

The phase structure of QCD from lattice simulations

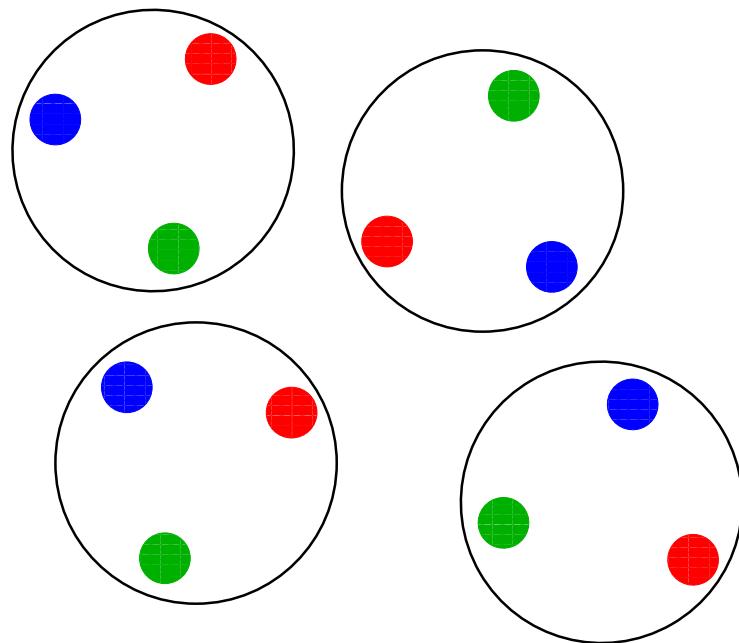
Philippe de Forcrand
ETH Zürich and CERN



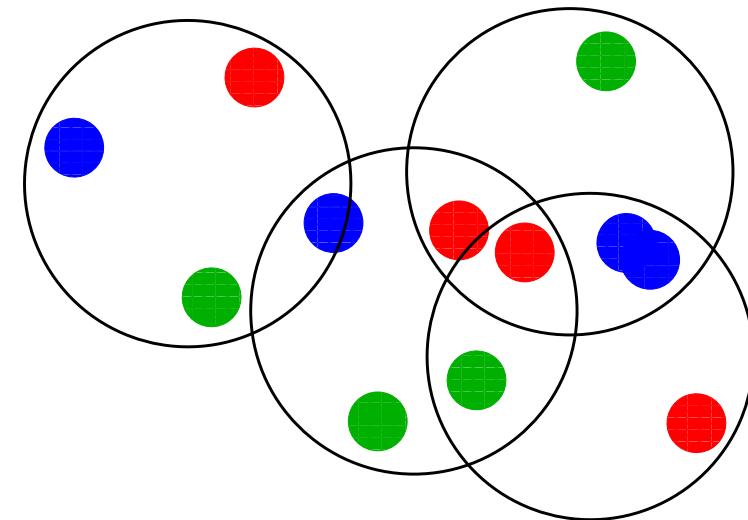
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Matter under extreme conditions: confinement and χ SB

- **Confinement** loses meaning when hadrons **overlap**



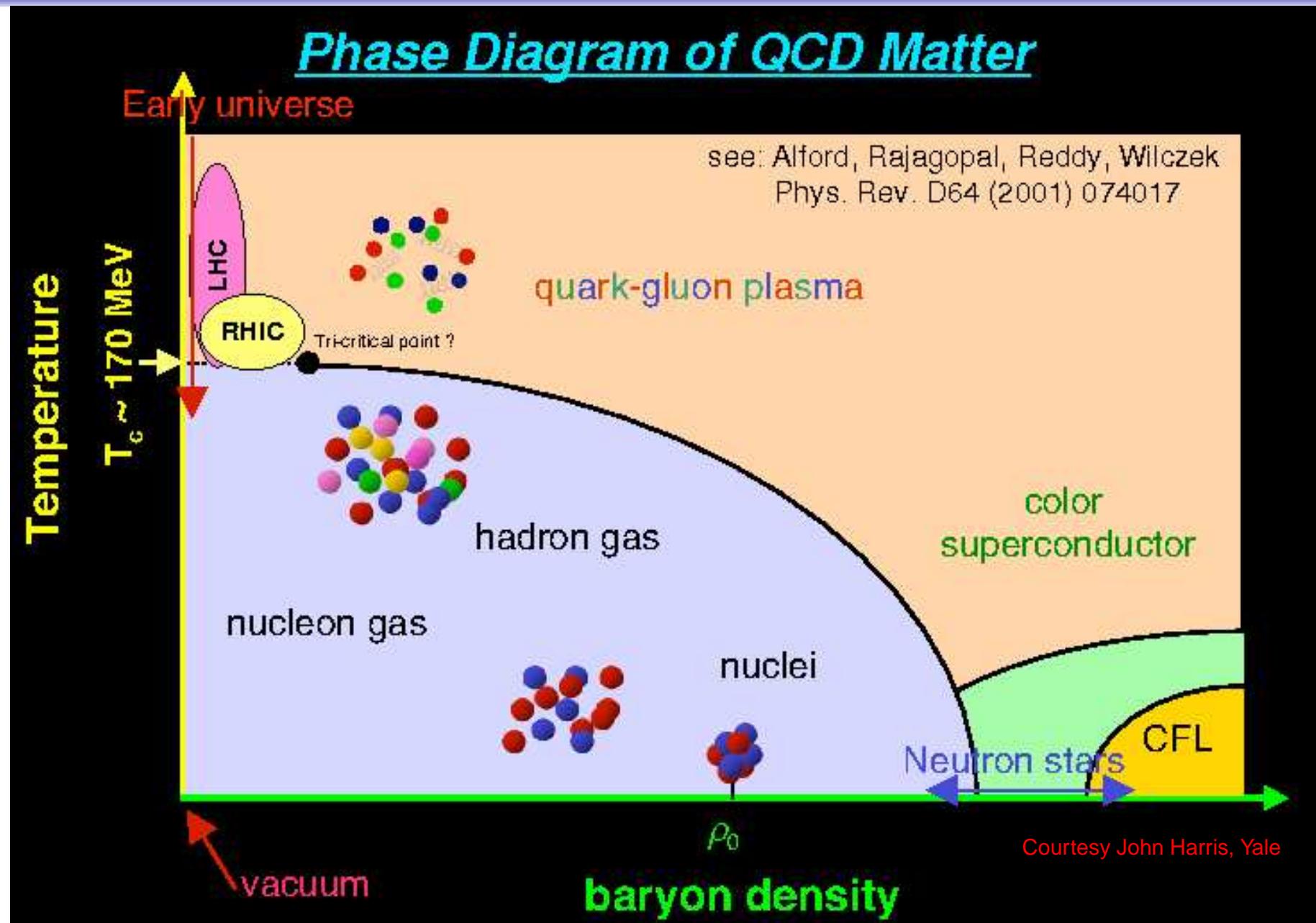
low density



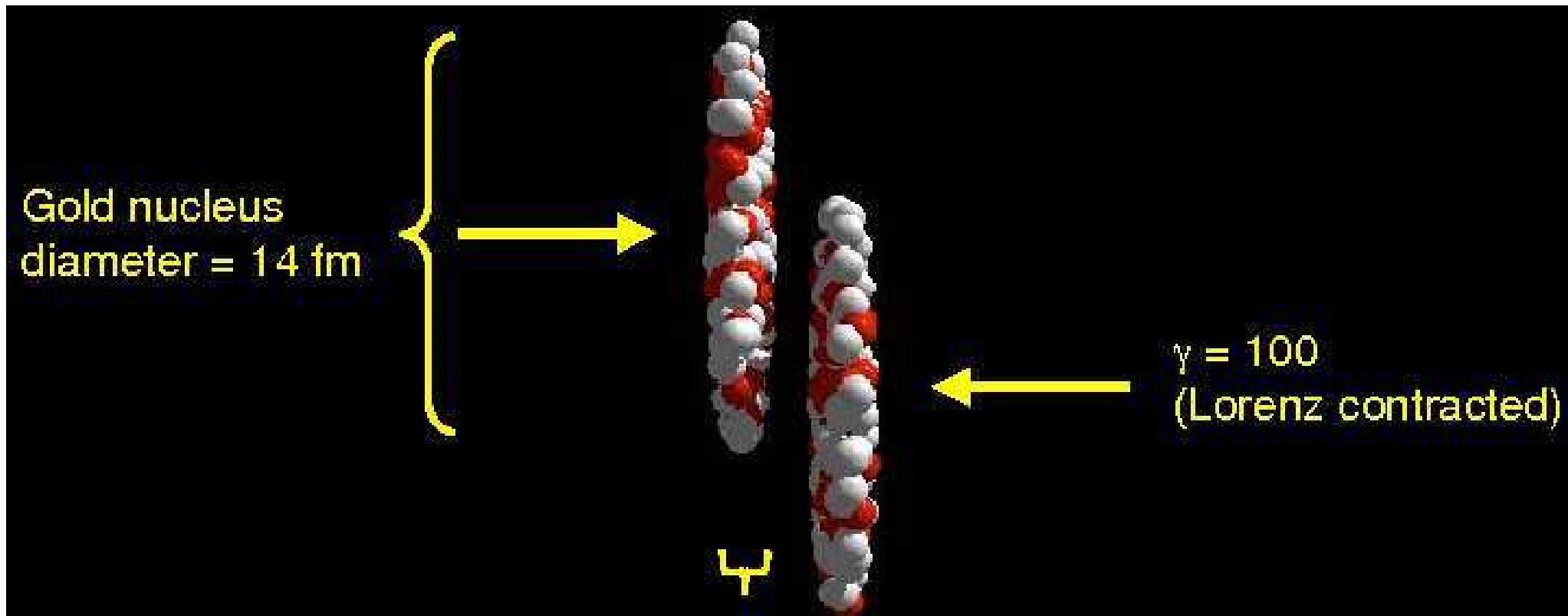
high density

- Same at high temperature (overlapping with pions)
- **Bag** picture: percolation → **chiral symmetry restoration**

