A Remnant of Supersymmetry on the Lattice

Georg Bergner

December 15, 2007

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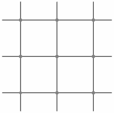
suggestion [Lüscher]; earlier attempts [So, Ukita]

Introduction

continuum
path integral



lattice
N (finite) integrations

symmetries

? (broken by the lattice?, anomalies?, realization?)

Introduction

continuum
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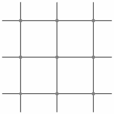
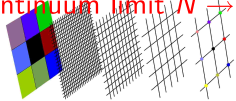


lattice
N (finite) integrations

continuum limit $N \to \infty$

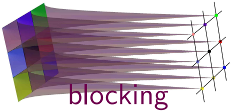


symmetry

? (broken by the lattice?, anomalies?, realization?)

should generate
fine tuning

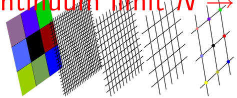
Introduction



blocking

continuum path integral $\longrightarrow$ lattice N (finite) integrations

continuum limit $N \to \infty$

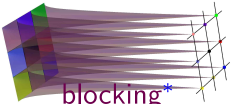


imply

symmetry $\rightleftarrows$ relations

condition for generation

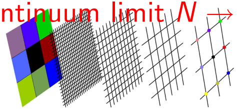
Introduction



blocking*

continuum path integral → lattice N (finite) integrations

continuum limit $N \to \infty$



imply

symmetry → relations

condition* for generation

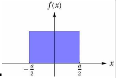
$*$ blocking not unique; not invertible (condition necessary?)

The blocking transformation

- averaging of the continuum field φ:
  $\Phi_t[\varphi] \equiv c \int f_t(x)\varphi(x)dx = c \int f(x - x_t)\varphi(x)dx$

dimension

$f(x)$ peaked around zero e. g.



lattice point $x_t = ta$

- associate the degrees of freedom on the lattice with averaged continuum fields e. g.
  $e^{-S_L(\{\phi_t\})} = \int \mathcal{D}\varphi \prod_t \delta(\phi_t - \Phi_t[\varphi])e^{-S[\varphi]}$
- "smeared" version[1] (α blocking matrix):

$$\boxed{e^{-S_L(\{\phi_t\})} = \int \mathcal{D}\varphi \, e^{-\frac{1}{2}(\phi_t - \Phi_t[\varphi])\alpha_{tt'}(\phi_{t'} - \Phi_{t'}[\varphi])} e^{-S[\varphi]}}$$

$$\exp(-\tfrac{1}{2}(\phi_t - \Phi_t[\varphi])\alpha_{tt'}(\phi_{t'} - \Phi_{t'}[\varphi])) \xrightarrow[\alpha_{tt} \to \infty]{} \prod_t \delta(\phi_t - \Phi_t[\varphi])$$

[1] used in [Ginsparg, Wilson]

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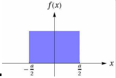

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  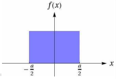

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Symmetry on the lattice

- global continuum symmetry ($\tilde{Q}_{\sigma,\rho}$):
  $S[\varphi + \delta\varphi] \equiv S[(1 + \varepsilon\tilde{Q})\varphi] = S[\varphi]$
  ($\varphi(x) \to \varphi_\sigma(x)$ (multiplet); $\Phi_i \equiv \Phi_{\sigma t} = c \int f_t(x)\varphi_\sigma(x)dx$)
- to translate the continuum symmetry transformations ($\tilde{Q}$) into a lattice transformations (Q):

$$\boxed{\varepsilon Q_{ij}\Phi_j = \varepsilon Q_{\sigma t,\rho t'}\Phi_{\rho t'} \equiv \varepsilon c \int f_t(x)\tilde{Q}_{\sigma,\rho}\varphi_\rho(x)dx}$$

- can not be defined for every $\tilde{Q}$ and $f \hookrightarrow$ additional condition
- lattice transformations: $(\delta\phi)_i = \varepsilon Q_{ij}\phi_j$
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Fermionic fields

- fermionic fields can be included in φ (real fields):

$$
\alpha_{ij} = \alpha_{\sigma t,\rho t'} = \begin{array}{c}
\overbrace{}^{\sigma \text{ bosonic}}\;\overbrace{}^{\sigma \text{ fermionic}} \\
\begin{pmatrix} \text{symmetric} & 0 \\ 0 & \text{antisymmetric} \end{pmatrix}
\end{array}
\begin{array}{l} \} \, \rho \text{ bosonic} \\ \} \, \rho \text{ fermionic} \end{array}
$$

