Lattice SUSY: $N = 2$ Super Yang-Mills Theory in Two Dimensions

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Δ-Meeting:

QCD, Nonequilibrium Dynamics, Complex Systems and Simulation Methods

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Motivation

Understanding supersymmetric theories is a challenging and fascinating problem

Of course, the motivation for studying supersymmetric theories is to see whether supersymmetry is a symmetry of Nature and experimental evidence for this will become available in the next years

Since supersymmetry predicts the existence of bosons and fermions of equal mass it must be broken in some way

There is however another motivation to study supersymmetry: as a theoretical laboratory to study strongly coupled gauge dynamics. In many respects they resemble Quantum Chromodynamics (confinement and chiral symmetry breaking)

While much is known analitically, the hope is that a discretized formulation of supersymmetric gauge theories would provide information about non-perturbative dynamics and additional information for supersymmetry

$\Longrightarrow$ lattice formulation

Recent reviews

1. J. Giedt, hep-lat/0701006, (Plenary talk at Lattice 2006), " Advances and applications of lattice supersymmetry".

2. S. Catterall, hep-lat/0509136, (Plenary talk at Lattice 2005), "Dirac-Kahler fermions and exact lattice supersymmetry".

3. A. F., hep-lat/0410012, (Review for MPLA), " Predictions and recent results in SUSY on the lattice'

4. D. B. Kaplan, hep-lat/0309099, (Plenary talk at Lattice 2003), "Recent developments in lattice supersymmetry".

5. A. F., hep-lat/0210015, (Plenary talk at Lattice 2002), "Supersymmetry on the lattice"

6. I. Montvay, hep-lat/0112007, (Review), "Supersymmetric Yang-Mills theory on the lattice"

7. I. Montvay, hep-lat/9709080, (Plenary talk at Lattice 1997), "SUSY on the lattice"

Plan of the Talk

Lattice Formulation of Super Yang-Mills theory

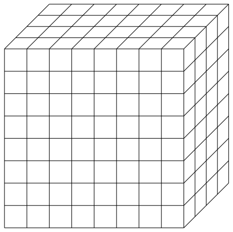
Two dimensional $N = 2$ Super Yang-Mills theory with one exact supercharge

Extension of Sugino's formulation using the other three supercharges and possible applications

A. F., Work in progress
comments/suggestions/criticism welcome!

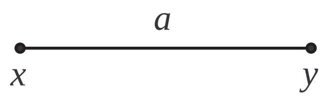
Lattice Gauge theory: General Remarks

Discretization of space-time is achieved introducing an euclidean space-time lattice with spacing a and volume $L^3 \cdot T$. The inverse lattice spacing a^{-1} acts as an UV cutoff.

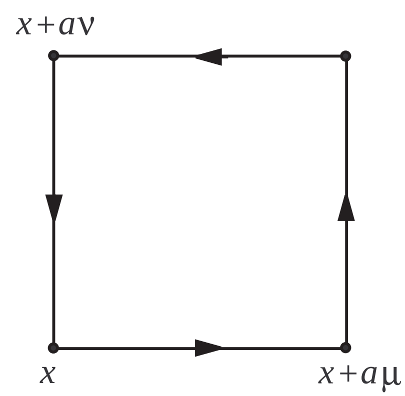


The quark and antiquark fields $\psi(x), \bar{\psi}(x)$ leave in the lattice sites x.

Gauge fields are represented by the link variable $U_\mu(x)$ which are group elements $\in SU(N)$ associated with straight-line path conecting nearest neighbour pairs of lattices sites.



Gauge invariant expressions on the lattice are traces of products of link variables along closed paths. The most elementary one is the *Plaquette variable* 1×1



$$P_{\mu\nu}(x) = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\nu}) U_\nu^\dagger(x)$$

that can be used to construct the lattice Yang-Mills action

There is no a unique way to discretize an observable on the lattice and the only request is that have to reduce to the classical value in the continuum limit ($a \to 0$).

Wilson propose the simplest one

$$S_W = \sum_P S_P = \frac{1}{2}\beta \sum_x \sum_{\mu\nu} \left(1 - \frac{1}{2N}\mathrm{Tr}(P_{\mu\nu}(x) + P^\dagger_{\mu\nu}(x))\right)$$

Introducing the gauge field variables by

$$U_\mu(x) \equiv \exp i g_0 a A^b_\mu(x) T^b$$

and using Baker-Campbell-Haussdorf

$$\begin{aligned} P_{\mu\nu} &\simeq e^{i g_o a^2 F_{\mu\nu}(x)} \\ &\simeq 1 + i g_o a F_{\mu\nu}(x) - \frac{g_o^2 a^2}{2} F_{\mu\nu}(x) F_{\mu\nu}(x) \end{aligned}$$

in the limit $a \to 0$

$$S_W = \sum_x \sum_{\mu\nu} \left(a^4 \frac{\beta g_o^2}{2N} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu} + O(a^6) \right).$$

So the continuum limit is

$$S_W = \int d^4x \frac{1}{2} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu},$$

if we define β to be

$$\beta = \frac{2N}{g_0^2}.$$

Fermions on the Lattice

Recently, following the rediscovery of the Ginsparg-Wilson relation (1982), it has emerged that chiral theories can be put on the lattice in a consistent way:

- The overlap (Narayanan-Neuberger 1993,1995,1998)
- Domain wall fermions (Kaplan-Shamir 1992, 1993, 1994)
- Perfect action (Hasenfratz-Niedermayer 1994, 1998).