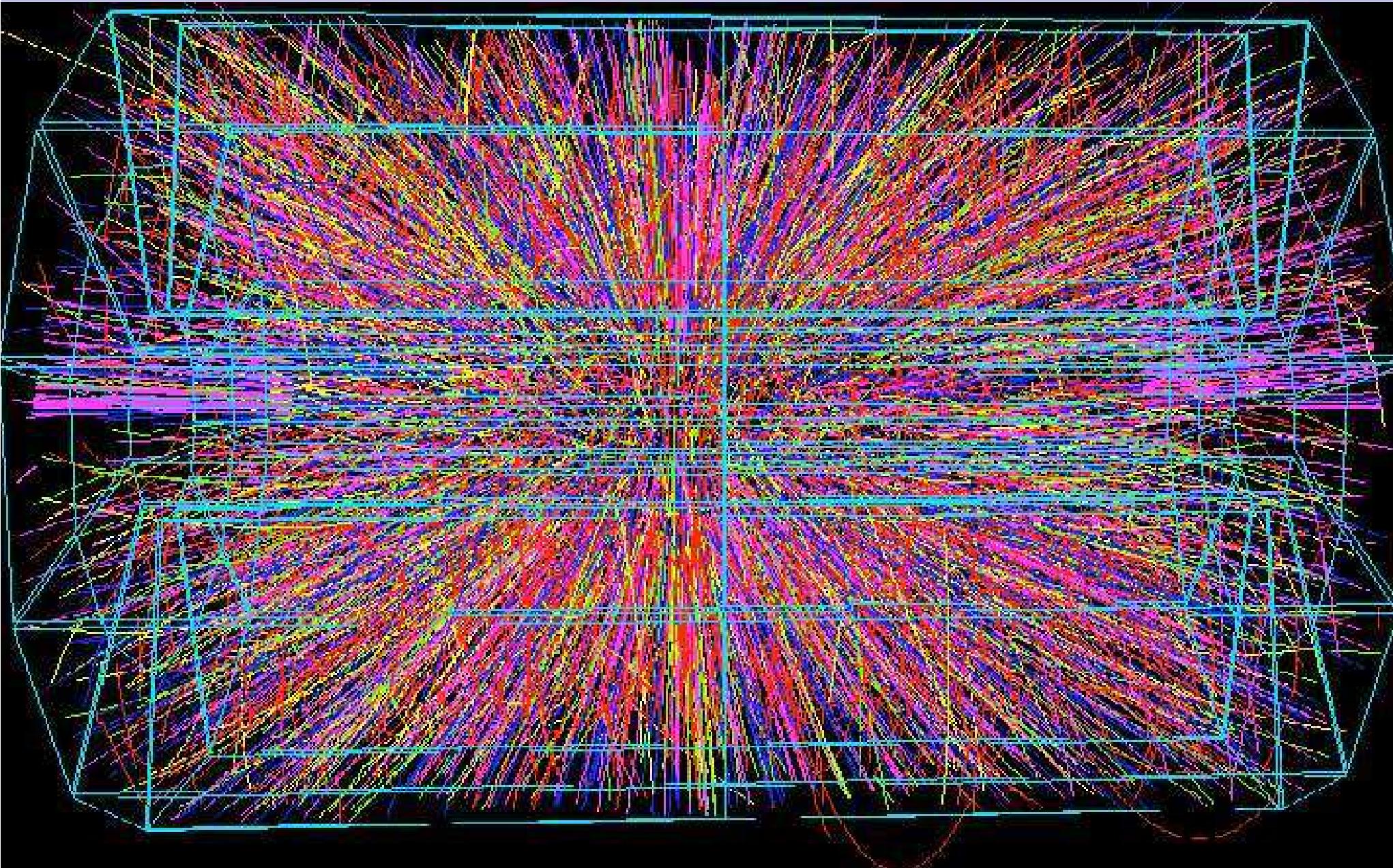
QCD Phase diagram



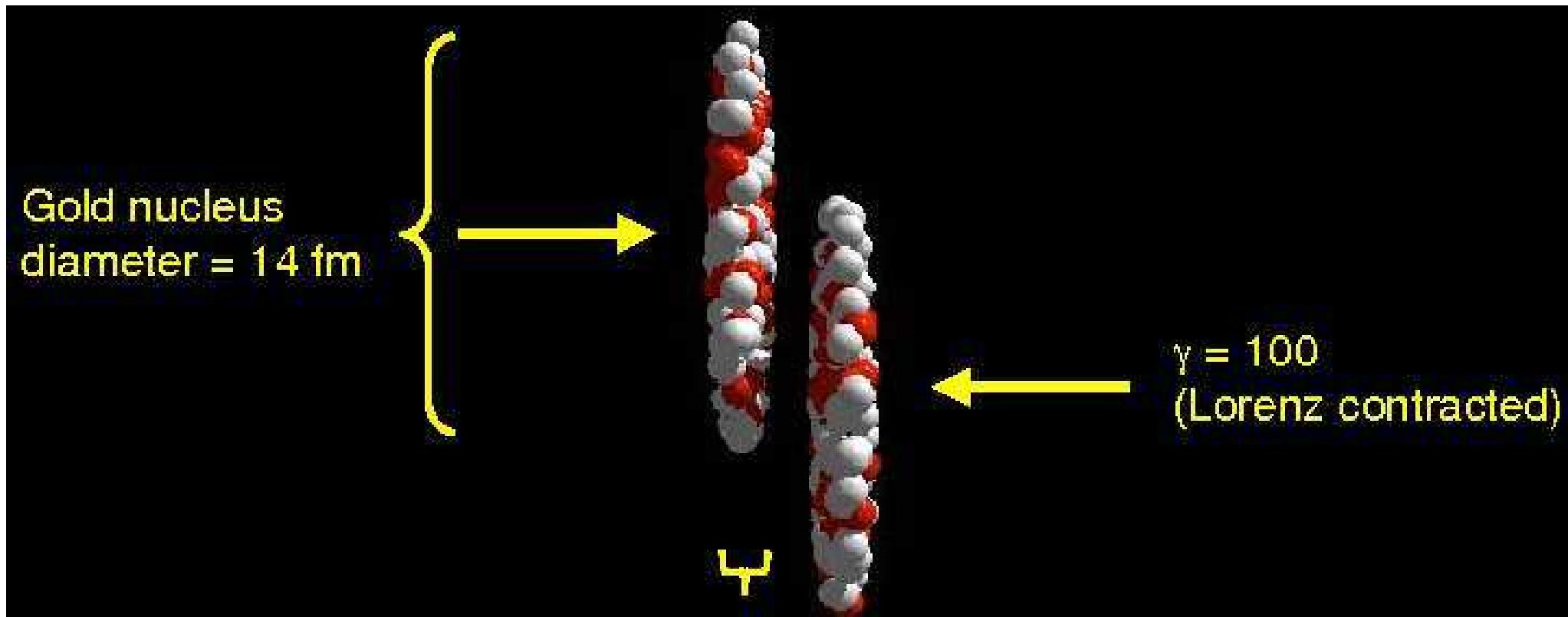
Heavy-ion collisions



Heavy-ion collisions



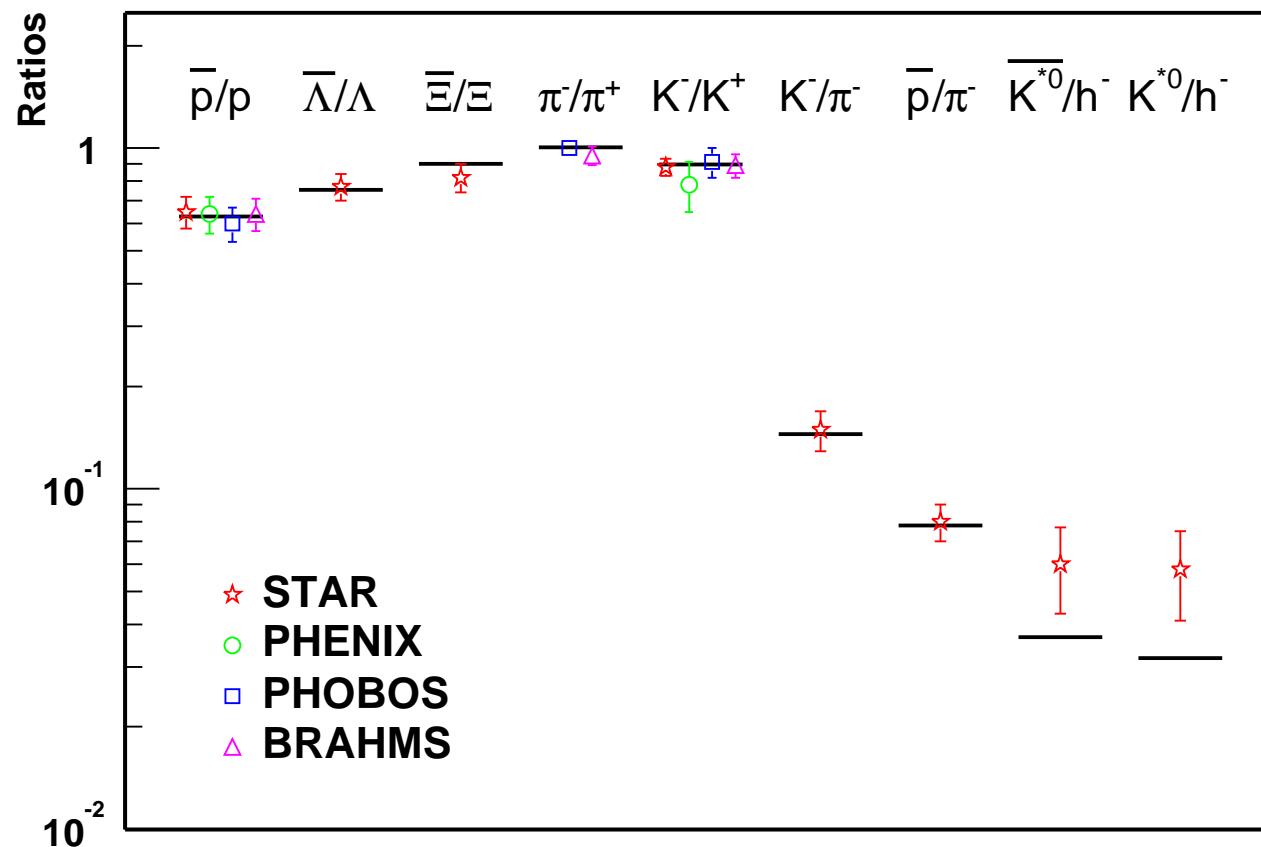
Heavy-ion collisions



- does not behave like superposition of $N - N$ collisions
- well described by relativistic hydrodynamic fluid

“Strongly Interacting Quark-Gluon Plasma” found

Phase boundary versus freeze-out temperature?

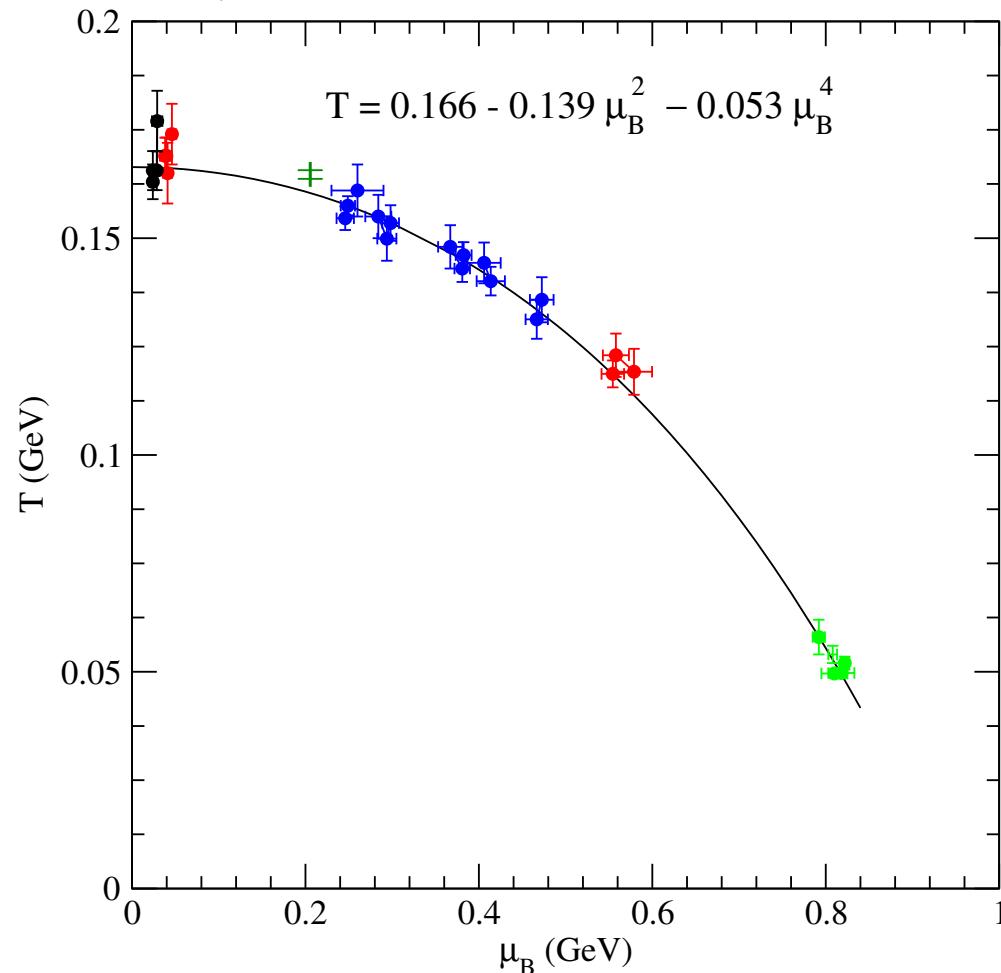


At fixed \sqrt{s} , relative abundances fitted well with Boltzmann (T, μ_B)

Phase boundary versus freeze-out temperature?