- ε can be fermionic (Graßmann)
- the additional minus sign if ε is commuted with a fermionic field can be absorbed in $\alpha \to \alpha^T$
  ($\alpha^{(T)}$ means α^T if ε fermionic)

Inherited symmetry of the blocked action

- a transformation inside the blocked action (use symmetry of the continuum action and additional condition):

$$e^{-S_L(\{\phi_i+\varepsilon Q_{ij}\phi_j\})} = \int \mathcal{D}\varphi\, e^{-\frac{1}{2}(1+\varepsilon Q)_{ij}(\phi-\Phi[\varphi])_j\alpha_{ik}(1+\varepsilon Q)_{kl}(\phi-\Phi[\varphi])_l}\, e^{-S[\varphi]}$$

- consider only terms linear in ε:

$$\varepsilon e^{-S_L} Q_{ij}\phi_j \frac{\partial}{\partial \phi_i} S_L = \varepsilon \int \mathcal{D}\varphi\, \frac{1}{2}(\phi-\Phi)(Q^T\alpha+\alpha^{(T)}Q)(\phi-\Phi)\times$$
$$\times\, e^{-(\phi-\Phi)\alpha(\phi-\Phi)}\, e^{-S}$$

- use $\frac{\partial}{\partial\phi} M \frac{\partial}{\partial\phi} e^{-\frac{1}{2}\phi\alpha\phi} = (\phi\alpha^T M\alpha\phi - (M\alpha)_{ii})e^{-\frac{1}{2}\phi\alpha\phi}$

$$\boxed{\begin{aligned}\varepsilon e^{-S_L}\phi_i Q^T_{ij}\left(\frac{\partial}{\partial\phi_j}S_L\right) = \varepsilon\frac{1}{2}\Bigg(&\frac{\partial}{\partial\phi_i}(\alpha^{-1})^T_{ij}(Q^T\alpha+\alpha^{(T)}Q)_{jk}(\alpha^{-1})_{kl}\frac{\partial}{\partial\phi_l}\\ &+\frac{1}{2}(\alpha^{-1})^T_{ij}(Q^T\alpha+\alpha^{(T)}Q)_{ji}\Bigg)e^{-S_L}\end{aligned}}$$

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Supersymmetry relation for the blocked action

- Supersymmetry: ε fermionic
  $(\frac{\partial}{\partial\phi}((\alpha^{-1})^T Q^T + Q\alpha^{-1})\frac{\partial}{\partial\phi} = 2\frac{\partial}{\partial\phi}(Q\alpha^{-1})\frac{\partial}{\partial\phi})$

$$\varepsilon\phi Q^T\left(\frac{\partial}{\partial\phi}S_L\right) = \varepsilon\left(\frac{\partial}{\partial\phi}S_L\right)(Q\alpha^{-1})\left(\frac{\partial}{\partial\phi}S_L\right) - (Q\alpha^{-1})_{ij}\left(\frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}S_L\right) \quad (+\mathrm{Tr}_B(Q)\text{ vanishes})$$

- α_S^{-1} drops out if $((\alpha_S^{-1})^T Q^T + Q\alpha_S^{-1}) = 0 \hookrightarrow$ invariant blocking
- α^{-1} equivalent to $\alpha^{-1} + \alpha_S^{-1}$ (true for any symmetry but not always possible to construct α_S^{-1})

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Matrix relation for a quadratic action

- consider a quadratic action: $S_L = \phi_i K_{ij} \phi_j$
- naive Symmetry: $\varepsilon \phi Q^T \left(\frac{\partial}{\partial \phi} S_L \right) = 2\varepsilon \phi Q^T K \phi = 0 \Leftrightarrow$
  $\boxed{Q^T K + K^T Q = 0}$ (like α_S)
- blocking induced symmetry:

$$2\varepsilon\phi Q^T K \phi = 4\varepsilon \phi K^T (Q\alpha^{-1}) K \phi \quad (-(Q\alpha^{-1})_{ij} K_{ij} \text{ vanishes})$$

- define a K dependent $\hat{Q}^T \equiv Q^T - 2K^T Q\alpha^{-1}$
  $\hookrightarrow \varepsilon\phi \hat{Q}^T K \phi = 0$
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The additional condition for supersymmetry

