Polyakov loops from Dirac spectra

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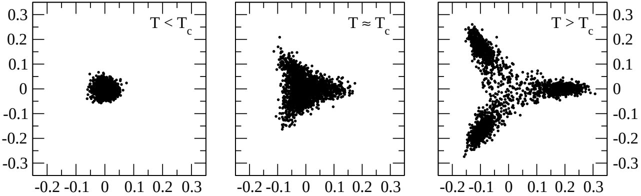
Motivation

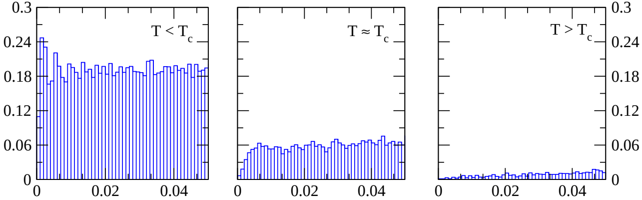
QCD (quenched $\sim$ Yang-Mills) at finite temperature

- Polyakov loop: $\mathcal{P}(\vec{x}) = \mathcal{P}\exp\left(i\int_0^\beta dx_0 A_0(x_0,\vec{x})\right), \quad \beta = 1/k_B T$



  order parameter for confinement: $\langle \mathrm{tr}_c\, \mathcal{P}\rangle \sim e^{-\beta F_{\text{quark}}}$

- spectral density $\rho(\lambda)$ of the Dirac operator (in background A_μ):



  order parameter of chiral symmetry: $\rho(0) \sim \langle\bar{\psi}\psi\rangle$ Banks-Casher

The idea

relate Polyakov loops to Dirac spectra, on the lattice

- Polyakov loop: $\mathcal{P}(x) \equiv \prod_{\tau=1}^{N} U_0(x_0+\tau,\vec{x})$ $\qquad N \equiv N_0$
- Dirac operator, here staggered $\qquad$ Kogut,Susskind

$$D(x,y) \equiv \frac{1}{a}\sum_{\mu} \eta_\mu(x)\left[U_\mu(x)\delta_{x+\hat{\mu},y} - h.c.\right]$$ hopping by one link

$D^N(x,x) \ni$ products of links along closed loops at x

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$$D(x,y) \equiv \frac{1}{a}\sum_{\mu} \eta_\mu(x)\left[U_\mu(x)\delta_{x+\hat{\mu},y} - h.c.\right] \quad \text{hopping by one link}$$

$D^N(x,x) \ni$ products of links along closed loops at x

how to distinguish Polyakov loops from 'trivially closed' loops?

- phase 'twisted' boundary conditions:

$$\psi_z(x_0+\beta,\vec{x}) = z\psi_z(x_0,\vec{x}), \quad z = e^{i\phi}$$

- realized by $U_0 \to zU_0,\ U_0^\dagger \to z^* U_0^\dagger$ at, say, the last time slice

  $\Rightarrow$ Polyakov loops: $\mathcal{P} \to z\mathcal{P},\ P^\dagger \to z^*\mathcal{P}^\dagger$ Gattringer '06

  while trivial loops do no change

$$D_z^N(x,x) = z\,\mathcal{P}(\vec{x}) + z^*\mathcal{P}^\dagger(\vec{x}) + \ldots \qquad (a=1)$$

linear system, extract $\mathcal{P}$ by three different bc.s, say center

$$\mathcal{P}(x) = D_1^N + z^* D_z^N + z D_{z^*}^N \qquad z = e^{i\frac{2\pi}{3}}$$

or by an integral over all bc.s

$$\mathcal{P}(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, z^* D_z^N(x,x) \qquad z = e^{i\phi}$$

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trace and space average $\rightarrow$ completeness of ψ_n:

$$\boxed{\frac{1}{V}\sum_x \mathrm{tr}_c\mathcal{P}(x) = \frac{1}{V}\sum_n \left[\lambda_{1,n}^N + z^*\lambda_{z,n}^N + z\lambda_{z^*,n}^N\right]}$$

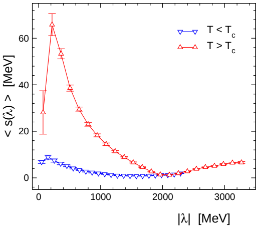
exact formula if all modes included ($n = 1\ldots 3NV$) IR dominated?!

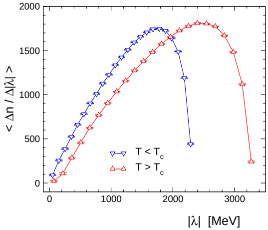
Results from Lattice calculations

aim: reconstruct $\sum_{\vec{x}} \mathrm{tr}_c \mathcal{P}/V \neq 0$ in the deconfined phase
from a finite number of eigenvalues

what counts:

- shift of λ_z with z:
- density of λ's:
- λ itself, even λ^N





[shown are $N = 4$]

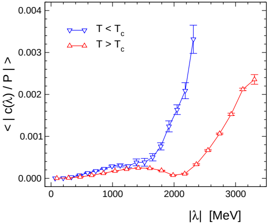
IR dominates

IR (and UV) suppressed

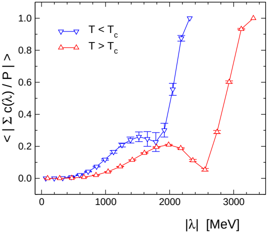
IR suppressed

altogether this results in . . .