Repeat for successive \sqrt{s} :

eg. J. Cleymans et al., hep-ph/0607164



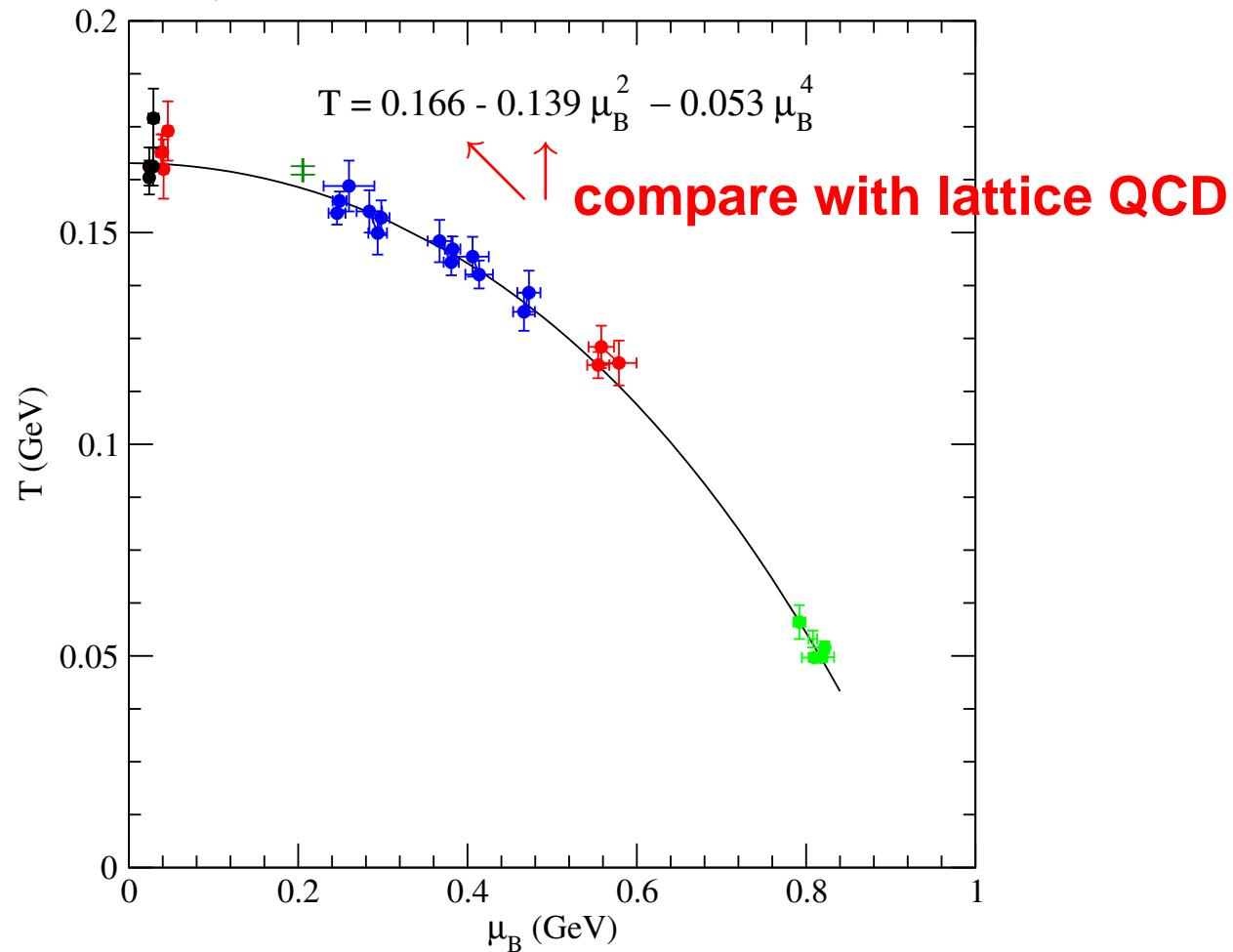
$T(\text{freeze-out}) \leq T_c$ but very close

Braun-Munzinger, Stachel & Wetterich, nucl-th/0311005

Phase boundary versus freeze-out temperature?

Repeat for successive \sqrt{s} :

eg. J. Cleymans et al., hep-ph/0607164



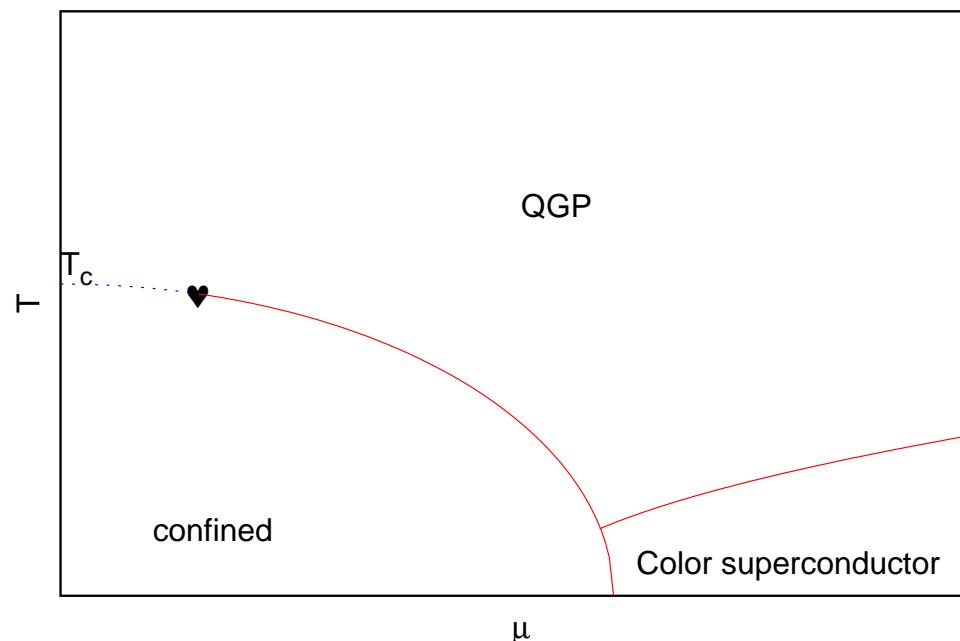
$T(\text{freeze-out}) \leq T_c$ but very close

Braun-Munzinger, Stachel & Wetterich, nucl-th/0311005

Scope of lattice QCD simulations

What can lattice QCD say about:

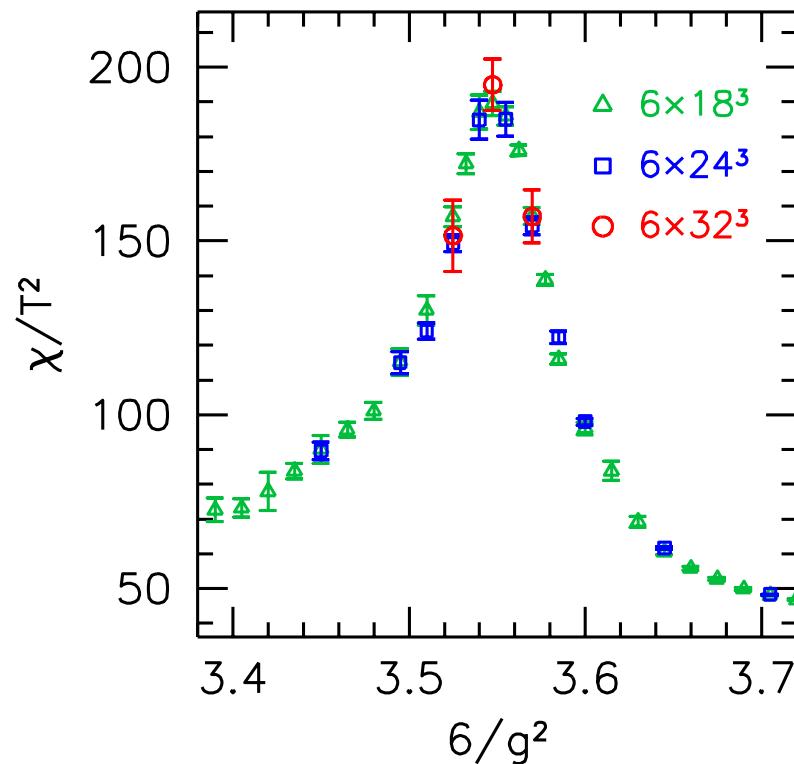
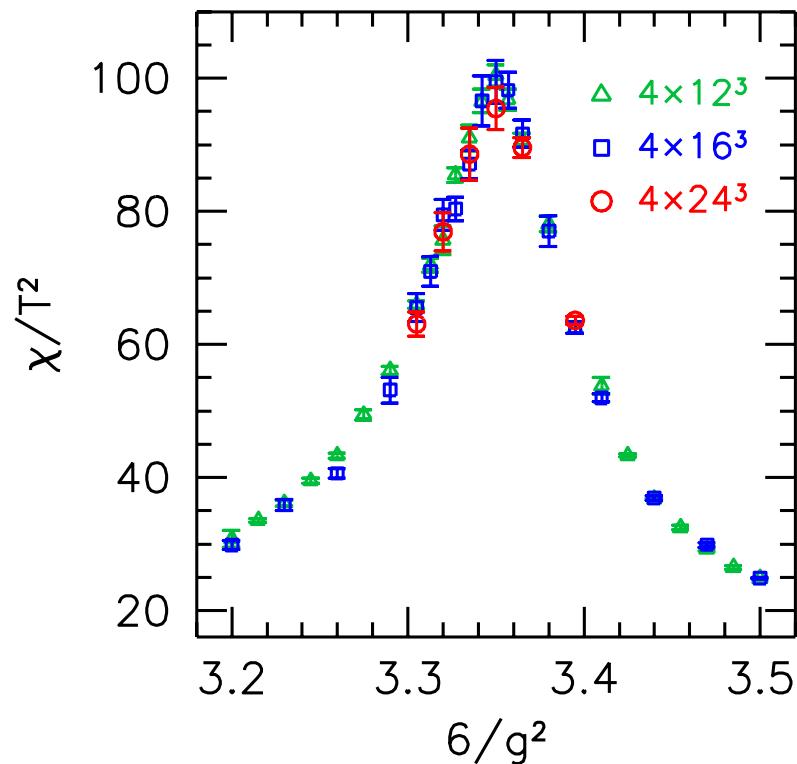
1. The $\mu = 0$ finite-temperature transition/crossover ?
2. The “phase” boundary $T_c(\mu)$?
3. The QCD critical point ?



1. The $\mu = 0$ finite-temperature transition/crossover

The **ultimate**: Fodor et al. (hep-lat/0611014 → Nature; hep-lat/0609068)
physical quark masses, **4** lattice spacings (but staggered fermions)

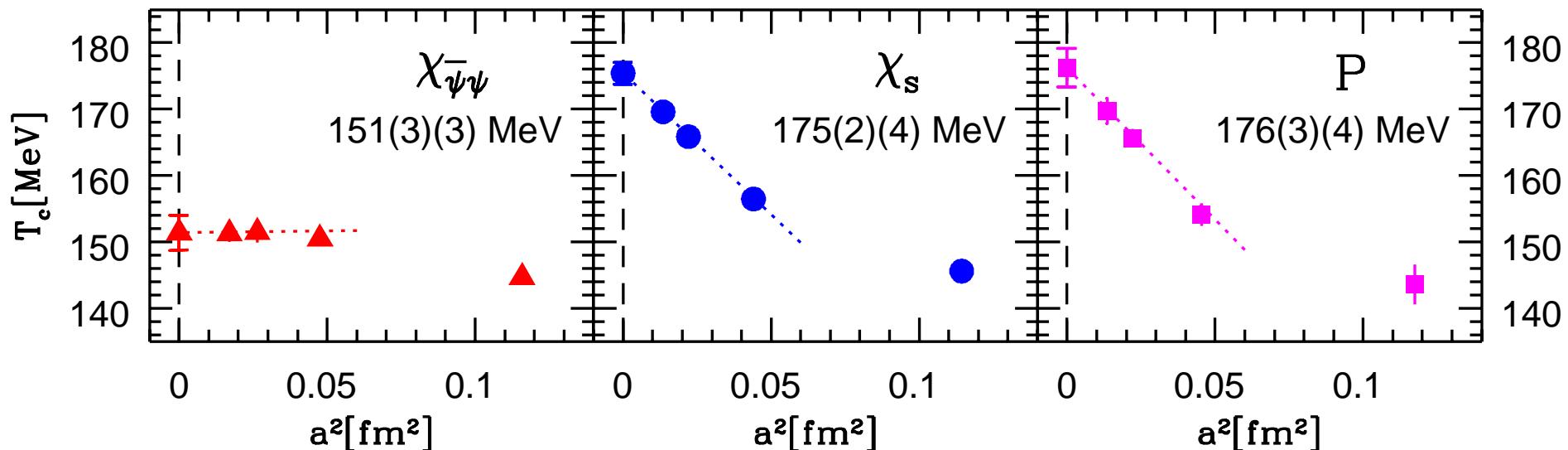
- No phase transition: **crossover**



1. The $\mu = 0$ finite-temperature transition/crossover

The **ultimate**: Fodor et al. (hep-lat/0611014 → Nature; hep-lat/0609068)
physical quark masses, **4** lattice spacings (but staggered fermions)

