Outline

1. Introduction
2. 2D discrete model
3. Boltzmann & Hydrodynamic approach in 2D
4. Simulations & reality
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1. Introduction
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What are self-propelled particles (SPP)?
- Particles, which have a constant velocity in a certain direction or an intrinsic propulsion force in the direction of velocity.

Why is their behaviour of interest?
- A flock of SPP is a simple model to describe the behaviour of fish schools, herds, swarms, bacteria, etc.
Important points of the model

- (random) placed point-like particles with constant velocity $v_0$ in (random) direction $\theta$
- Aligning the velocities of near particles
## Comparison to the ferromagnetic model

<table>
<thead>
<tr>
<th>SPP</th>
<th>ferromagnetic</th>
</tr>
</thead>
<tbody>
<tr>
<td>dynamic ( (v_0 = \text{const}) )</td>
<td>equilibrium ( (v_0 \to 0) )</td>
</tr>
<tr>
<td>rule of aligning of velocities pertubations</td>
<td>rule of aligning of spin directions temperature</td>
</tr>
</tbody>
</table>
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Conditions

- particles on a plane with periodic boundary conditions
- location $\vec{x}_i$
- velocity $\vec{v}_i$

→ There are more than one possible models
Model by Czirók and Vicsek:

- \( \vec{v}_i \) in direction \( \theta_i \) and \( |v_i| = v_0 \), e.g. \( \vec{v}_i = v_0 \cdot \left( \begin{array}{c} \cos \theta_i \\ \sin \theta_i \end{array} \right) \)
- \( \theta_i(t + \Delta t) = \langle \theta(t) \rangle_{S(i)} + \eta \)
- \( \rightarrow \vec{x}_i(t + \Delta t) = \vec{x}_i(t) + \vec{v}_i(t)\Delta t \)

**Figure:** Aligning of velocities from E. Bertin et al. in comparison to the aligning of bacteria (I. Aranson et al.)
Figure: Local neighbourhood $S(i)$ from A. Czirók and T. Vicsek
Model by H. Levine and W.J. Rappel:

\[ m_i \cdot \delta_t \vec{v}_i = \alpha \vec{f}_i - \beta \vec{v}_i - \nabla U \]

where

- \( \vec{f}_i = \sum_{i \neq j} \vec{v}_j \exp \left( -\frac{|\vec{x}_i - \vec{x}_j|}{l_c} \right) \)
- \( U = u_a + u_r = \sum_{i \neq j} C_a \exp \left( -\frac{|\vec{x}_i - \vec{x}_j|}{l_a} \right) - \sum_{i \neq j} C_r \exp \left( -\frac{|\vec{x}_i - \vec{x}_j|}{l_r} \right) \)
**Order parameter & Correlation function**

average momentum: $\Phi \equiv \frac{1}{N} \left| \sum_i \vec{v}_i \right|$

- disordered phase, $\Phi = 0$
- ordered phase, $\Phi \geq 0$
- $\Phi(t = 0) \approx 0$

Numerical results show the presence of a phase transition described by:

$$\Phi(\eta) \sim \begin{cases} 
\left( \frac{\eta_c(\rho) - \eta}{\eta_c(\rho)} \right)^\beta & \text{for } \eta < \eta_c(\rho) \\
0 & \text{for } \eta > \eta_c(\rho) 
\end{cases}$$

with $\beta = 0.42 \pm 0.03$.
Order parameter & Critical exponents

Figure: Order parameter $\Phi$ over noise $\eta$ from A. Czirók and T. Vicsek
Correlation function

Beside the order parameter, it is useful to characterize the correlation function:

\[ C(\vec{r}) = \langle \vec{v}(\vec{r} + \vec{r}' , t)\vec{v}(\vec{r}' , t) \rangle \]

Toner & Tu predicted a decay like \( C(\vec{r}) \sim r_\perp^{-2/5} \)
Figure: Correlation function parallel and perpendicular to the average direction of the velocity by A. Czirók and T. Vicsek
Simulation results

Figure: Snapshots of the time development of a system with $N = 4000$, $L = 40$ and $v_0 = 0.01$ from A. Czirók and T. Vicsek
Simulation results

Figure: Transient multiple vortex state from A. Czirók and T. Vicsek
Simulation results

Figure: Snapshots of a system with 200 particles from H. Levine and W.J. Rappel
Simulation results

Figure: Average density of a rotating vortex state from H. Levine and W.J. Rappel
Comments on 1D & 3D

1D:
- particles cannot get around each other
- slowing down before changing direction has to be taken into account
- high density regions can evolve
- emerging ordered phases were observed through second order phase transition
Comments on 1D & 3D

Figure: Density distribution in 1D from H. Levine and W.J. Rappel
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Comments on 1D & 3D

3D:
- particles have lower chance to get close each other
- weak interaction acts against ordering
- ordered phases appear like in the 2D case → vanishing order by constant noise and decreasing density
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Analytical approach

- Analytical approach of SSP is still lacking
- Use hydrodynamic equations for density and velocity fields, within a Boltzmann approach

Model like Czirók and Vicsek:

- Point-like particle with velocity $\vec{v}$ in certain direction $\theta$ and fixed magnitude $v_0$
- Evolve ballistically until an event of self diffusion or binary collision
- Order parameter $\Phi \equiv \frac{1}{N} \sum_i |\vec{v}_i|$
Boltzmann equation