This was believed to be impossible for a long time [Nielsen-Ninomiya, 1981] the no-go theorem.

A naive formulation of fermions on the lattice fails

$$S_F = \frac{1}{2} \sum_x \sum_\mu \bar{\psi}(x)(\gamma_\mu \Delta_\mu + m)\psi(x) + h.c.$$

and the resulting propagator is

$$\tilde{\Delta}(k) = \frac{-i \sum_\mu \gamma_\mu \mathrm{sin} k_\mu + m}{\sum_\mu \mathrm{sin}^2 k_\mu + m^2}$$

There is a pole for small k_μ representing the physical particle, but additional poles near $k_\mu = \pm\pi$ appears. S_F describes 16 instead of 1 particle. $\rightarrow$ Doubling problem.

Two popular choices introduced in order to deal with this problem:

- Wilson fermions: Get rids of the doubling species but breaks chiral symmetry explicitily by the Wilson term.

- Staggered fermions (Kogut-Susskind): Reduce from 16 to 4 fermions and for massless fermions a remnant chiral symmetry remains.

Supersymmetry

Such a symmetry makes:

$$Q|boson\rangle = |fermion\rangle \quad Q|fermion\rangle = |boson\rangle$$

the symmetry generator Q (and its hermitian conjugate $Q^\dagger$) carry spin $\frac{1}{2}$

There is essential one possibility for the SUSY algebra:

$$\{Q_\alpha, Q^\dagger_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu$$
$$\{Q_\alpha, Q_\beta\} = \{Q^\dagger_\alpha, Q^\dagger_\beta\} = 0$$
$$[P_\mu, Q_\alpha] = [P_\mu, Q^\dagger_\alpha] = 0$$

Also,

- Q, $Q^\dagger$ transform as spinors under the Lorentz group
- Q, $Q^\dagger$ commute with gauge symmetry generators

We may have more than one Q: $Q^i, i = 1, \cdots N$ (extended supersymmetry)

Lattice formulation of Super Yang-Mills theory

- The major obstacle in formulating a supersymmetric theory on the lattice arises from the fact that the supersymmetry algebra, which is actually an extension of the Poincaré algebra, is explicitly broken on the lattice [Dondi and Nicolai 1977] .

  In particular the Super Poincaré algebra is given by the anti-commutator of a supercharge Q_α and its conjugate Q_β yields the generator of infinitesimal translations P_μ. Schematically,

  $$\left\{ Q_\alpha, Q_\beta^\dagger \right\} = 2\sigma^\mu_{\alpha\beta} P_\mu$$

  On the lattice there are no infinitesimal translations and therefore the supersymmetry algebra must be broken.

Ordinary Poincaré algebra is also broken by the lattice but the hyper-cubic crystal symmetry forbids relevant operators which could spoil the Poincaré symmetry in the continuum limit $\rightarrow$

The Poincaré invariance is achieved automatically in the continuum limit without fine tuning since operators that violate Poincaré invariance are all irrelevant.

However, in the case of the super Poincaré algebra, the lattice crystal group is not enough to guarantee the absence of supersymmetry violating operators.

Failure of the Leibniz rule

On the lattice the Leibniz rule does not hold anymore. [Fujikawa, hep-th/0205095]

$$
\begin{aligned}
&\frac{1}{a}(f(x+a)g(x+a) - f(x)g(x)) = \\
&= \frac{1}{a}(f(x+a) - f(x))g(x) + \frac{1}{a}f(x)(g(x+a) - g(x)) \\
&+ a\frac{1}{a}(f(x+a) - f(x))\frac{1}{a}(g(x+a) - g(x)) \\
&= (\nabla f(x))g(x) + f(x)(\nabla g(x)) + a(\nabla f(x))(\nabla g(x))
\end{aligned}
$$

the breaking of supersymmetry is of order $O(a)$.

- If the supersymmetric theory contains scalar mass terms they break supersymmetry. Since these operators are relevant fine tuning is needed in order to cancel their contributions.

- A naive regularization of fermions results in the doubling problem [Nielsen and Ninomiya, 1981] $\longrightarrow$ wrong number of fermions and violation of the balance between bosons and fermions

Without exact lattice supersymmetry one might hope to construct non-supersymmetric lattice theories with a supersymmetric continuum limit.

This is the case of the Wilson fermion approach for the $4d$ $N = 1$ supersymmetric Yang-Mills theory where the only operator that violates supersymmetry is a fermion mass term.

By tunning the fermion mass to the supersymmetric limit one recovers supersymmetry in the continuum limit [Curci and Veneziano, 1987; I. Montvay, hep-lat/0112007, hep-lat/9510042; Feo, hep-lat/0305020]

Alternatively, using domain wall fermions [Kaplan and Schmaltz hep-lat/0002030] or overlap fermions [Huet, Narayanan, Neuberger, hep-th/9602176] this fine tunning is not required.

In the case of theories with extended supersymmetries the fine tuning of coupling constant is neither feasible nor theoretically practical.

Due to difficulties in realizing exact supersymmetry on the lattice, all that remain us it to realize part of the supercharges as an exact symmetry on the lattice:

This exact lattice supersymmetry is expected to play a key role to restore continuum supersymmetry without (or with less) fine tuning of the parameters of the action.