- “ T_c ” depends **a lot** on the observable

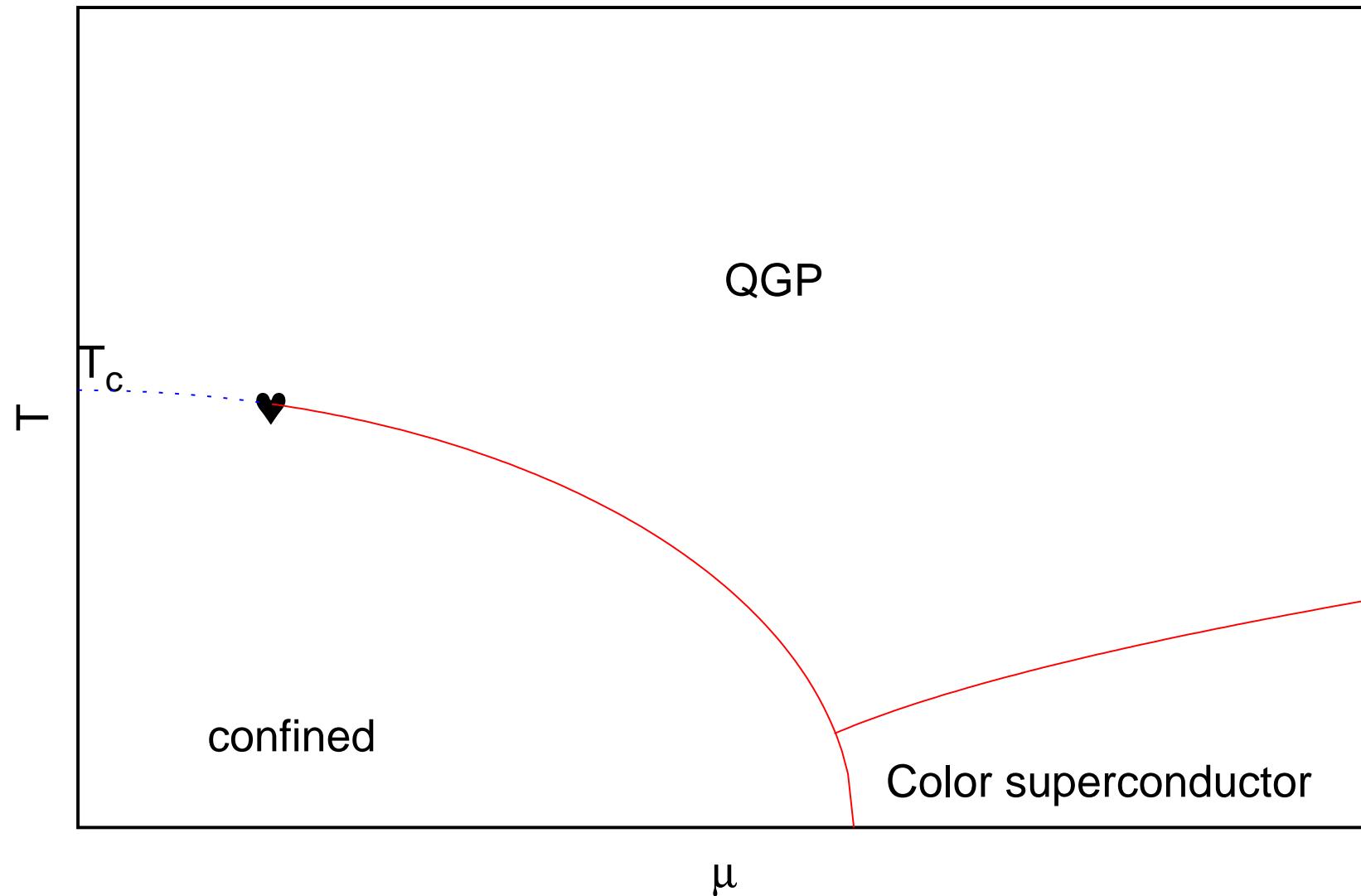


But: - “ $T_c(\bar{\psi}\psi)$ ” $<$ $T_{\text{freeze-out}} \approx 166$ MeV ?

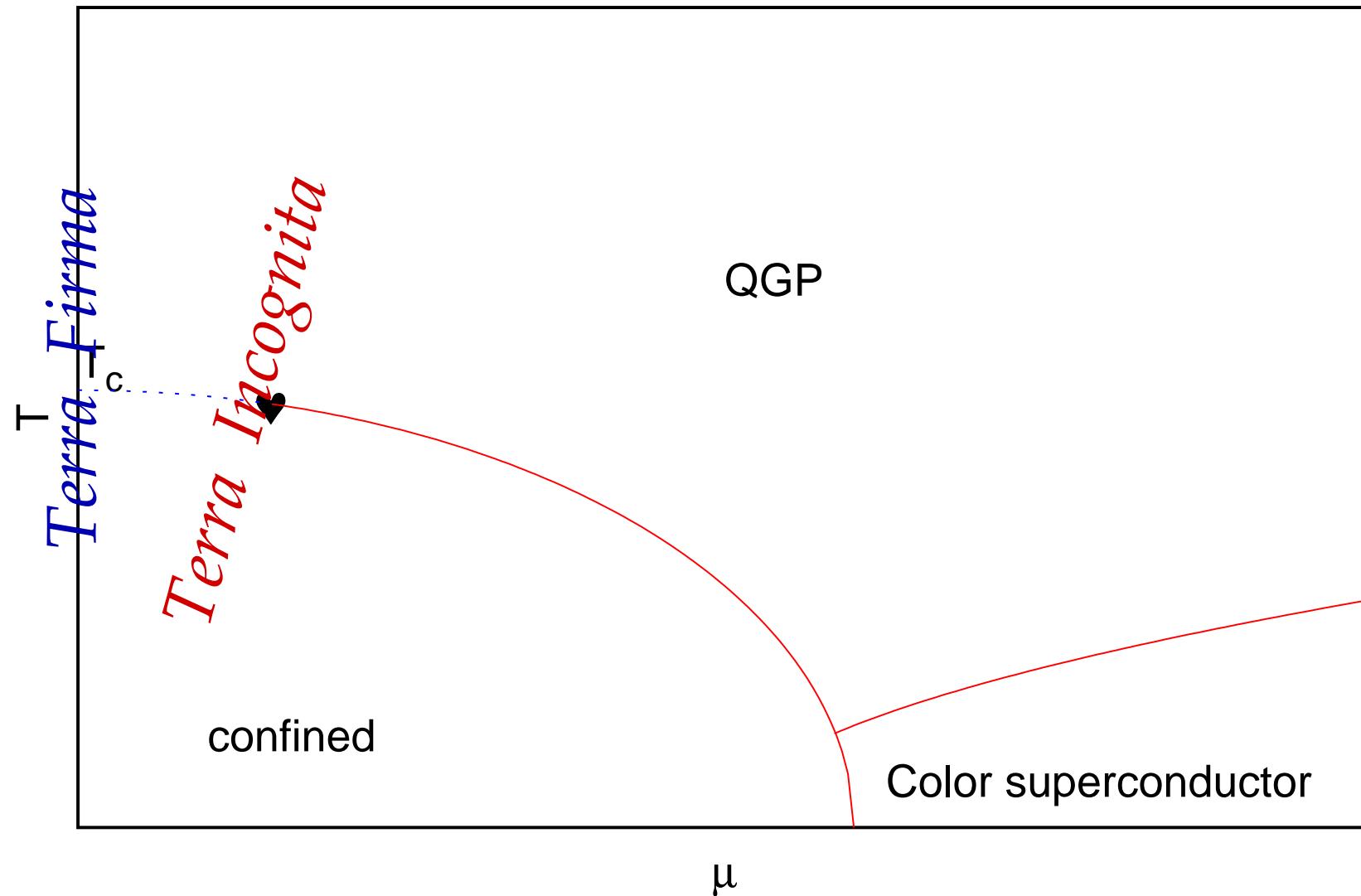
- $T_c = 192(7)(4)$ MeV (Karsch et al.), from $N_t = 4$ and 6

The dust should settle soon.. ($N_t = 8$, two actions from HotQCD)

2. The “phase” boundary $T_c(\mu)$



Phase diagram: to be checked by lattice QCD simulations

2. The “phase” boundary $T_c(\mu)$ 

Phase diagram: to be checked by lattice QCD simulations

The difficulty: “sign” problem

- quarks anti-commute → integrate analytically: $\det(\not{D}(U) + m + \mu \gamma_0)$

$$\gamma_5(i\not{p} + m + \mu \gamma_0)\gamma_5 = (-i\not{p} + m - \mu \gamma_0) = (i\not{p} + m - \mu^* \gamma_0)^\dagger$$

$$\det \not{D}(\mu) = \det^* \not{D}(-\mu^*)$$

\det complex unless $\mu = 0$ (or $i\mu_I$)

The difficulty: “sign” problem

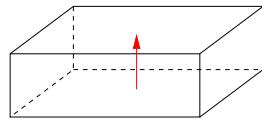
- quarks anti-commute → integrate analytically: $\det(\not{D}(U) + m + \mu \gamma_0)$

$$\gamma_5(i\not{p} + m + \mu \gamma_0)\gamma_5 = (-i\not{p} + m - \mu \gamma_0) = (i\not{p} + m - \mu^* \gamma_0)^\dagger$$

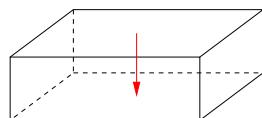
$$\det \not{D}(\mu) = \det^* \not{D}(-\mu^*)$$

\det complex unless $\mu = 0$ (or $i\mu_I$)

- Corollary: measure $\bar{\omega}$ must be complex when $\mu \neq 0$



$$\langle \text{Tr Polyakov} \rangle = \exp(-\frac{1}{T} F_q) = \langle \text{Re Pol} \times \text{Re} \bar{\omega} - \text{Im Pol} \times \text{Im} \bar{\omega} \rangle$$



$$\langle \text{Tr Polyakov}^\dagger \rangle = \exp(-\frac{1}{T} F_{\bar{q}}) = \langle \text{Re Pol} \times \text{Re} \bar{\omega} + \text{Im Pol} \times \text{Im} \bar{\omega} \rangle$$

$$F_q \neq F_{\bar{q}} \Rightarrow \text{Im} \bar{\omega} \neq 0$$

The difficulty: “sign” problem

- quarks anti-commute → integrate analytically: $\det(\not{D}(U) + m + \mu \gamma_0)$
 $\gamma_5(i\not{\phi} + m + \mu \gamma_0)\gamma_5 = (-i\not{\phi} + m - \mu \gamma_0) = (i\not{\phi} + m - \mu^* \gamma_0)^\dagger$

$$\det \not{D}(\mu) = \det^* \not{D}(-\mu^*)$$

\det complex unless $\mu = 0$ (or $i\mu_I$)

- Need auxiliary partition function for Monte Carlo sampling

$$Z(\mu) = \int \mathcal{D} U e^{-S_g} \det \not{D}(\mu) \rightarrow \text{no Monte Carlo}$$

$$Z_{MC} = \int \mathcal{D} U e^{-S_g} |\det \not{D}(\mu)| \quad (\text{or } \det(\mu = 0) \text{ or ...})$$

$$Z(\mu)/Z_{MC} = \langle \frac{\det \not{D}(\mu)}{|\det \not{D}(\mu)|} \rangle = \langle e^{i\phi} \rangle, \text{ average “sign”}$$

$$\langle e^{i\phi} \rangle \sim \exp(-V \frac{\delta F(\mu)}{T}), \text{ but each measurement } o(1)$$

⇒ Need statistics $\propto \exp(+V)$ for constant accuracy

Numerical approaches: I. Conservative

I. Conservative: evaluate **coefficients of Taylor series about $\mu = 0$**

No sign problem \implies can control thermodynamic/continuum limits

Numerical approaches: I. Conservative

I. Conservative: evaluate **coefficients of Taylor series about $\mu = 0$**

No sign problem \implies can control thermodynamic/continuum limits

- Simple-minded: simulate at $\mu = 0$, measure **susceptibilities**

MILC, ..., TARO, Bielefeld-Swansea, Gavai & Gupta

ie. Taylor expand $\frac{P(\mu) - P(\mu=0)}{T^4} = \sum_{n=1} c_{2n}(T) \left(\frac{\mu}{\pi T}\right)^{2n}$, measure $c_{2n}(T)$

A few Taylor coeffs (max. 4); expect convergence for $|\frac{\mu}{\pi T}| \lesssim 1$

Numerical approaches: I. Conservative

I. Conservative: evaluate **coefficients of Taylor series about $\mu = 0$**

No sign problem \implies can control thermodynamic/continuum limits

- Simple-minded: simulate at $\mu = 0$, measure **susceptibilities**

MILC, ..., TARO, Bielefeld-Swansea, Gavai & Gupta
 ie. Taylor expand $\frac{P(\mu) - P(\mu=0)}{T^4} = \sum_{n=1} c_{2n}(T) \left(\frac{\mu}{\pi T}\right)^{2n}$, measure $c_{2n}(T)$
 A few Taylor coeffs (max. 4); expect convergence for $|\frac{\mu}{\pi T}| \lesssim 1$

- [Much] better: simulate at $\mu = i\mu_I$ **imaginary**

PdF & Philipsen, D'Elia & Lombardo, Chen & Luo, Azcoiti et al.,..