- individual contributions:



- accumulated $|\mathcal{P}|$:



$\Rightarrow$ Polyakov loop dominated by UV modes

(same for higher *N* and larger volumes)

unphysical! these modes do not reflect the continuum well!

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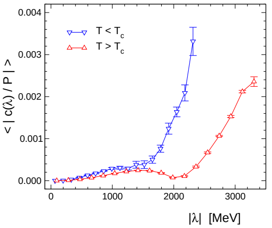

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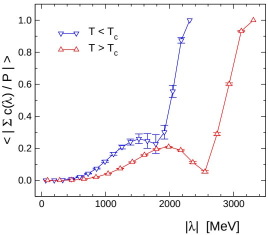
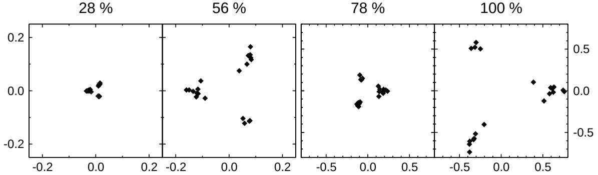


⇒ Polyakov loop dominated by UV modes

(same for higher *N* and larger volumes)

unphysical! these modes do not reflect the continuum well!

in addition, the smallest λ's generate the wrong sign:



Explanation

- staggered eigenvalues λ are purely imaginary
  $\lambda^{N=4} > 0$
- the twist in the boundary condition lifts the lowest eigenvalue by roughly the same amount for $z = e^{i\frac{2\pi}{3}}$ and $z^* = e^{-i\frac{2\pi}{3}}$

lowest contribution:

$$\begin{aligned} \lambda^N_{1,0} + z^* \lambda^N_{z,0} + z \lambda^N_{z^*,0} &= p_1 + (z^* + z) p_2 \qquad \text{with } p_2 > p_1 \\ &= p_1 - p_2 < 0 \end{aligned}$$

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- argument does not hold for $N = 6$ since $\lambda^{N=6} < 0$
- indeed the lowest contribution there comes with the correct sign, but later the sign changes to the wrong one

Summary I

Polyakov loops can be obtained from powers of one-link operators . . .

(cf. links from Laplace operator FB, Ilgenfritz, '05

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continuum limit:

$\sum_n \lambda_n^N$ is crazy, since:

- $\lambda \in i\,[0,\infty)$: continuous spectrum
- $N \to \infty$: finer (in x_0)
- well, could be cancelled by dependence of $\lambda_{z,n}$ on bc. z

Other approaches

- consider instead of D^N other functions of D:

Synatschke, Wipf, Wozar, '07:

$$\frac{1}{D}, \ \frac{1}{D^2}, \ e^{-D}, \ e^{-D^\dagger D}$$

all summed over center boundary conditions
(Wilson-Dirac operator, small lattices)

$\oplus$ IR dominated
better continuum behaviour!?

$\ominus$ no direct relation to Polyakov loop, however:
empirically still order parameters: $\langle$spectral sums$\rangle \sim \langle \mathrm{tr}_c \, \mathcal{P} \rangle$
hopping expansion: becomes $\langle \mathrm{tr}_c \, \mathcal{P} \rangle$ in leading order

Dressed Polyakov loops

definition (color trace included):

$$\tilde{\mathcal{P}}_\kappa \equiv \frac{1}{V} \sum_{l \in \mathcal{L}_l^{(1)}} \kappa^{|l|} \, \mathrm{tr}_c \prod_{(x,\mu) \in l} U_\mu(x)$$

$\mathcal{L}_l^{(1)}$: all (lattice) loops l of length $|l|$ winding 1 time in x_0

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not really feasible, since in principle arbitrarily long loops; convergent?!