Two ways to study SUSY on the lattice

- Construct non-SUSY lattice theories with a SUSY continuum limit.
  - N=1 Super-Yang Mills
    (with Wilson fermions – Curci and Veneziano formulation)
- Keep some exact algebra of SUSY on the lattice in order to recover the continuum limit with no (or less) fine tuning of parameters of the action.
  - N=1 Super-Yang Mills
    (with Domain Wall-fermions – Kaplan formulation) $\rightarrow$ the zero gluino mass term is achieved without find tuning

Exact Supersymmetry on the Lattice

It is possible to obtain exact supersymmetry respect to the supersymmetric transformations. An incomplete list for Wess-Zumino type models is ...

Golterman and Petcher, 1989
Bietenholz, hep-lat/9807010
Catterall and Karamov, hep-lat/0110071, hep-lat/0305002
Fujikawa and Ishibashi, hep-lat/0112050
Fujikawa, hep-lat/0208015

while for the $N = 1$ Wess-Zumino model in 4 dimensions an exact lattice formulation have been achieved (also checked the Ward-Takahashi identity to one-loop in the continuum limit) [Bonini and Feo, hep-lat/0402034, hep-lat/0504010]

The two dimensional $N = 2$ Super Yang-Mills theory

The two dimensional continuum theory

The two dimensional $N = 2$ Super Yang-Mills theory can be written as a topological field theory form or Q-exact form [Witten, 1988]

$$S_{SYM}^{N=2,d=2} = Q\frac{1}{2g_0^2}\int d^2x \mathrm{Tr}\left[\frac{1}{4}\eta[\phi,\bar{\phi}] - i\chi_{12}\Phi + \chi_{12}B_{12} - i\psi_\mu D_\mu \bar{\phi}\right],$$

where μ is the index for the two dimensional space-time

The bosonic fields are represented by two scalar fields ϕ and $\bar{\phi}$, a vector field A_μ and another commuting field B_{12}, which is an auxiliary field

The fermionic fields are represented by a vector ψ_μ, an anticommuting scalar field η and a field χ_{12} conjugate to B_{12}

Φ is a function of the field strength $F_{\mu\nu}$ and for two dimensions is given by

$$\Phi \equiv 2F_{12}\,.$$

Q is one of the supercharges of $N = 2$ Super Yang-Mills theory and its transformation rule over the fields is given by the following rule,

$$\begin{aligned} &QA_\mu = \psi_\mu \\ &Q\psi_\mu = iD_\mu\phi \\ &Q\phi = 0 \\ &Q\chi_{12} = B_{12} \\ &QB_{12} = [\phi, \chi_{12}] \\ &Q\bar{\phi} = \eta \\ &Q\eta = [\phi, \bar{\phi}]\,. \end{aligned}$$

Q is nilpotent up to infinitesimal gauge transformations with parameter ϕ, i.e., the square of Q yields an infinitesimal gauge transformations, $Q^2 = \delta_G^\phi$, with parameter ϕ. Carrying out the Q-variation leads to the more explicit form for the $N = 2$ Super Yang-Mills action,

$$\begin{aligned} S_{SYM}^{N=2,d=2} = \frac{1}{2g_0^2}\int d^2x \mathrm{Tr}\bigg[&\frac{1}{4}[\phi,\bar{\phi}]^2 + B_{12}^2 - iB_{12}\Phi \\ &+ D_\mu\phi D_\mu\bar{\phi} - \frac{1}{4}\eta[\phi,\eta] - \chi_{12}[\phi,\chi_{12}] + \psi_\mu[\bar{\phi},\psi_\mu] \\ &+ i\chi_{12}Q\Phi + i\psi_\mu D_\mu\eta\bigg]\,. \end{aligned}$$

and integrate out the field B_{12} gives

$$S_{SYM}^{N=2,d=2} = \frac{1}{2g_0^2}\int d^2x \,\mathrm{Tr}\left[\frac{1}{4}[\phi,\bar{\phi}]^2 + F_{12}^2 \right.$$
$$+ D_\mu\phi D_\mu\bar{\phi} - \frac{1}{4}\eta[\phi,\eta] - \chi_{12}[\phi,\chi_{12}] + \psi_\mu[\bar{\phi},\psi_\mu]$$
$$\left. + i\chi_{12}Q\Phi + i\psi_\mu D_\mu\eta \right].$$

Lattice Formulation with One Exact Supercharge

F.Sugino, hep-lat/0311021, hep-lat/0401017, hep-lat/0410035;
S. Catterall, hep-lat/0410052, hep-lat/0503036, hep-lat/0602004;
D'Adda, et al., hep-lat/0406029, hep-lat/0507029.

Start with a formulation of the theory on a two dimensional hypercubic lattice where the gauge field $A_\mu(x)$ is represented by the unitary link variable $U_\mu(x) = e^{iaA_\mu(x)}$ [Sugino]

the Q-transformation can be generalized on the lattice preserving the property that $Q^2 =$ (is an infinitesimal gauge transformation with the parameter ϕ)

A possible solution is

$$QU_\mu(x) = i\psi_\mu(x)U_\mu(x)$$

$$Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) - i\left(\phi(x) - U_\mu(x)\phi(x+\widehat{\mu})U_\mu^\dagger(x)\right)$$

$$\begin{aligned}
&Q\phi(x) = 0 \\
&Q\chi_{12}(x) = B_{12}(x) \\
&QB_{12}(x) = [\phi(x), \chi_{12}(x)] \\
&Q\bar{\phi}(x) = \eta(x) \\
&Q\eta(x) = [\phi(x), \bar{\phi}(x)]
\end{aligned}$$

where the dependence on the lattice spacing for each field variable is the following,

$$
\begin{aligned}
&Q = O(a^{1/2})\,,\\
&\psi_\mu(x), \chi_{12}(x), \eta(x) = O(a^{3/2})\,,\\
&\phi(x), \bar{\phi}(x) = O(a)\,,\\
&B_{12}(x) = O(a^2)\,,
\end{aligned}
$$

All transformations are the same in the continuum except for $QU_\mu(x)$ and $Q\psi_\mu(x)$.