- **two** control parameters: β and μ_I
- fit with truncated Taylor expansion, then analytically continue $\mu_I^2 \rightarrow -\mu^2$
- systematics: can check significance of higher-order terms
- limited by **singularity** at $\mu_I = \frac{\pi}{3} T$

$$\frac{T_c(\mu)}{T_c(\mu=0)} = 1 - c(N_f, m_q) \left(\frac{\mu}{\pi T}\right)^2 + \dots$$

$$c \approx 0.500(34), 0.602(9), 0.93(10) \text{ for } N_f = 2, 3, 4, \frac{m_q}{T} \ll 1 \quad (c \propto \frac{N_f}{N_c})$$

Numerical approaches: II. Adventurous

II. Adventurous: evaluate **full result at finite μ**

Sign problem \implies small, coarse lattices \rightarrow crosscheck essential

Numerical approaches: II. Adventurous

II. Adventurous: evaluate **full result at finite μ**

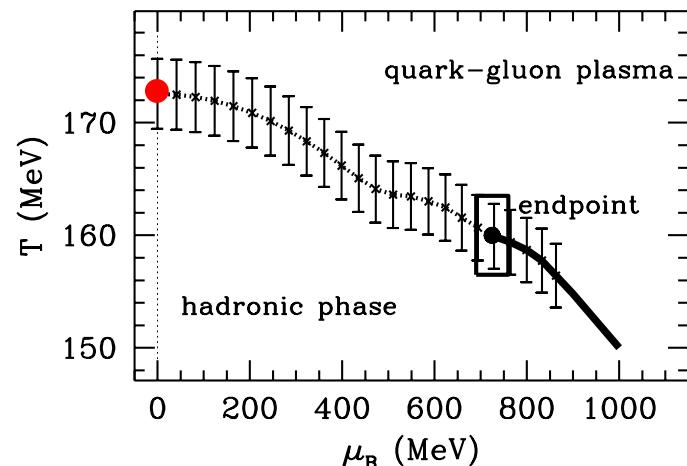
Sign problem \implies small, coarse lattices \rightarrow crosscheck essential

- Double reweighting in (μ, β) from $(\mu = 0, \beta_c)$

Fodor & Katz

$$Z(\mu, \beta) = \langle \frac{\exp(-\beta S_g) \det M(\mu)}{\exp(-\beta_c S_g) \det M(\mu=0)} \rangle Z_{MC}(\mu=0, \beta_c)$$

Sign problem? Overlap problem?



Numerical approaches: II. Adventurous

II. Adventurous: evaluate **full result at finite μ**

Sign problem \implies small, coarse lattices \rightarrow crosscheck essential

- Double reweighting in (μ, β) from $(\mu = 0, \beta_c)$

Fodor & Katz

$$Z(\mu, \beta) = \langle \frac{\exp(-\beta S_g) \det M(\mu)}{\exp(-\beta_c S_g) \det M(\mu=0)} \rangle Z_{MC}(\mu=0, \beta_c)$$

Sign problem? Overlap problem?

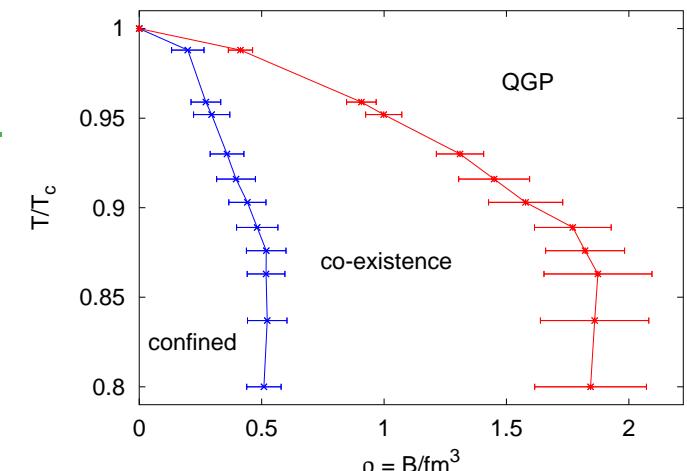
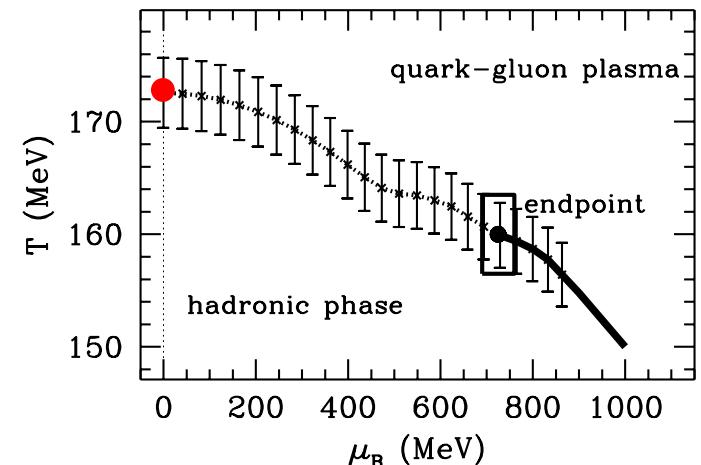
- Canonical ensemble:

$$Z_C(B) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\left(\frac{\mu_I}{T}\right) e^{-i3B\frac{\mu_I}{T}} Z_{GC}(\mu = i\mu_I)$$

Hasenfratz & Toussaint; Alford & Wilczek; PdF & Krajcovich; Alexandru et al.

Sign problem *pushed to observable* $\langle Z_C(B)/Z_{GC} \rangle$

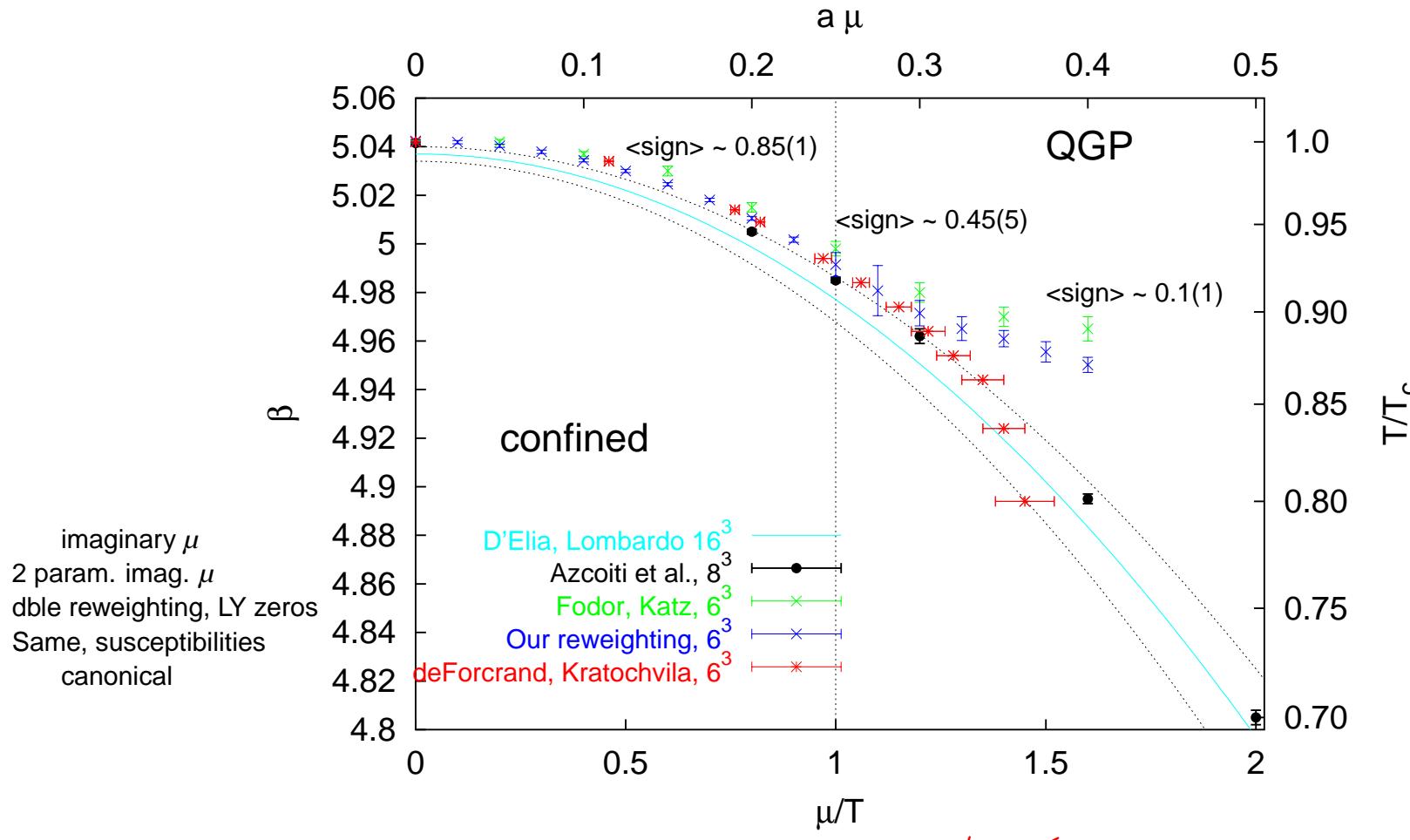
Coexistence region: cf. nuclear density $0.17/\text{fm}^3$



Phase Diagram $T - \mu$: comparing apples with apples

All with $N_f = 4$ staggered fermions, $am_q = 0.05, N_t = 4$ ($a \sim 0.3$ fm)