- must solve
  $Q_{\sigma t,\rho t'}\Phi_{\rho t'} \equiv Q_{\sigma t,\rho t'} c \int f_t(x)\varphi_\rho(x)dx = c\int f_t(x)\tilde{Q}_{\sigma,\rho}\varphi_\rho(x)dx$
  for Q and $f_t(x) = f(x - x_t)$
- off-shell supersymmetry: $\tilde{Q}$ is a matrix acting on σ, ρ ($\surd$) but contains derivative operators ($\hookrightarrow$ solve)
- ∇ is the lattice derivative inside Q:
  $\nabla_{tt'}\int f_{t'}(x)\varphi(x)dx = \int f_t(x)\partial_x\varphi(x)dx \; \forall\, \varphi \Leftrightarrow$
  $\boxed{\nabla_{tt'} f_{t'}(x) + \partial_x f_t(x) = 0}$
- if λ_n are the eigenvalues of $\nabla_{tt'}$ the solution is :
  $$f_t(x) = B_t^{(0)} + \sum_{n=-(N-1)/2}^{(N-1)/2} B_n\, e^{\mathrm{i}\frac{2\pi}{N}nt - \lambda_n x} \stackrel{!}{=} f(x-t)$$
- fulfilled if $\hat{\lambda}_n = \mathrm{i}\frac{2\pi}{N}n$ if $\hat{\lambda}_n$ are the λ_n with $B_n \neq 0$
- this ensures the periodic boundary conditions for lattice and continuum

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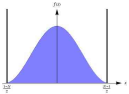
Solution of the additional condition

- maximum number of $\hat{\lambda}_n$ is N (SLAC-derivative); minimum to get "sensible" $f(x)$ is $\hat{\lambda}_n = \{0, \pm \mathrm{i}2\pi/N\}$ 
- Hermiticity: $\nabla(N, p) = \mathrm{i}\sum_l C_l(N)\sin(lp)$ $(\lambda_n = \nabla(N, \frac{2\pi}{N}n))$
- continuum limit: $\partial_p \nabla(N, p = 0) = \mathrm{i}$ for all N (more general $\lim_{a\to 0} 1/a\nabla(1/a, ap) = \mathrm{i}p$)
- if $\hat{\lambda}_i = 0, \pm\mathrm{i}2\pi/N, \pm\mathrm{i}4\pi/N, \ldots, \pm\mathrm{i}L2\pi/N$ then $\mathrm{i}2\pi/N = \nabla(N, 2\pi/N); \ldots; \mathrm{i}L2\pi/N = \nabla(N, L2\pi/N)$
- $L+1$ conditions to solve use $C_1, \ldots, C_{L+1}$
- in position space: $\mathrm{i}\sin(lp) \to 1/2(\delta_{t,t'+l} - \delta_{t,t'-l})$ (the larger L the more off diagonal elements needed until the whole matrix is covered)

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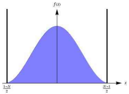
- maximum number of $\hat{\lambda}_n$ is N (SLAC-derivative); minimum to get "sensible" $f(x)$ is $\hat{\lambda}_n = \{0, \pm \mathrm{i}2\pi/N\}$ 

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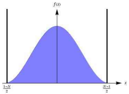

- Hermiticity: $\nabla(N,p) = \mathrm{i}\sum_l C_l(N)\sin(lp)$ $(\lambda_n = \nabla(N, \frac{2\pi}{N}n))$
- continuum limit: $\partial_p \nabla(N, p=0) = \mathrm{i}$ for all N (more general $\lim_{a\to 0} 1/a\nabla(1/a, ap) = \mathrm{i}p$)
- if $\hat{\lambda}_i = 0, \pm\mathrm{i}2\pi/N, \pm\mathrm{i}4\pi/N, \ldots, \pm\mathrm{i}L2\pi/N$ then $\mathrm{i}2\pi/N = \nabla(N, 2\pi/N); \ldots; \mathrm{i}L2\pi/N = \nabla(N, L2\pi/N)$
- $L+1$ conditions to solve use $C_1, \ldots, C_{L+1}$
- in position space: $\mathrm{i}\sin(lp) \to 1/2(\delta_{t,t'+l} - \delta_{t,t'-l})$ (the larger L the more off diagonal elements needed until the whole matrix is covered)