In fact,

$$
\begin{aligned}
Q^2 U_\mu(x) &= Q(i\psi_\mu(x)U_\mu(x))\\
&= (\phi(x)U_\mu(x) - U_\mu(x)\phi(x+\widehat{\mu}))
\end{aligned}
$$

then we have

$$
\begin{aligned}
Q^2\psi_\mu(x) &= Q[i\psi_\mu(x)\psi_\mu(x) - i(\phi(x) - U_\mu(x)\phi(x+\widehat{\mu})U_\mu(x)^\dagger)]\\
&= [\phi(x), \psi_\mu(x)]
\end{aligned}
$$

Once one have the Q-transformation rule closed among all the lattice variables, it is easy to write the lattice action with the exact supersymmetry Q,

$$S_{SYM}^{N=2} = Q\frac{1}{2g_0^2}\sum_x \mathrm{Tr}\Big[\frac{1}{4}\eta(x)[\phi(x),\bar{\phi}(x)] - i\chi_{12}(x)\Phi(x) + \chi_{12}(x)B_{12}(x) + i\sum_{\mu=1}^{4}\psi_\mu(x)\Big(\phi(x) - U_\mu(x)\phi(x+\hat{\mu})U_\mu^\dagger(x)\Big)\Big]$$

where $\Phi(x) \equiv -i(P_{12}(x) - P_{21}(x))$ and $P_{12}(x) = U_1(x)U_2(x+1)U_1^\dagger(x+2)U_2^\dagger(x)$ and

$$\lim_{a\to 0}\Phi(x) = 2F_{12}(x)\,.$$

that originates the lattice $N = 2$ SYM action

$$
\begin{aligned}
S_{SYM}^{N=2} = \frac{1}{g_0^2} \int \mathrm{Tr} \Bigg[& \frac{1}{4}[\phi(x), \bar{\phi}(x)]^2 + B_{12}^2 - iB_{12}\Phi(x) \\
& + \sum_{\mu} \left(\phi(x) - U_\mu(x)\phi(x+\widehat{\mu})U_\mu(x)^\dagger \right) \left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\widehat{\mu})U_\mu(x)^\dagger \right) \\
& - \frac{1}{4}\eta(x)[\phi(x), \eta(x)] - \chi_{12}(x)[\phi(x), \chi_{12}(x)] + i\chi_{12}(x)Q\Phi(x) \\
& - i\sum_{\mu} \psi_\mu(x) \left(\eta(x) - U_\mu(x)\eta(x+\widehat{\mu})U_\mu(x)^\dagger \right) \\
& - \sum_{\mu} \psi_\mu(x)\psi_\mu(x) \left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\widehat{\mu})U_\mu(x)^\dagger \right) \Bigg]
\end{aligned}
$$

and reduces to the continuum $N = 2$ SYM action in the limit $a \to 0$ without any fine tuning of the parameters of the action.

In fact, the fermionic kinetic term,

$$i\chi_{12}(x)Q\Phi(x) - i\sum_{\mu}\psi_{\mu}(x)\Big(\eta(x) - U_{\mu}(x)\eta(x+\widehat{\mu})U_{\mu}(x)^{\dagger}\Big)$$

has the correct continuum naive limit and contains no doublers. The continuum naive limit is

$$\lim_{a\to 0} i\mathrm{Tr}(\chi_{12}Q\Phi(x)) = 2i\mathrm{Tr}[\chi_{12}(D_1\psi_2(x) - D_2\psi_1(x))]$$

is of order $O(1)$ and is exactly the continuum value, while the second term gives $i\psi_{\mu}(x)D_{\mu}\eta(x)$. Moreover, the second term of the lattice action gives $D_{\mu}\phi(x)D_{\mu}\bar{\phi}(x)$, while the last term gives $\psi_{\mu}(x)[\bar{\phi}(x), \psi_{\mu}(x)]$.

After integrating out the auxiliary field $B_{12}(x)$, one is left with a gauge kinetic term of the form

$$\frac{1}{2g_0^2}\sum_x \sum_{\mu<\nu} \mathrm{Tr}\Big[-\left(U_{\mu\nu}(x) - U_{\nu\mu}(x)\right)^2 \Big]$$

which is slightly different to the one corresponding to the Wilson action

$$\frac{1}{2g_0^2}\sum_x \sum_{\mu<\nu} \mathrm{Tr}\Big[2 - U_{\mu\nu}(x) - U_{\nu\mu}(x) \Big] .$$

As has been discussed in [Sugino], while the term here gives a unique minimun $U_{\mu\nu}(x) = 1$, the piece above contains many classical vacua ± 1. This problem was resolved later on where and admissibility condition on the plaquette variable was included, similar to the one used for the Ginsparg-Wilson operator without spoiling the exact supersymmetry on the lattice.

Lattice Action for the Other Three Supercharges

We extended Sugino's formulation [A.F., work in preparation] and showed that it is possible to construct other three supercharges that are nilpotent up to infinitesimal gauge transformations and we write the lattice action as an exact
$\tilde{Q}, Q_1, Q_2$-form.