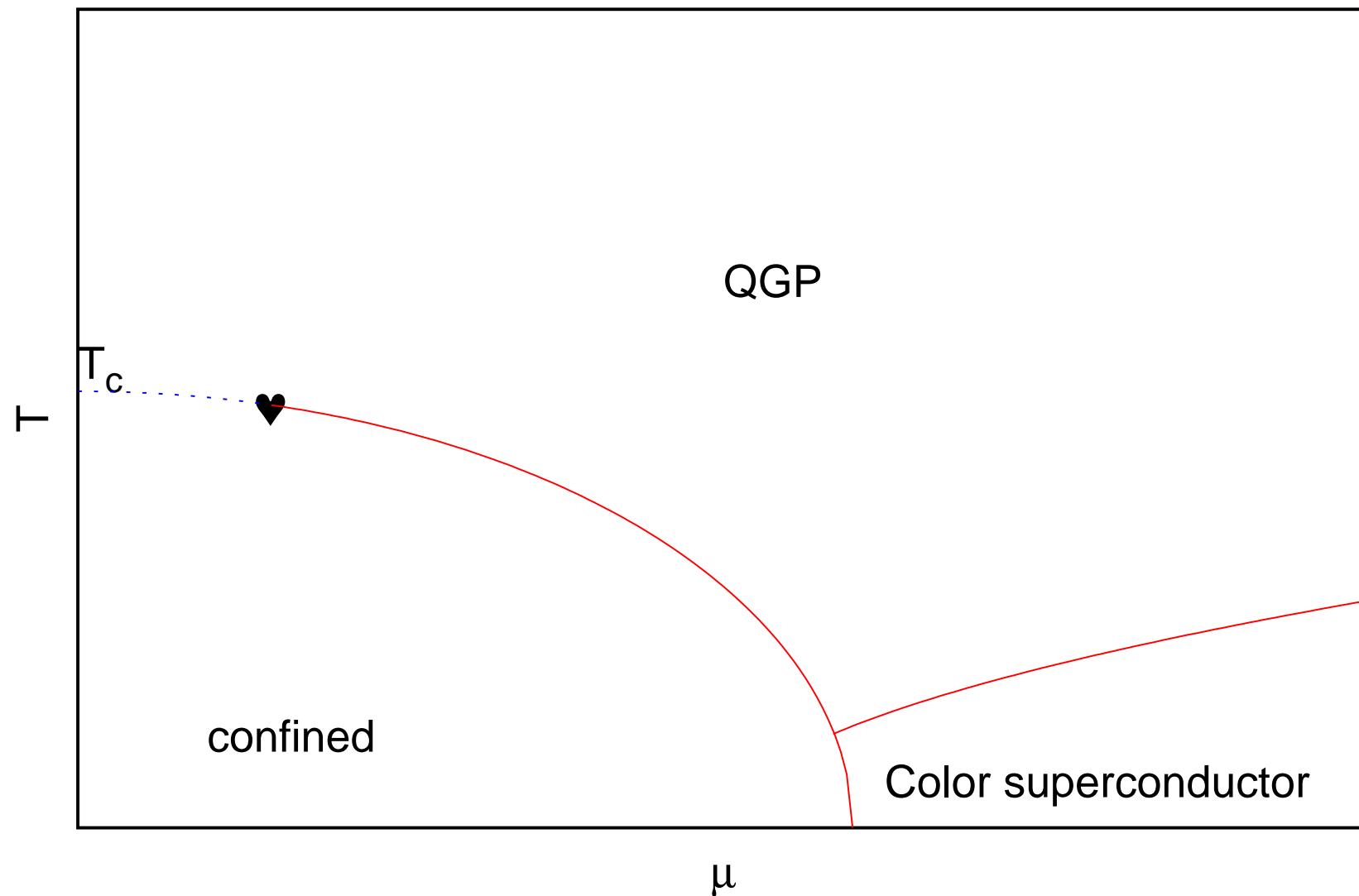
PdF & Kratochvila



Summary for phase boundary

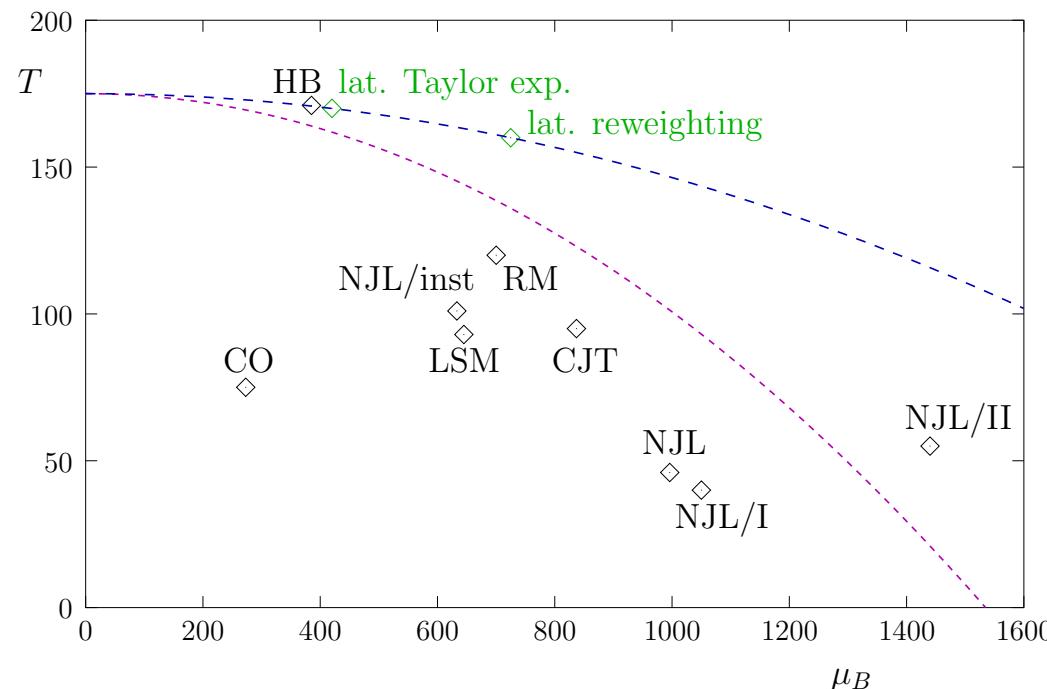
- Under control for $\mu/T \lesssim 1$
- Well described by parabola
- Curvature **about 1/3** freeze-out parabola (using pert. scaling)
- **Can study continuum limit for physical masses** (\sim susceptibility)

3. The QCD critical point



Can one locate the **critical point** (μ_E, T_E) ?

Locating the critical point

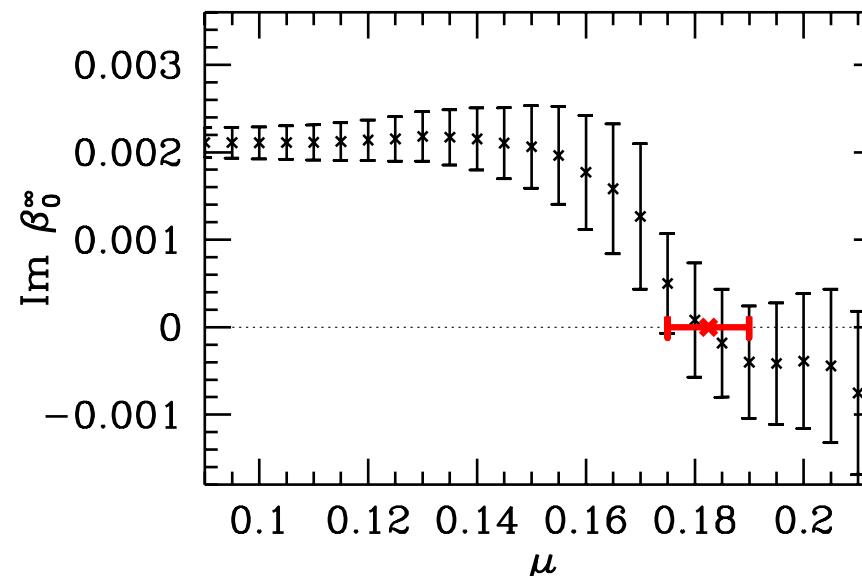


M. Stephanov, hep-ph/0402115

- Much harder task:
detect divergence of correlation length on small lattice (??)

Already determined, but...

- Fodor & Katz: $(T_E, \mu_E) = (162(2), 120(13))$ MeV ?



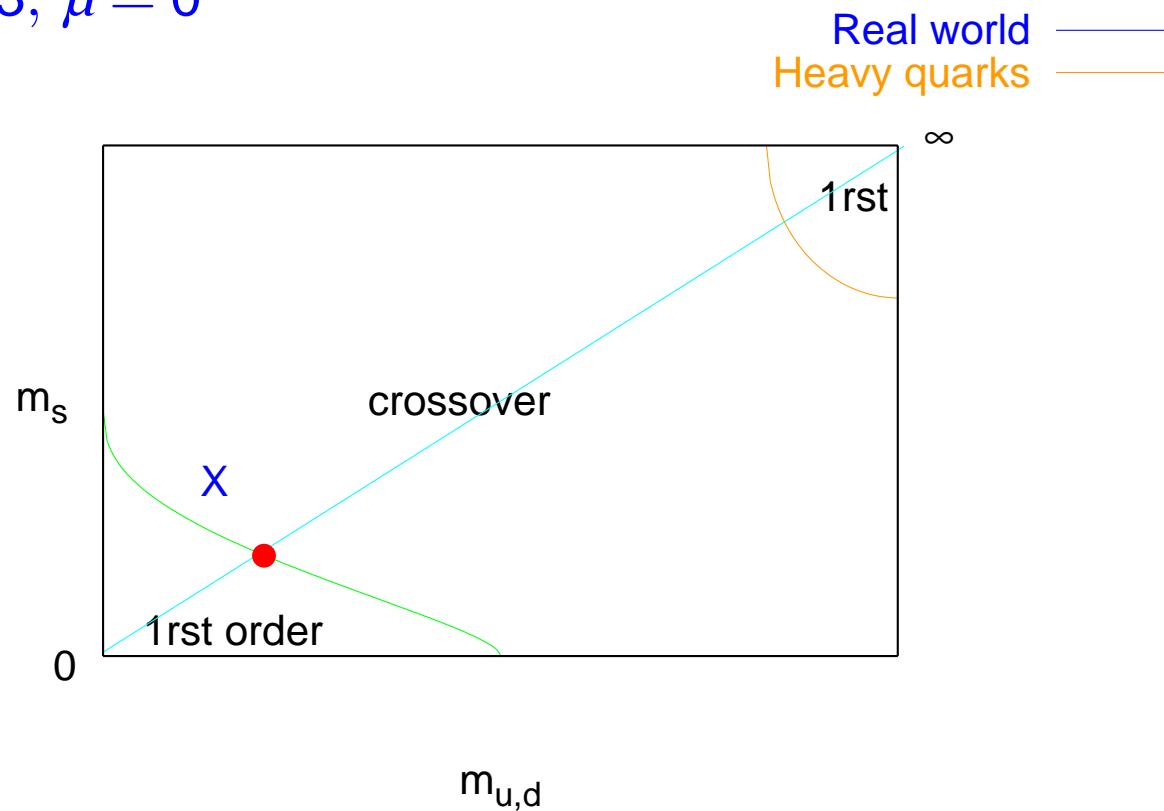
- Overlap of $\mu = 0$ MC ensemble with target $\mu = \mu_E$ ensemble?

- Discretization error ($N_t = 4 \Rightarrow a \sim 0.3$ fm) ?

Error on dynamics of phase transition may be much larger than on, e.g., hadron masses

Case study of discretization error

- Learn about QCD by generalizing to arbitrary $(m_{u,d}, m_s)$ quark masses
- Start with $N_f = 3$, $\mu = 0$



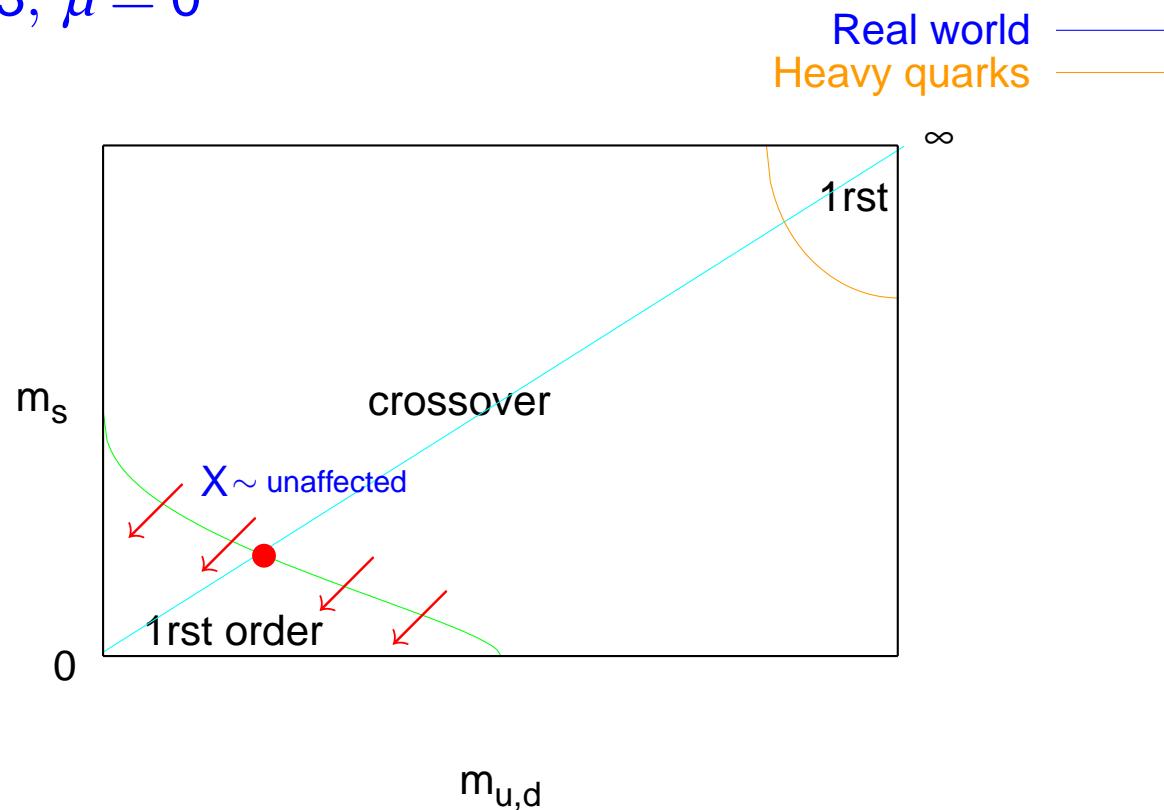
Tune $m_{u,d} = m_s \rightarrow m_c \equiv$ second order phase transition