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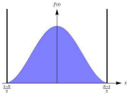
- maximum number of $\hat{\lambda}_n$ is N (SLAC-derivative); minimum to get "sensible" $f(x)$ is $\hat{\lambda}_n = \{0, \pm\mathrm{i}2\pi/N\}$ 
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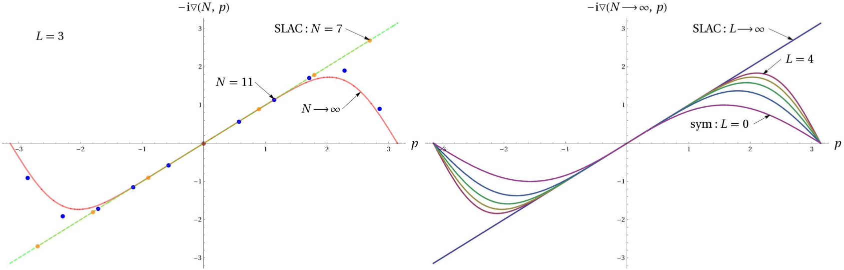
Derivatives that solve the additional condition

Interpolation between SLAC-derivative ($L = (N-1)/2$) and symmetric derivative ($L = 0$):



other solution: $C(N)\nabla^S(p)$ where $C(N \to \infty) \to \frac{1}{2}$

Transformations for supersymmetric quantum mechanics

For a quadratic action only the matrices Q, K and α need to be specified.
Transformations in the continuum ($\varphi_\sigma = (\tilde{\varphi}, F, \psi, \bar{\psi})$):

$$\delta\tilde{\varphi} = -\bar{\varepsilon}\psi + \varepsilon\bar{\psi} \quad \delta F = -\bar{\varepsilon}\partial\psi - \varepsilon\partial\bar{\psi}$$
$$\delta\psi = -\varepsilon\partial\tilde{\varphi} - \varepsilon F \quad \delta\bar{\psi} = \bar{\varepsilon}\partial\tilde{\varphi} - \bar{\varepsilon}F$$

Transformations on the lattice (∇ solution to additional condition, $\phi_{\sigma,t} = (\hat{\varphi}_t, \hat{F}_t, \hat{\psi}_t, \hat{\bar{\psi}}_t)$):

$$\delta \begin{pmatrix} \hat{\varphi} \\ \hat{F} \\ \hat{\psi} \\ \hat{\bar{\psi}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\bar{\varepsilon} & \varepsilon \\ 0 & 0 & -\bar{\varepsilon}\nabla & -\varepsilon\nabla \\ -\varepsilon\nabla & -\varepsilon & 0 & 0 \\ \bar{\varepsilon}\nabla & -\bar{\varepsilon} & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\varphi} \\ \hat{F} \\ \hat{\psi} \\ \hat{\bar{\psi}} \end{pmatrix} = (\varepsilon Q + \bar{\varepsilon}\bar{Q})\phi$$

Ansatz for the lattice action in supersymmetric quantum mechanics

in the continuum:

$$S = \frac{1}{2}(\partial\tilde{\varphi})^2 - mF\varphi - \frac{1}{2}F^2 + \bar{\psi}(\partial + m)\psi$$

on the lattice:

$$S = \phi_i K_{ij} \phi_j =$$

$$\frac{1}{2}\begin{pmatrix} \hat{\varphi}_t \\ \hat{F}_t \\ \hat{\psi}_t \\ \hat{\bar{\psi}}_t \end{pmatrix}^T \begin{pmatrix} -\Box_{tt'} & -m_{b,tt'} & 0 & 0 \\ -m_{b,tt'} & -\mathbb{1}_{tt'} & 0 & 0 \\ 0 & 0 & 0 & (\tilde{\nabla} - m_f)_{tt'} \\ 0 & 0 & (\tilde{\nabla} + m_f)_{tt'} & 0 \end{pmatrix} \begin{pmatrix} \hat{\varphi}_{t'} \\ \hat{F}_{t'} \\ \hat{\psi}_{t'} \\ \hat{\bar{\psi}}_{t'} \end{pmatrix}$$