A mass dependent observable

consider as observable the integrated propagator with mass m:

$$\mathcal{O}(m) \equiv \frac{1}{V} \int dx \, \mathrm{tr}_{c(,\gamma)} \frac{1}{D(x,x)+m}$$

relation to Polyakov loop: lattice and introduce z again

$$\mathcal{O}_z(m) = \frac{1}{mV} \sum_x \sum_k \Big(- \frac{D_z(x,x)}{m} \Big)^k$$

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$$= \frac{1}{mV} \sum_x \Big\{ \ldots + z \Big[\frac{\mathrm{tr}_c \mathcal{P}(x)}{(2am)^N} + \frac{\mathrm{tr}_c p^{(2)}}{(2am)^{N+2}} + \ldots \Big] + z^0 \big[\ldots \big] + \ldots \Big\}$$

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projection on z-term gives the dressed Polyakov loop!

namely:

$$\int_0^{2\pi} \frac{d\phi}{2\pi}\, z^* \mathcal{O}_z(m) = \frac{1}{m}\, \tilde{\mathcal{P}}_{\kappa=1/am}$$

hence for large mass:

$$\boxed{\int_0^{2\pi} \frac{d\phi}{2\pi}\, z^* \mathcal{O}_z(m) \stackrel{m\to\infty}{\longrightarrow} const\, \frac{1}{V} \sum_x \mathrm{tr}_c \mathcal{P}(x)}$$

$\Rightarrow$ approaches conventional deconfinement order parameter namely:

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on the other hand for small mass:

$$\boxed{\lim_{V\to\infty} \mathcal{O}_z(m) \overset{m\to 0}{\longrightarrow} \pi\rho(0)}$$

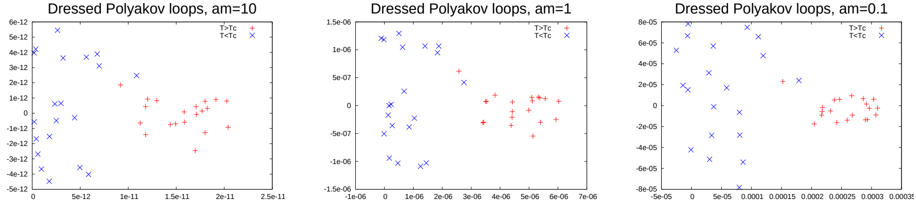
$\Rightarrow$ approaches chiral condensate

Numerical findings (preliminary)

- dressed Polyakov loops as a function of dressing coefficient:

$$\int_0^{2\pi} \frac{d\phi}{2\pi}\, z^* \mathcal{O}_z(m) \sim \tilde{\mathcal{P}}_{\kappa=1/am}$$

$12^3 \cdot 6$, integral by 16 values, for $T > T_c$ only real Polyakov loops



even for enhancement of smeared loops ($\kappa > 1,\ am < 1$)

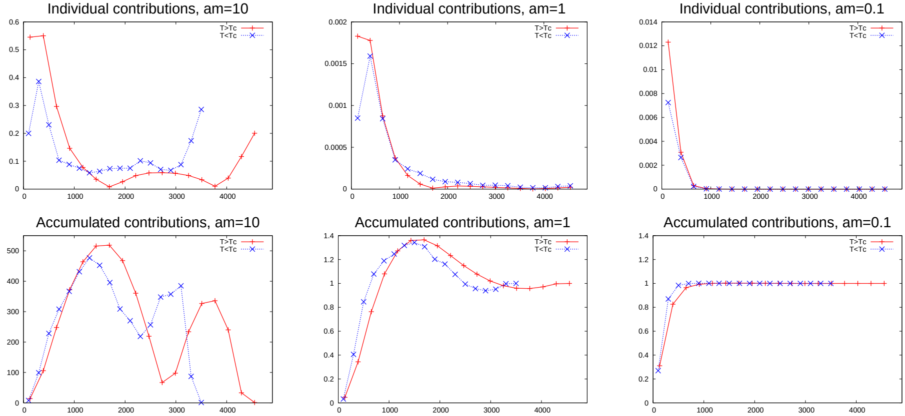
correlated to thin Polyakov loop configuration-wise (averaged)

$\Rightarrow$ still an order parameter `to be made more quantitative`

- individual and accumulated contributions:

$$\int_0^{2\pi} \frac{d\phi}{2\pi}\, z^* \mathcal{O}_z(m) \sim \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_n \frac{z^*}{\lambda_{z,n} + m}$$

as a function of $|\lambda|$:



$\Rightarrow$ IR dominated, probes chiral condensate

Loose ends

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but integrated over twist z, dependence of $\rho_z(0)$ on z:

confined phase:

$$\rho_z(0) \text{ indep. of } z \quad \Rightarrow \quad \int_0^{2\pi} \frac{d\phi}{2\pi}\, z^* \mathcal{O}_z(m) = 0, \quad \text{as is } \langle \mathrm{tr}_c \mathcal{P} \rangle$$

deconfined phase (real Polyakov loop):

$$\rho_z(0) \sim \delta(\phi) \quad \Rightarrow \quad \int_0^{2\pi} \frac{d\phi}{2\pi}\, z^* \mathcal{O}_z(m) = \text{finite}, \quad \text{as is } \langle \mathrm{tr}_c \mathcal{P} \rangle$$

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- full QCD:

lattice simulations may suggest a crossover with $T_c^{\mathrm{deconf}} \neq T_c^{\chi\mathrm{sb}}$

Aoki& Wuppertal vs. RBC-Bielefeld (staggered fermions)

what could go ‘wrong’ in our connection between the Polyakov loop and the chiral condensate?