Measure $m_\pi(T=0)/T_c$ with quarks of mass m_c :
as $a \rightarrow 0$, reduction by **factor ~ 4 !!**

PdF & Philipsen 0711.0262, Bielefeld, MILC

Case study of discretization error

- Learn about QCD by generalizing to arbitrary $(m_{u,d}, m_s)$ quark masses
- Start with $N_f = 3$, $\mu = 0$



Tune $m_{u,d} = m_s \rightarrow m_c \equiv$ second order phase transition

Measure $m_\pi(T=0)/T_c$ with quarks of mass m_c :
as $a \rightarrow 0$, reduction by **factor ~ 4 !!**

PdF & Philipsen 0711.0262, Bielefeld, MILC

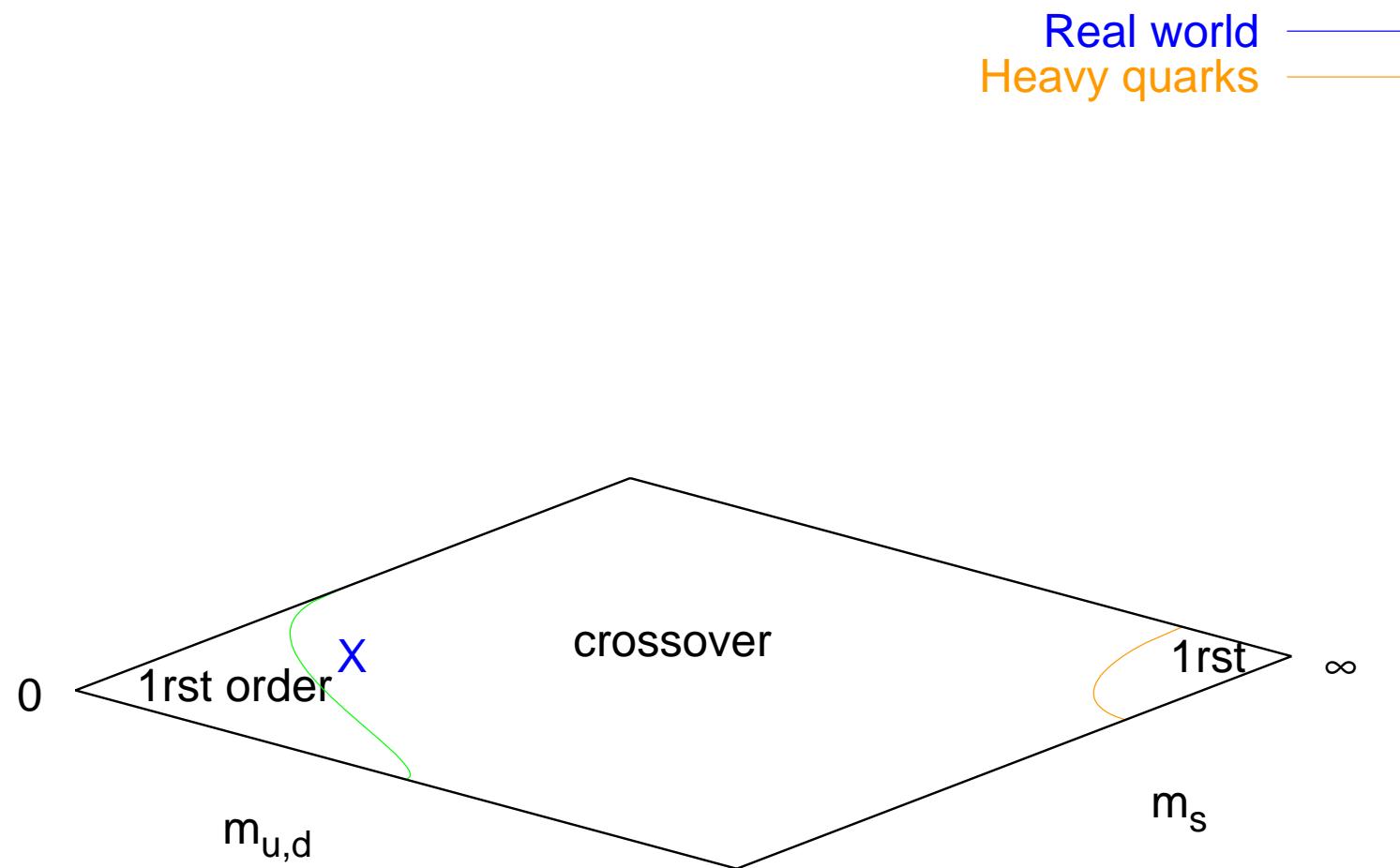
Asking a simpler question (PdF & Philipsen)

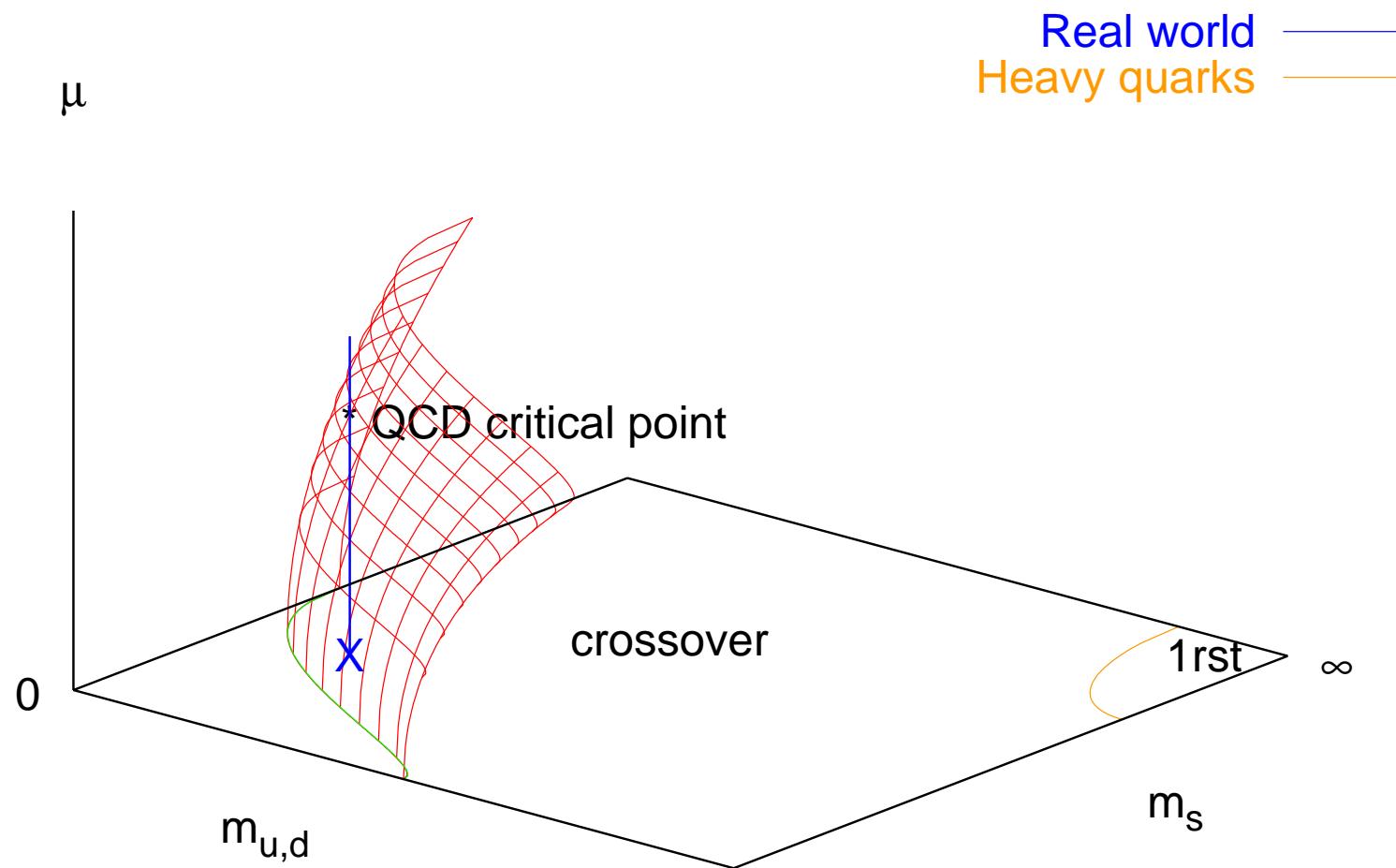
Lattice $N_t = 4 \Rightarrow a \sim 0.3$ fm **too coarse**

- Perhaps ok for “phase” boundary $T_c(\mu)$?
- Surely not good enough for critical point (T_E, μ_E) in physical units
→ ask a simpler question:

What is the sign of $\frac{dm_c}{d\mu^2} |_{\mu=0}$?

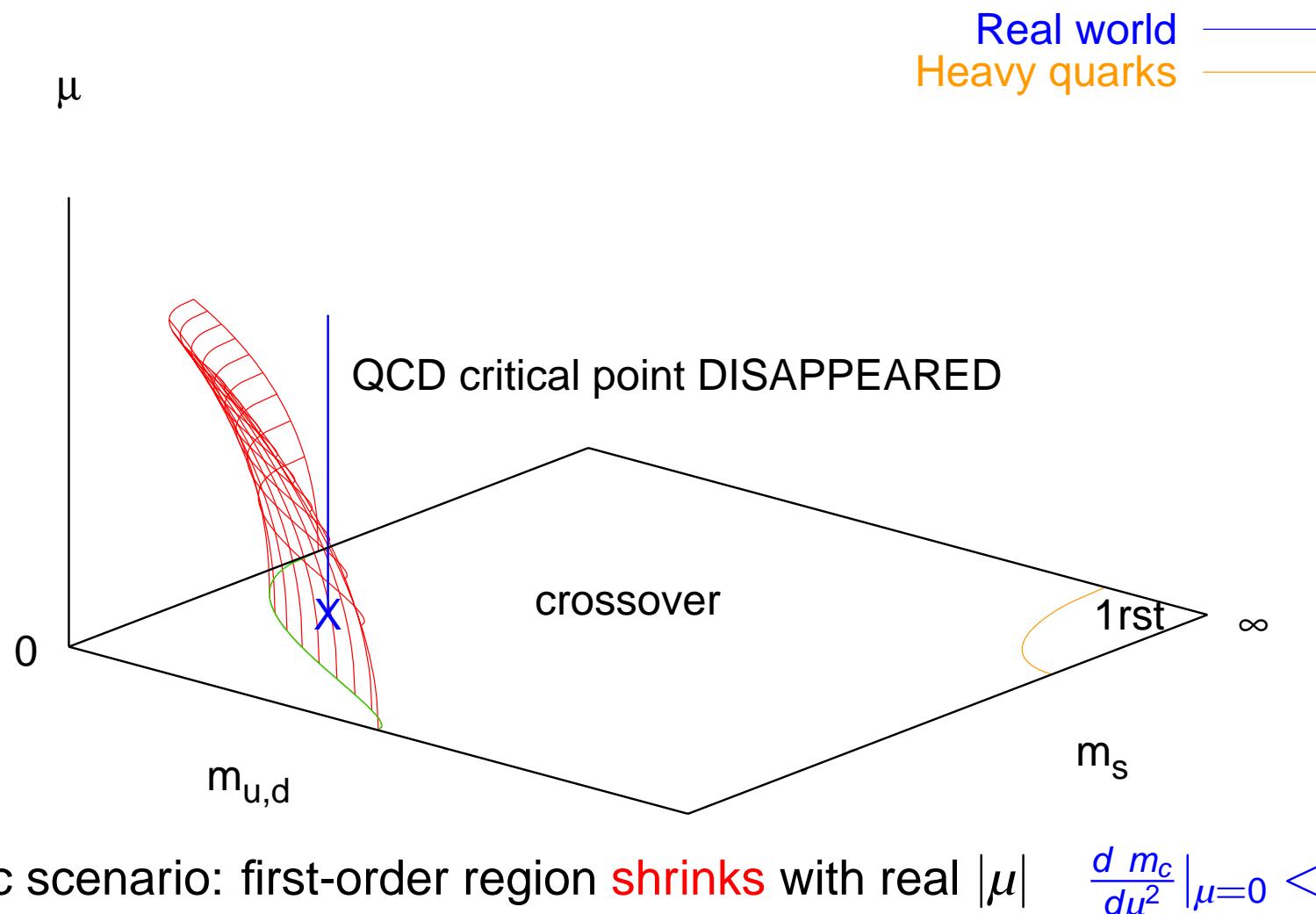
Discretization error is μ -independent → less cutoff sensitive ?