The continuum $\tilde{Q}, Q_1, Q_2$ supercharges are given in
Kato, Kawamoto, Miyake, hep-th/0502119

Supersymmetry $\tilde{Q}$

Using the same naive discretization for the derivative,

$$\tilde{Q}\psi_\mu(x) = i\epsilon_{\mu\nu}\Big(\phi(x) - U_\nu(x)\phi(x+\widehat{\nu})U_\nu^\dagger(x)\Big) - i\epsilon_{\mu\nu}\psi_\mu(x)\psi_\mu(x)$$

$$\tilde{Q}U_\mu(x) = i\epsilon_{\mu\nu}\psi_\nu(x)U_\mu(x)$$
$$\tilde{Q}\phi(x) = 0$$
$$\tilde{Q}\bar{\phi}(x) = 2\chi_{12}(x)$$
$$\tilde{Q}B_{12}(x) = \frac{1}{2}[\phi(x), \eta(x)]$$
$$\tilde{Q}\eta(x) = -2B_{12}(x)$$
$$\tilde{Q}\chi_{12}(x) = \frac{1}{2}[\phi(x), \bar{\phi}(x)]$$

$\tilde{Q}$ is nilpotent up to infinitesimal gauge transformations, in fact,

$$\begin{aligned}\tilde{Q}^2U_\mu(x) &= \tilde{Q}(i\epsilon_{\mu\nu}\psi_\nu(x)U_\mu(x)) \\ &= i\epsilon_{\mu\nu}[i\epsilon_{\nu\rho}(\phi(x) - U_\rho(x)\phi(x+\widehat{\rho})U_\rho^\dagger(x)) - i\epsilon_{\nu\rho}\psi_\nu(x)\psi_\nu(x)]U_\mu(x) \\ &\quad - i\epsilon_{\mu\nu}\psi_\nu(x)(i\epsilon_{\mu\rho}\psi_\rho(x)U_\mu(x)) \\ &= (\phi(x)U_\mu(x) - U_\mu(x)\phi(x+\widehat{\mu}))\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}^2\psi_\mu(x) &= \tilde{Q}[i\epsilon_{\mu\nu}(\phi(x) - U_\nu(x)\phi(x+\widehat{\nu})U_\nu^\dagger(x)) - i\epsilon_{\mu\nu}\psi_\mu(x)\psi_\mu(x)] \\ &= [\phi(x), \psi_\mu(x)]\end{aligned}$$

The action can be written as a $\tilde{Q}$-variation of

$$\begin{aligned}S_{SYM}^{N=2} = \tilde{Q}\frac{1}{2g_0^2}\sum_x \mathrm{Tr}\Bigg[&\frac{1}{2}\chi_{12}(x)[\phi(x),\bar{\phi}(x)] - \frac{1}{2}\eta(x)B_{12}(x) + \frac{i}{2}\eta(x)\Phi(x) \\ &+ i\sum_{\mu,\rho}\epsilon_{\mu\rho}\psi_\rho(x)\left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\widehat{\mu})U_\mu^\dagger(x)\right)\Bigg]\end{aligned}$$

and is $\tilde{Q}$-invariant since it is a $\tilde{Q}$-exact form.

Applying the $\tilde{Q}$-variation over the different pieces we get

$$S_{SYM}^{N=2} = \frac{1}{g_0^2}\int \mathrm{Tr}\Big[\frac{1}{4}[\phi(x),\bar{\phi}(x)]^2 + B_{12}^2 - iB_{12}\Phi(x)$$
$$- \frac{1}{4}\eta(x)[\phi(x),\eta(x)] + \chi_{12}(x)[\phi(x),\chi_{12}(x)] - \frac{i}{2}\eta(x)\tilde{Q}\Phi(x)$$
$$+ \sum_{\mu}\left(\phi(x) - U_\nu(x)\phi(x+\hat{\nu})U_\nu(x)^\dagger\right)\left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right)$$
$$- 2i\sum_{\mu,\rho}\epsilon_{\mu\rho}\psi_\rho(x)\left(\chi_{12}(x) - U_\mu(x)\chi_{12}(x+\hat{\mu})U_\mu(x)^\dagger\right)$$
$$- \sum_{\mu,\rho}\psi_\rho(x)\psi_\rho(x)\left(\bar{\phi}(x) + U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right)(1-\delta_{\mu\rho})\Big]$$

which is exactly the lattice $N = 2$ Super Yang-Mills with the change of variables

$$\psi_1(x) \to -\psi_2(x)\,,\, \psi_2(x) \to \psi_1(x)\,,\, \chi_{12}(x) \to \frac{1}{2}\eta(x)\,,\, \frac{1}{2}\eta(x) \to -\chi_{12}(x)$$

which corresponds to a transformation $\Psi \to \sigma_1\sigma_2\Psi$

where the fermionic fields components can be combined in a two-components Dirac spinor as

$$\Psi = -i \begin{pmatrix} \psi_1 + i\psi_2 \\ \chi_{12} + i\frac{\eta}{2} \end{pmatrix},$$

After applying these change of variables we get

$$\lim_{a\to 0} \left[-\frac{i}{2} \mathrm{Tr}(\eta(x)\tilde{Q}\Phi(x)) \right] = 2i\mathrm{Tr}[\chi_{12}(D_1\psi_2(x) - D_2\psi_1(x))],$$

and the action reduces to the continuum $N = 2$ SYM without fine tuning of any parameters of the action.

