

The hadron states and this observable have in common that they are invariant under a SU(3) transformation in colour space

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$$q(x) \longrightarrow \mathcal{U}\, q(x) \,, \tag{4.7}$$

or in components

$$q_\alpha(x) \longrightarrow \mathcal{U}_{\alpha\beta}\, q_\beta(x)$$

$$(\alpha, \beta \in \{1,2,3\}) \,, \tag{4.8}$$

with

$$\mathcal{U} = (\mathcal{U}_{\alpha\beta}) \in SU(3) \,. \tag{4.9}$$

That is, $\mathcal{U}$ satisfies

$$\mathcal{U}^\dagger \mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = \mathbb{1}_3 \,,$$
$$\det \mathcal{U} = 1 \,. \tag{4.10}$$

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At this point we should say a few words on the mathematics of the SU(3) group, a close friend of the SU(2) group which you know from the discussion of spin and isospin. To characterise the structure of a group it is convenient to consider infinitesimal transformations

$$\mathcal{U} = \mathbb{1}_3 + i\delta\varphi\, H \, . \tag{4.10a}$$

Inserting this in (4.10) we find that H must be hermitian and traceless

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$$H = H^{\dagger},$$
$$\mathrm{Tr}\, H = 0 . \qquad (4.106)$$

There are eight linearly independent 3×3 matrices satisfying (4.106) and a convenient basis is given by the Gell-Mann λ matrices. These matrices are the analogues of the three Pauli matrices of $SU(2)$. We say, that $SU(3)$ has eight generators, $SU(2)$ three generators.

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The λ_a ($a = 1, \ldots, 8$) are the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{C.44}$$

Relations:

$$\mathrm{Tr}\, \lambda_a = 0, \tag{C.45}$$

$$\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}, \tag{C.46}$$

$$[\lambda_a, \lambda_b] = 2\mathrm{i} f_{abc} \lambda_c, \tag{C.47}$$

$$\{\lambda_a, \lambda_b\} = \tfrac{4}{3}\delta_{ab} + 2d_{abc}\lambda_c, \tag{C.48}$$

where f_{abc} is totally antisymmetric, and d_{abc} is totally symmetric with values for the independent components as given in Table C.1.

(From O. Nachtmann, Elementary Particle Physics)

Any infinitesimal transformation $\mathcal{U} \in SU(3)$ can thus be written as follows

$$\mathcal{U} = \mathbb{1}_3 + i\,\delta\varphi_a \frac{\lambda_a}{2}, \qquad \delta\varphi_a \text{ real}, \tag{4.10c}$$

where summation over $a = 1, \ldots, 8$ is implied. The λ matrices satisfy a number of relations, e.g.

$$\mathrm{Tr}(\lambda_a \lambda_b) = 2\,\delta_{ab}, \tag{4.10d}$$

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc} \frac{\lambda_c}{2}. \tag{4.10e}$$

The constants f_{abc} occurring in eq. (4.10e) are real and totally antisymmetric in a, b, c. They are called the <u>structure constants</u> of SU(3). For the anticommutator of two λ matrices one finds

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab}\,\mathbb{1}_3 + 2\, d_{abc}\, \lambda_c\,, \tag{4.10f}$$

where the constants d_{abc} are real and totally symmetric in a, b, c. The f_{abc} and d_{abc} are given in table 4.1.

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Table 4.1 The independent, nonvanishing components of f_{abc} and d_{abc}.

(a, b, c)	f_{abc}	(a, b, c)	d_{abc}
123	1	118	$1/\sqrt{3}$
147	1/2	146	1/2
156	$-1/2$	157	1/2
246	1/2	228	$1/\sqrt{3}$
257	1/2	247	$-1/2$
345	1/2	256	1/2
367	$-1/2$	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	1/2
678	$\sqrt{3}/2$	355	1/2
		366	$-1/2$
		377	$-1/2$
		448	$-1/(2\sqrt{3})$
		558	$-1/(2\sqrt{3})$
		668	$-1/(2\sqrt{3})$
		778	$-1/(2\sqrt{3})$
		888	$-1/\sqrt{3}$

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Now we go back to the unitary transformation, eq. (4.7), in the colour space for quarks. Under this transformation we have

$$\begin{aligned} \pi^+ &\longrightarrow \pi^+ , \\ \Omega^- &\longrightarrow \Omega^- , \\ J_\mu^{hadr.}(x) &\longrightarrow J_\mu^{hadr.}(x). \end{aligned} \qquad (4.10g)$$

Clearly, also the free Lagrangian, eq. (4.2) is invariant under the transformation, eq. (4.7), as long as $\mathcal{U}$ is a constant matrix

$$\mathcal{L}^{(0)}_{quarks}(x) \longrightarrow \mathcal{L}^{(0)}_{quarks}(x). \qquad (4.11)$$

In fact, nobody has so far been able to design an experiment which would single out any direction in colour space. So we should have the freedom in the theory to choose these directions

in colour space as we wish.
The invariance, eq. (4.7), allows us to do this, but, so far, only if we make the <u>same</u> reorientation in colour space everywhere in the space-time manifold.
From a physics point of view this is clearly unsatisfactory.
Following H. Weyl's argument for the free phase changes in QED we can say also here that it would be much nicer if we could choose the unobservable directions in colour

space freely at each space-time point. That is, we are looking for an invariance of the form

$$\underset{\sim}{q}(x) \longrightarrow \mathcal{U}(x)\, \underset{\sim}{q}(x) \tag{4.12}$$

where $\mathcal{U}(x) \in SU(3)$ for every x but otherwise arbitrary. Our Lagrange density for free quarks, eq. (4.2), is then not invariant. But we can add to our theory vector fields, <u>called here gluons</u>, in order to have a theory with the invariance, eq. (4.12).