Hermiticity: $\Box$, m_b, m_f symmetric; $\tilde{\nabla}$ antisymmetric (undetermined)

translation invariance: all circulant matrices → commutative algebra

A blocking matrix α

general ($\alpha \sim \delta_{tt'}$):

$$\alpha_G = \begin{pmatrix} b_1 & b_4 & 0 & 0 \\ b_4 & b_3 & 0 & 0 \\ 0 & 0 & 0 & -b_2 \\ 0 & 0 & b_2 & 0 \end{pmatrix} \delta_{tt'}; \quad \alpha_G^{-1} = \begin{pmatrix} \beta_1 & \beta_4 & 0 & 0 \\ \beta_4 & \beta_3 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 \\ 0 & 0 & -\beta_2 & 0 \end{pmatrix} \delta_{tt'}$$

α^{-1} and $\alpha^{-1} + \alpha_S^{-1}$ lead to the same relations (are equivalent)
$\hookrightarrow$ Use α_S^{-1} to reduce matrix entries ($\beta_4 = 0$ by shift in β_2):

$$\alpha_S^{-1} = \begin{pmatrix} 0 & \beta_4' & 0 & 0 \\ \beta_4' & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_4' \\ 0 & 0 & -\beta_4' & 0 \end{pmatrix} \delta_{tt'}; \quad \alpha_A^{-1} = \begin{pmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & \beta_3 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 \\ 0 & 0 & -\beta_2 & 0 \end{pmatrix} \delta_{tt'}$$

Solution for supersymmetric quantum mechanics with a quadratic action

- pure matrix identity, $\hat{Q}^T K + K^T \hat{Q} = 0$ $(\hat{Q}^T = Q^T - 2K^T Q\alpha^{-1})$, leads to:

$$\begin{aligned} \nabla\tilde{\nabla} - \Box + \nabla(m_f - m_b) &= -(\beta_1 \Box \nabla + \beta_2(\Box + m_b\nabla))(m_f + \tilde{\nabla}) \\ m_f - m_b + \tilde{\nabla} - \nabla &= -(\beta_1 m_b \nabla + \beta_2(\Box + m_b\nabla))(m_f + \tilde{\nabla}) \end{aligned}$$

- same conditions from $\bar{Q}$
- solution:

$$\begin{aligned} \Box &= \frac{(1 - \beta_1 {m_b}^2)\nabla^2}{1 - \beta_1 \nabla^2} \\ (\tilde{\nabla} + m_f) &= \frac{\nabla + m_b}{1 + \beta_2 m_b + \beta_2 \nabla + \beta_1 m_b \nabla} \end{aligned}$$

- m_b not specified – could use $m_b = m$
- if there are doublers in ∇ then a Wilson-term is needed: $m_b = m + m_W \hookrightarrow$ for small β_1 no doubler and sensible twopointfunctions (F integrated out)

Continuum limit

- so far all in lattice units ($a = 1$)
- according to dimension $\lim_{a\to 0} \frac{1}{a^2}\Box(a, m_{ph}a, ap_{ph})$ ($m_{ph} = m/a$ fixed; $\Box(m, p)$)
- dimension of parameter b_{1ph} ($\varphi_t b_1 \varphi_t$); b_{2ph} ($\bar{\psi}_t b_2 \psi_t$) should be the same as m_{ph}^2 and m_{ph}
- but keep b_1 (b_2) and not b_{1ph} (b_{1ph}) fixed $\hookrightarrow$ b_{1ph} (b_{1ph}) diverges in the continuum limit and $\exp\{-(\phi_{ph}-\Phi_{ph})b_{1ph}(\phi_{ph}-\Phi_{ph})\} \xrightarrow[a\to 0]{} \prod \delta(\phi_{ph}-\Phi_{ph})$

$\hookrightarrow$ β_1, β_2, m_{ph} fixed

$$\begin{aligned}
\frac{1}{a^2}\Box(a, m_{ph}a, p_{ph}a, \beta_1, \beta_2) &= -p^2 + O(a) \\
\frac{1}{a}(\tilde{\nabla} + m_f)(a, m_{ph}a, p_{ph}a, \beta_1, \beta_2) &= (\mathrm{i}p + m) + O(a)
\end{aligned}$$

Conclusions

- symmetry of a continuum action implies the fulfillment of certain relations for the lattice action which ensure a symmetric continuum limit
- requirement: definition of a lattice transformation by the "averaged" continuum symmetry transformation (additional condition)
- if α_S^{-1} exists the relations can be reduced to the naive lattice symmetry – otherwise the naive symmetry is modified by terms proportional to α^{-1}
- in case of off-shell supersymmetry the additional condtion can be fulfilled and it is possible to find α_S^{-1}
- susy. with quadratic action: it is possible to fulfill the naive symmetry if an α_S^{-1} is used as well as the modfied symmetry if a more general α^{-1} is taken into account