Supersymmetry Q_μ

We now show the algebra associated with the supercharge Q_μ, which can be naively discretized as,

$$
\begin{aligned}
&Q_\mu U_\nu(x) = i\varepsilon_{\mu\nu}\chi_{12}(x)U_\nu(x) - \frac{i}{2}\delta_{\mu\nu}\eta(x)U_\nu(x)\\
&Q_\mu \eta(x) = -2i\Big(\bar\phi(x) - U_\mu(x)\bar\phi(x+\hat\mu)U^\dagger_\mu(x)\Big) - \frac{1}{2}i\delta_{\mu\nu}\eta^2(x)\\
&Q_\mu \chi_{12}(x) = i\varepsilon_{\mu\nu}\Big(\bar\phi(x) - U_\nu(x)\bar\phi(x+\hat\nu)U^\dagger_\nu(x)\Big) + i\varepsilon_{\mu\nu}\chi^2_{12}(x)\\
&Q_\mu \psi_\nu(x) = \varepsilon_{\mu\nu}B_{12} + \frac{1}{2}\delta_{\mu\nu}[\phi(x),\bar\phi(x)]\\
&Q_\mu B_{12}(x) = [\varepsilon_{\mu\nu}\psi_\nu(x),\bar\phi(x)]\\
&Q_\mu \bar\phi(x) = 0\\
&Q_\mu \phi(x) = 2\psi_\mu(x)\,.
\end{aligned}
$$

The terms $\frac{1}{2}\eta^2$ and χ^2_{12} are $O(a)$ improved respect to the other ones thus, in the continuum limit they disappear and these lattice transformation goes to the continuum one.

One can close the algebra associated with Q_1 and Q_2, separately,

$$
\begin{aligned}
&Q_1 U_1(x) = -\frac{i}{2}\eta(x)U_1(x)\,, \qquad Q_1 U_2(x) = i\chi_{12}(x)U_2(x)\\
&Q_1\eta(x) = -2i\Big(\bar\phi(x) - U_1(x)\bar\phi(x+1)U_1^\dagger(x)\Big) - \frac{1}{2}i\eta^2(x)\\
&Q_1\chi_{12}(x) = i\Big(\bar\phi(x) - U_2(x)\bar\phi(x+2)U_2^\dagger(x)\Big) + i\chi_{12}^2(x)\\
&Q_1\psi_1(x) = \frac{1}{2}[\phi(x),\bar\phi(x)]\,, \qquad Q_1\psi_2(x) = B_{12}(x)\\
&Q_1 B_{12}(x) = [\psi_2(x),\bar\phi(x)]\\
&Q_1\bar\phi(x) = 0\\
&Q_1\phi(x) = 2\psi_1(x)\,,
\end{aligned}
$$

where the following rules for Q_1^2 are satisfied,

$$\begin{aligned} Q_1^2\eta(x) &= Q_1\Big[-2i(\bar{\phi}(x) - (U_1(x)\bar{\phi}(x+1)U_1^\dagger(x)) - \frac{1}{2}i\eta^2(x)\Big] \\ &= [\eta(x), \bar{\phi}(x)] \end{aligned}$$

and

$$\begin{aligned} Q_1^2\chi_{12}(x) &= Q_1\Big[i(\bar{\phi}(x) - (U_2(x)\bar{\phi}(x+2)U_2^\dagger(x)) + i\chi_{12}^2(x)\Big] \\ &= [\chi_{12}(x), \bar{\phi}(x)]\,. \end{aligned}$$

Then we also have,

$$\begin{aligned} Q_1^2U_1(x) &= Q_1(-\frac{1}{2}i\eta(x)U_1(x)) \\ &= -(\bar{\phi}(x)U_1(x) - U_1(x)\bar{\phi}(x+1)) \end{aligned}$$

and similarly,

$$\begin{aligned} Q_1^2U_2(x) &= Q_1(i\chi_{12}(x)U_2(x)) \\ &= -(\bar{\phi}(x)U_2(x) - U_2(x)\bar{\phi}(x+2))\,. \end{aligned}$$

Since Q_1 is nilpotent up to infinitesimal gauge transformations, we can write the action as a Q_1-variation of,

$$S_{SYM}^{N=2} = Q_1 \frac{1}{2g_0^2} \sum_x \mathrm{Tr}\Bigg[\frac{1}{2}\psi_1(x)[\phi(x), \bar{\phi}(x)] + \psi_2(x)B_{12}(x) - i\psi_2(x)\Phi(x) \\ + \frac{i}{2}\eta(x)\Big(\phi(x) - U_1(x)\phi(x+1)U_1^\dagger(x)\Big) \\ - i\chi_{12}(x)\Big(\phi(x) - U_2(x)\phi(x+2)U_2^\dagger(x)\Big)\Bigg].$$

Applying the Q_1-variation over the different fields we obtain

$$
\begin{aligned}
S_{SYM}^{N=2} = \frac{1}{2g_0^2} \sum_x \mathrm{Tr}\Bigg[& \frac{1}{4}[\phi(x), \bar{\phi}(x)]^2 + B_{12}^2 - iB_{12}\Phi(x) + i\psi_2(x)Q_1\Phi(x) \\
& - \psi_1(x)[\psi_1(x), \bar{\phi}(x)] - \psi_2(x)[\psi_2(x), \bar{\phi}(x)] \\
& + \sum_\mu \left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\widehat{\mu})U_\mu^\dagger(x)\right)\left(\phi(x) - U_\mu(x)\phi(x+\widehat{\mu})U_\mu^\dagger(x)\right) \\
& + \frac{1}{4}\eta^2(x)\left(\phi(x) - U_1(x)\phi(x+1)U_1^\dagger(x)\right) \\
& + \chi_{12}^2(x)\left(\phi(x) - U_2(x)\phi(x+2)U_2^\dagger(x)\right) \\
& - i\eta(x)\left(\psi_1(x) - U_1(x)\psi(x+1)U_1^\dagger(x)\right) \\
& + 2i\chi_{12}(x)\left(\psi_1(x) - U_2(x)\psi_1(x+2)U_2^\dagger(x)\right) \\
& + \frac{1}{2}\eta^2(x)U_1(x)\phi(x+1)U_1^\dagger(x) \\
& + 2\chi_{12}(x)U_2(x)\phi(x+2)U_2^\dagger(x)\Bigg].
\end{aligned}
$$