The prescription how to do this was given in a different context by C.N. Yang and R. Mills in 1954. For our case the result is the

<u>Lagrange density of QCD</u>

$$\mathcal{L}_{QCD}(x) = -\frac{1}{4} G^{a}_{\lambda\rho}(x) G^{a\lambda\rho}(x) + \sum_{q} \bar{q}(x) \left[\frac{i}{2} \gamma^{\lambda} \overrightarrow{D}_{\lambda} - \frac{i}{2} \gamma^{\lambda} \overleftarrow{D}_{\lambda} - m_q \right] q(x) .$$

(4.13)

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In addition to the quarks we have here eight vector fields $G_\lambda^a(x)$ $(a=1,\dots,8)$, the gluons, corresponding to the eight generators of the group $SU(3)$. The $G_\lambda^a(x)$ are the analogues of the vector potential $A_\lambda(x)$ of QED. The gluonic field strength tensor, the analogue of $F_{\lambda\rho}(x)$ in QED, is given here as

$$G_{\lambda\rho}^a(x) = \partial_\lambda G_\rho^a(x) - \partial_\rho G_\lambda^a(x) - g_s f_{abc}\, G_\lambda^b(x)\, G_\rho^c(x)$$

$$(a=1,\dots,8). \qquad (4.14)$$

The gluonic field strength tensor has again eight components. Note the nonlinear term in the gluon potentials on the r.h.s. of (4.14)!

Furthermore, g_s is a dimensionless coupling parameter, the analogue of e in QED. Also here we define

$$\alpha_s = \frac{g_s^2}{4\pi}, \qquad (4.15)$$

the strong "fine structure constant".

In eq. (4.13) D_λ is again the covariant derivative

$$D_\lambda q(x) = \left(\partial_\lambda + i g_s G_\lambda^a(x) \frac{\lambda_a}{2}\right) q(x),$$

$$\bar{q}(x) \overleftarrow{D}_\lambda \equiv \overline{\left(D_\lambda q(x)\right)}.$$

(4.16)

As parameters in the Lagrange density of QCD, eq. (4.13), we have thus one coupling parameter g_s and the quark masses:

$$g_s, \; m_u, \; m_d, \; m_s, \; m_c, \; m_b, \; m_t.$$

(4.17)

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It is now a nice exercise to show that $\mathcal{L}_{QCD}$, eq. (4.13), is indeed invariant under the transformations eq. (4.12) for quarks, supplemented by a suitable transformation for the gluons. This is best written down with the help of the gluon potential and field-strength matrices. We define

$$G_\lambda(x) = G_\lambda^a(x)\,\frac{\lambda_a}{2},$$

$$G_{\lambda\rho}(x) = G_{\lambda\rho}^a(x)\,\frac{\lambda_a}{2}.$$

(4.18)

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Furthermore we define a SU(3) gauge transformation for quarks and gluons as follows:

$$q(x) \longrightarrow \mathcal{U}(x)\, q(x)\,,$$

$$G_\lambda(x) \longrightarrow \mathcal{U}(x)\, G_\lambda(x)\, \mathcal{U}^\dagger(x) - \frac{i}{g_s}\, \mathcal{U}(x)\, \partial_\lambda\, \mathcal{U}^\dagger(x)\,. \tag{4.19}$$

Here we require $\mathcal{U}(x) \in SU(3)$ for all x, but otherwise $\mathcal{U}(x)$ can be arbitrary. Note that the gluon potential matrix is transformed including an inhomogenous term.

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We find then easily that the gluon field strength matrix transforms homogenously

$$G_{\lambda\rho}(x) \longrightarrow \mathcal{U}(x)\, G_{\lambda\rho}(x)\, \mathcal{U}^{\dagger}(x)$$

(4.20)

and that the Lagange density is invariant

$$\mathcal{L}_{QCD}(x) \longrightarrow \mathcal{L}_{QCD}(x).$$

(4.21)

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QCD is called a <u>non abelian gauge theory</u> since the gauge group, SU(3), is a non-abelian, that is non-commutative, group. This implies also the non linear term on the r.h.s. of (4.14). Looking at eq. (4.20) we see that the gluon field strength tensor has a non-trivial transformation under a gauge change, contrary to what we have for the field strength tensor in QED. <u>Gluons carry colour charge!</u> We shall see

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in the following sections that this has far reaching consequences for the strong interaction physics.