Outlook

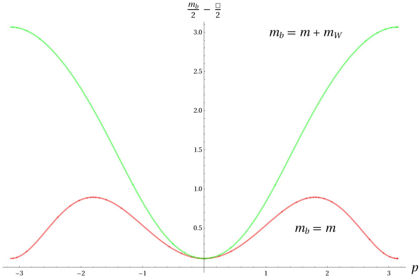
- In a non quadratic (local) supersymmetric lattice model at most a part of the naive supersymmetry can be preserved.
- In general there are two different kinds of problems:
  - chiral symmetry: no α_S^{-1} can be used
  - supersymmetry: no (non quadratic) local action found that preserves all naive symmetries – but α_S^{-1} possible
- hope for supersymmetry: find an action that is not naively symmetric but local and fulfills the modified symmetry
- a quadratic action is only a simple model to show the modification of the symmetry on the lattice $\hookrightarrow$ intresting case: apply the procedure to a non quadratic action

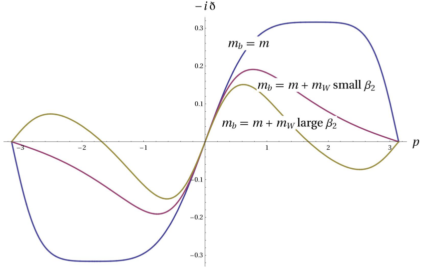
H. So and N. Ukita
Ginsparg-Wilson relation and lattice supersymmetry.
Phys. Lett., B457:314-318, 1999

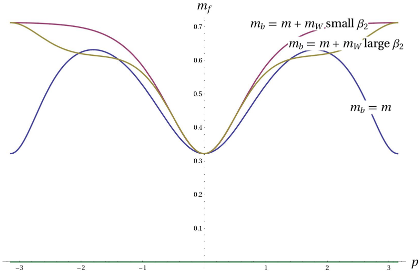
M. Lüscher
Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation.
Phys. Lett., B428:342-345, 1998

P. H. Ginsparg and K. G. Wilson
A Remnant of Chiral Symmetry on the Lattice.
Phys. Rev., D25:2649, 1982

Dispersion relations of the operators

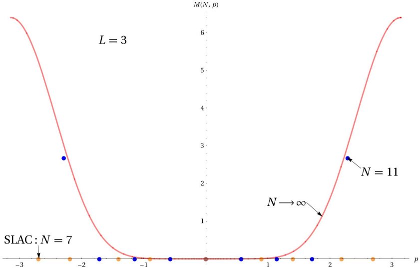


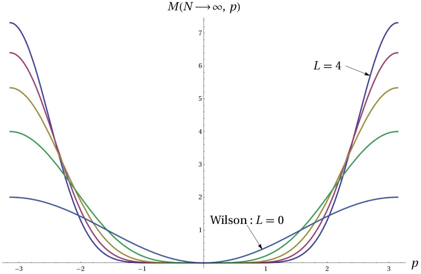




Spectral improved Wilson terms

Introduce a Wilson term with $2L+1$ eigenvalues $=0$
SLAC: $L=(N-1)/2$





Dispersion relations of the operators

$$
\begin{aligned}
\Box &= \frac{(1-\beta_1 {m_b}^2)\nabla^2}{1-\beta_1\nabla^2} \\
\tilde{\nabla} &= \frac{(1-\beta_1 {m_b}^2)\nabla}{(1+\beta_2 m_b)^2-(\beta_2+\beta_1 m_b)^2\nabla^2} \\
m_f &= \frac{m_b(1+\beta_2 m_b)-(\beta_2+\beta_1 m_b)\nabla^2}{(1+\beta_2 m_b)^2-(\beta_2+\beta_1 m_b)^2\nabla^2} \\
m_b &= m+m_W = m+(1-cos(p)) \\
L=1 \hookrightarrow \nabla &= \mathrm{i}C_1\sin(p)+\mathrm{i}\frac{1}{2}(1-C_1)\sin(2p) \\
\text{with } C_1(N) &= \frac{\frac{2\pi}{N}-\frac{1}{2}\sin(\frac{4\pi}{N})}{\sin(\frac{2\pi}{N})-\frac{1}{2}\sin(\frac{4\pi}{N})}
\end{aligned}
$$