This expression is the lattice $N = 2$ SYM action after a change variables,

$$\psi_1 \to \frac{1}{2}\eta\,, \psi_2 \to -\chi_{12}\,, \chi_{12} \to -\psi_2\,, \frac{1}{2}\eta \to \psi_1\,,$$

that corresponds a transformation $\Psi \to \sigma_2 \Psi$ if simultaneously change

$$\phi \leftrightarrow -\bar{\phi}$$

It reduces to the continuum supersymmetric limit without any fine tuning of the parameters of the action.

Supersymmetry Q_2

$$Q_2U_1(x) = -i\chi_{12}(x)U_1(x)\,, \qquad Q_2U_2(x) = -\frac{i}{2}\eta(x)U_2(x)$$
$$Q_2\eta(x) = -2i\left(\bar\phi(x) - U_2(x)\bar\phi(x+2)U_2^\dagger(x)\right) - \frac{1}{2}i\eta^2(x)$$
$$Q_2\chi_{12}(x) = -i\left(\bar\phi(x) - U_1(x)\bar\phi(x+1)U_1^\dagger(x)\right) - i\chi_{12}^2(x)$$
$$Q_2\psi_1(x) = -B_{12}(x)\,, \qquad Q_2\psi_2(x) = \frac{1}{2}[\phi(x),\bar\phi(x)]$$
$$Q_2B_{12}(x) = -[\psi_1(x),\bar\phi(x)]$$
$$Q_2\bar\phi(x) = 0$$
$$Q_2\phi(x) = 2\psi_2(x)\,,$$

and close the algebra in the following way,

$$
\begin{aligned}
Q_2^2\eta(x) &= Q_2[-2i(\bar\phi(x) - (U_2(x)\bar\phi(x+2)U_2^\dagger(x)) - \frac{1}{2}i\eta^2(x)] \\
&= [\eta(x), \bar\phi(x)]
\end{aligned}
$$

and

$$
\begin{aligned}
Q_2^2\chi_{12}(x) &= Q_2[-i(\bar\phi(x) - (U_1(x)\bar\phi(x+1)U_1^\dagger(x)) - i\chi_{12}^2(x)] \\
&= [\chi_{12}(x), \bar\phi(x)]\,.
\end{aligned}
$$

Then we also have,

$$
\begin{aligned}
Q_2^2 U_1(x) &= Q_2(-i\chi_{12}(x)U_1(x)) \\
&= -(\bar\phi(x)U_1(x) - U_1(x)\bar\phi(x+1))
\end{aligned}
$$

and

$$
\begin{aligned}
Q_2^2 U_2(x) &= Q_2(-\frac{1}{2}i\eta(x)U_2(x)) \\
&= -(\bar\phi(x)U_2(x) - U_2(x)\bar\phi(x+2))\,.
\end{aligned}
$$

The action can be written as an exact Q_2-variation of

$$S_{SYM}^{N=2} = Q_2 \frac{1}{2g_0^2} \sum_x \mathrm{Tr}\left[\frac{1}{2}\psi_2(x)[\phi(x), \bar{\phi}(x)] - \psi_1(x) B_{12}(x) + i\psi_1(x)\Phi(x) \right.$$
$$+ \frac{i}{2}\eta(x)\Big(\phi(x) - U_2(x)\phi(x+2)U_2^\dagger(x)\Big)$$
$$\left. + i\chi_{12}(x)\Big(\phi(x) - U_1(x)\phi(x+1)U_1^\dagger(x)\Big)\right].$$

and applying the transformations rule we have,

$$
\begin{aligned}
S_{SYM}^{N=2} = \frac{1}{2g_0^2}\sum_x \mathrm{Tr}\Bigg[&\frac{1}{4}[\phi(x),\bar{\phi}(x)]^2 + B_{12}^2 - iB_{12}\Phi(x) - i\psi_1(x)Q_2\Phi(x) \\
&- \psi_2(x)[\psi_2(x),\bar{\phi}(x)] - \psi_1(x)[\psi_1(x),\bar{\phi}(x)] \\
&+ \sum_\mu \left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\widehat{\mu})U_\mu^\dagger(x)\right)\left(\phi(x) - U_\mu(x)\phi(x+\widehat{\mu})U_\mu^\dagger(x)\right) \\
&+ \frac{1}{4}\eta^2(x)\left(\phi(x) - U_2(x)\phi(x+2)U_2^\dagger(x)\right) \\
&+ \chi_{12}^2(x)\left(\phi(x) - U_1(x)\phi(x+1)U_1^\dagger(x)\right) \\
&- i\eta(x)\left(\psi_2(x) - U_2(x)\psi_2(x+2)U_2^\dagger(x)\right) \\
&- 2i\chi_{12}(x)\left(\psi_2(x) - U_1(x)\psi_2(x+1)U_1^\dagger(x)\right) \\
&+ \frac{1}{2}\eta^2(x)U_2(x)\phi_2(x+2)U_2^\dagger(x) \\
&+ 2\chi_{12}^2(x)U_1(x)\phi_1(x+1)U_1^\dagger(x)\Bigg].
\end{aligned}
$$

This expression is again the lattice $N = 2$ super Yang-Mills action with the change of variables,

$$\psi_1 \to \chi_{12}\,, \psi_2 \to \frac{1}{2}\eta\,, \chi_{12} \to \psi_1\,, \frac{1}{2}\eta \to \psi_2$$

and simultaneously,

$$\phi \leftrightarrow -\bar{\phi}\,,$$

that corresponds to a transformation, $\Psi \to \sigma_1 \Psi$,

and reduces to the continuum supersymmetric action without any fine tuning of the parameters of the action.