Take the Boltzmann transport equation for the one-particle phase-space distribution \( f(r, \theta, t) \)

\[
\left( \frac{\partial}{\partial t} + \vec{e}(\theta) \cdot \nabla \right)f(r, \theta, t) = I_{\text{dif}}(f) + I_{\text{col}}(f)
\]

and define the hydrodynamic density and velocity fields

\[
\rho(r, t) = \int_{-\pi}^{\pi} d\theta f(r, \theta, t)
\]

\[
w(r, t) = \rho(r, t)u(r, t) = \int_{-\pi}^{\pi} d\theta f(r, \theta, t)e(\theta)
\]
Introduce Fourier series: \( \hat{f}_k(r, t) = \int_{-\pi}^{\pi} d\theta \ f(r, \theta, t) e^{ik\theta} \)

Inserting in Boltzmann eq. leads to infinite set of coupled equations.

Identify \( \hat{f}_0(r, t) = \rho(r, t) \) and \( \hat{f}_1(r, t) = w(r, t) \)

Assume \( f(r, \theta, t) \) is slightly non-isotropic and hydrodynamic fields vary on length scales larger than \( d_0 \).

Expansion to leading order.
Hydrodynamic equation

\[ \frac{\partial \vec{w}}{\partial t} + \gamma (\vec{w} \nabla) \vec{w} = -\frac{1}{2} \nabla (\rho - \kappa \vec{w}^2) + (\mu - \xi \vec{w}^2) \vec{w} + \nu \nabla^2 \vec{w} - \kappa (\nabla \vec{w}) \vec{w} \]
Hydrodynamic equation

\[ \frac{\partial \bar{w}}{\partial t} + \gamma (\bar{w} \nabla) \bar{w} = -\frac{1}{2} \nabla (\rho - \kappa \bar{w}^2) + (\mu - \xi \bar{w}^2) \bar{w} + \nu \nabla^2 \bar{w} - \kappa (\nabla \bar{w}) \bar{w} \]

advection
Hydrodynamic equation

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\[ \text{advection} \]

\[ \text{pressure gradient} \]

with \( p_{eff} = \rho - \kappa \vec{w}^2 \)
Hydrodynamic equation

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\frac{\partial \vec{w}}{\partial t} + \gamma (\vec{w} \nabla) \vec{w} = -\frac{1}{2} \nabla (\rho - \kappa \vec{w}^2) + (\mu - \xi \vec{w}^2) \vec{w} + \nu \nabla^2 \vec{w} - \kappa (\nabla \vec{w}) \vec{w}
\]

- advection
- pressure gradient
- local relaxation

with \( p_{eff} = \rho - \kappa \vec{w}^2 \)
Hydrodynamic equation

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- **advection**
- **pressure gradient**
- **local relaxation**
- **viscosity**

with \( p_{eff} = \rho - \kappa \vec{w}^2 \)
Hydrodynamic equation

\[ \frac{\partial \vec{w}}{\partial t} + \gamma (\vec{w} \nabla) \vec{w} = -\frac{1}{2} \nabla (\rho - \kappa \vec{w}^2) + (\mu - \xi \vec{w}^2) \vec{w} + \nu \nabla^2 \vec{w} - \kappa (\nabla \vec{w}) \vec{w} \]

- advection
- pressure gradient with \( p_{eff} = \rho - \kappa \vec{w}^2 \)
- local relaxation
- feedback from the compressibility of the flow
- viscosity

Manuel Pietsch  Physics of self-propelled particles
This hydrodynamic equation gives analytical solutions for simple geometries and can be used to analyze their stabilities against perturbations.

Remarks:
- coefficients depend on density and noise
- $\gamma \neq 1 \rightarrow$ not Galilean invariant
- $\nu, \gamma, \kappa > 0$
- $\xi > 0$ when $\mu > 0$
Consider a uniform non-zero field, so that $\nabla \vec{w}$ vanishes:

$$\rightarrow \frac{\partial \vec{w}}{\partial t} = (\mu - \xi \vec{w}^2) \vec{w}$$

Solutions are e.g.:

- $\vec{w} = 0$
- $\vec{w} = \sqrt{\frac{\mu}{\xi}} \vec{e}$
- $\vec{w} = \pm \sqrt{\frac{\mu}{\xi}} \frac{e^{\mu t}}{\sqrt{e^{\mu t} + 1}}$
Hydrodynamic equation

Figure: Phase diagram with transition line from E. Bertin et al.
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- Simulation of random placed particles with periodic boundaries (www.youtube.com/watch?v=J_ZPYaEyyuk)
- Simulation of bacteria colonies (www.youtube.com/watch?v=UgIwz-xZil)
- Flock of Starlings (www.youtube.com/watch?v=ctMty7av0jc)
Appendix

Literature

Literature I

Aranson et al.
Model for dynamical coherence in thin films of self-propelled microorganisms

Bertin et al.
Boltzmann and hydrodynamic description for self-propelled particles

Czirók, András, and Tamás Vicsek
Collective behavior of interacting self-propelled particles

Manuel Pietsch
Physics of self-propelled particles
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Self-organization in systems of self-propelled particles.

Toner, John, and Yuhai Tu
Flocks, herds, and schools: A quantitative theory of flocking

Toner, John, Yuhai Tu, and Sriram Ramaswamy
Hydrodynamics and phases of flocks

Vicsek, Tamás, et al.
Novel type of phase transition in a system of self-driven particles
*Physical review letters*, 75.6 (1995): 1226.