Lattice Action as a $Q\tilde{Q}$-form

A natural question that can be analyzed is whether more than one supercharge can be preserved exactly on the lattice using this formulation.

It is possible to write the $N=2$ Super Yang-Mills action as a product of two supercharges Q and $\tilde{Q}$, which are separately exact on the lattice,

$$S_{SYM}^{N=2} = Q\tilde{Q}\frac{1}{2g_0^2}\sum_x \mathrm{Tr}\Big[-\frac{1}{2}\eta(x)\chi_{12}(x) - \frac{i}{2}\bar{\phi}(x)\Phi(x)\Big] .$$

Applying $\tilde{Q}$ we get,

$$= Q\frac{1}{2g_0^2}\sum_x \mathrm{Tr}\Big[B_{12}(x)\chi_{12}(x) + \frac{1}{4}\eta(x)[\phi(x),\bar{\phi}(x)] - i\chi_{12}(x)\Phi(x) - \frac{i}{2}\bar{\phi}(x)\tilde{Q}\Phi(x)\Big] .$$

The first three pieces are OK, while the last term should be investigated more carefully and gives,

$$
\sum_x \mathrm{Tr}\Big[-\frac{i}{2}\bar{\phi}(x)\tilde{Q}\Phi(x)\Big] =
$$

$$
-\frac{1}{2}\sum_x \mathrm{Tr}\bar{\phi}(x)\tilde{Q}\Big[U_1(x)U_2(x+1)U_1^\dagger(x+2)U_2^\dagger(x)
$$

$$
- U_2(x)U_1(x+2)U_2^\dagger(x+1)U_1^\dagger(x)\Big]
$$

$$
\equiv -\frac{i}{2}\sum_x \mathrm{Tr}\bar{\phi}(x)F_1(x)\,.
$$

Now applying Q we have,

$$
\sum_x \mathrm{Tr}\Big[-\frac{i}{2}\bar{\phi}(x)F_1(x)\Big] = -\frac{i}{2}\sum_x \mathrm{Tr}\Big[\eta(x)F_1(x) + \bar{\phi}(x)QF_1(x)\Big]\,.
$$

Let us investigate its continuum limit:

In the limit $a \to 0$ the first piece gives,

$$-\frac{i}{2}\sum_x \mathrm{Tr}\Big[\eta(x)F_1(x)\Big] \approx_{a\to 0} i\sum_x \mathrm{Tr}\Big[\eta(x)D_\mu\psi_\mu(x)\Big]$$

which is order $O(1)$. Integrating by part we obtain

$$i\psi_\mu(x)D_\mu\eta(x)\,.$$

While the second piece

$$-\frac{i}{2}\sum_x \mathrm{Tr}\bar{\phi}(x)QF_1(x) \approx_{a\to 0}\,,$$

gives two pieces:

$$\sum_x \mathrm{Tr}\psi_\mu(x)[\bar{\phi}(x),\psi_\mu(x)]$$

and

$$a^2\sum_x \mathrm{Tr}(\partial_\mu\bar{\phi}(x)+i[A_\mu(x),\phi(x)])(\partial_\mu\bar{\phi}(x)+i[A_\mu(x),\phi(x)]) = a^2\sum_x \mathrm{Tr}D_\mu\bar{\phi}(x)D_\mu\phi(x)\,.$$

Collecting all terms we obtain the classical continuum action without fine tuning even if on the lattice we do not satisfy the condition,

$$\left\{Q, \tilde{Q}\right\} = 0\,.$$

(two dimensions?)

Outlook

We start with a lattice formulation of the two dimensional $N = 2$ Super Yang-Mills theory proposed by Sugino where the gauge fields are represented by unitary link variables and use a naive discretization of fermions.

This formulation preserve exactly a single supercharge at finite lattice spacing and the action as a Q-exact form. In the continuum limit this lattice supersymmetry is enough to guarantee continuum supersymmetry without fine tuning of any parameters of the action.

We then show that it is possible to construct other three supercharges that are nilpotent up to infinitesimal gauge transformations and we write a lattice action as an exact $(\tilde{Q}, Q_1, Q_2)$-form.

At finite lattice spacing they define four different lattice models and in each model only one supersymmetry is realized. In the continuum limit they all flow to the same continuum supersymmetric theory without any fine tuning of the parameters of the action.

As an application of this procedure we write the lattice action as a $Q\tilde{Q}$-form (where Q and $\tilde{Q}$ are separately nilpotent) and we show that the continuum limit is realized without any fine tuning.

Future work

Write the action as a $Q\tilde{Q}Q_1Q_2$-form.

Use another lattice formulation for the fermion part of the action.

Extend this formulation for 4 dimensions

Our final goal is to write an exact lattice action for the 4 dimensional $N = 1$ Super Yang-Mills theory.