Phase diagram vs $(m_{u,d}, m_s)$, T and μ $\mu = 0$ **Now turn on μ**

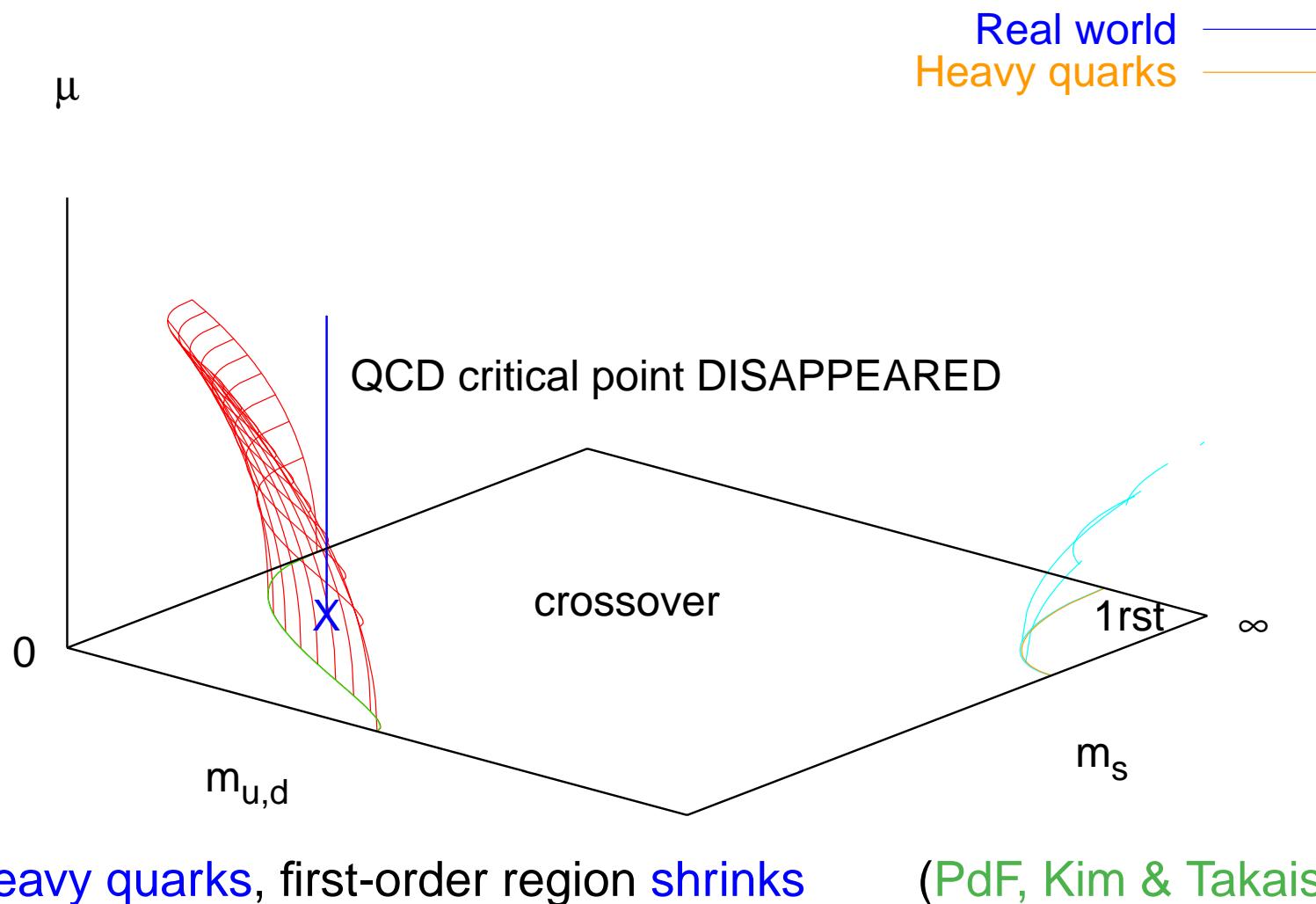
Phase diagram vs $(m_{u,d}, m_s)$, T and μ $\mu \neq 0$ 

Conventional wisdom: first-order region **expands** with real $|\mu|$

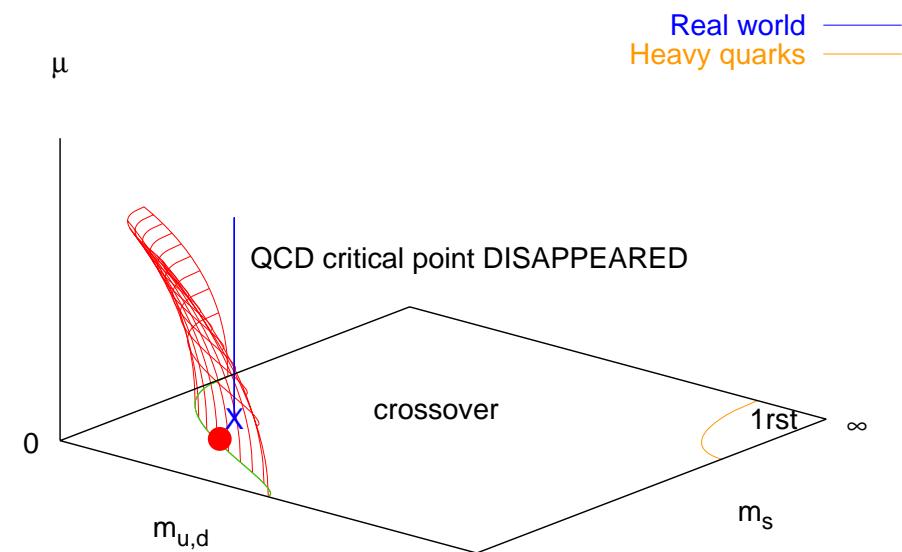
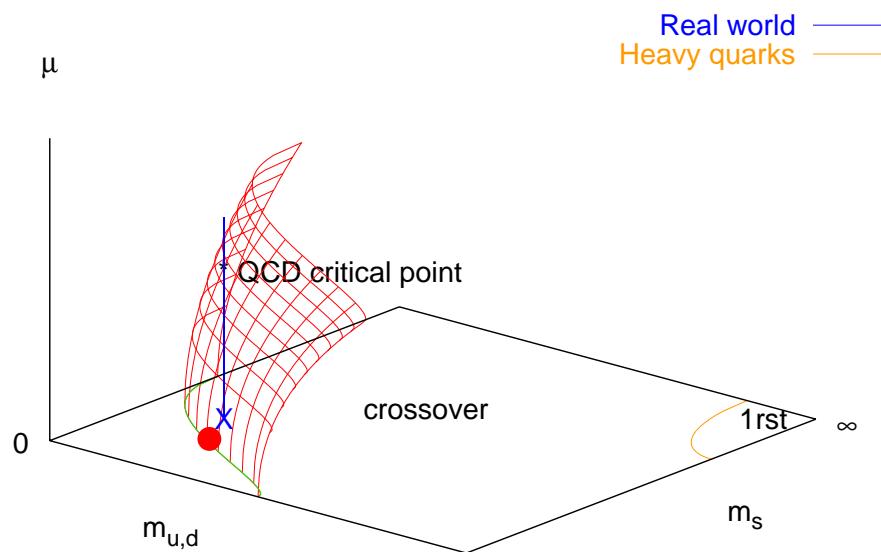
Phase diagram vs $(m_{u,d}, m_s)$, T and μ



Phase diagram vs $(m_{u,d}, m_s)$, T and μ

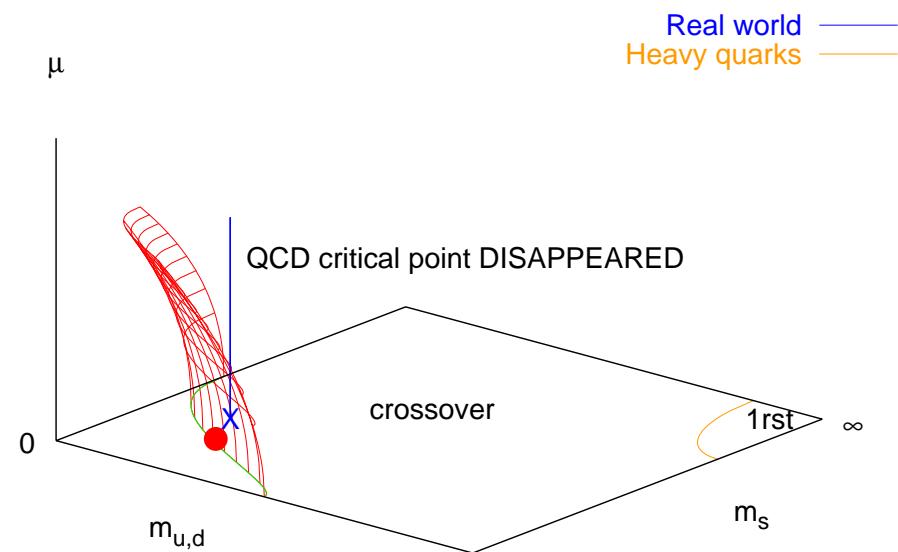
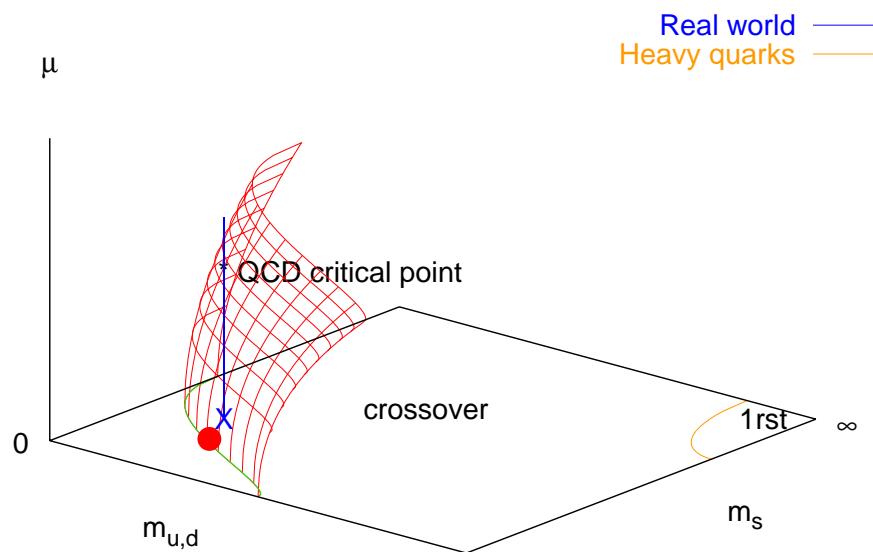


Lattice study with Owe Philipsen (hep-lat/0607017)



Strategy: tune m_q for 2nd-order P.T. at $\mu = 0$, then turn on infinitesimal μ
 Does the transition become 1rst-order (**left**) or crossover (**right**)?

Lattice study with Owe Philipsen (hep-lat/0607017)

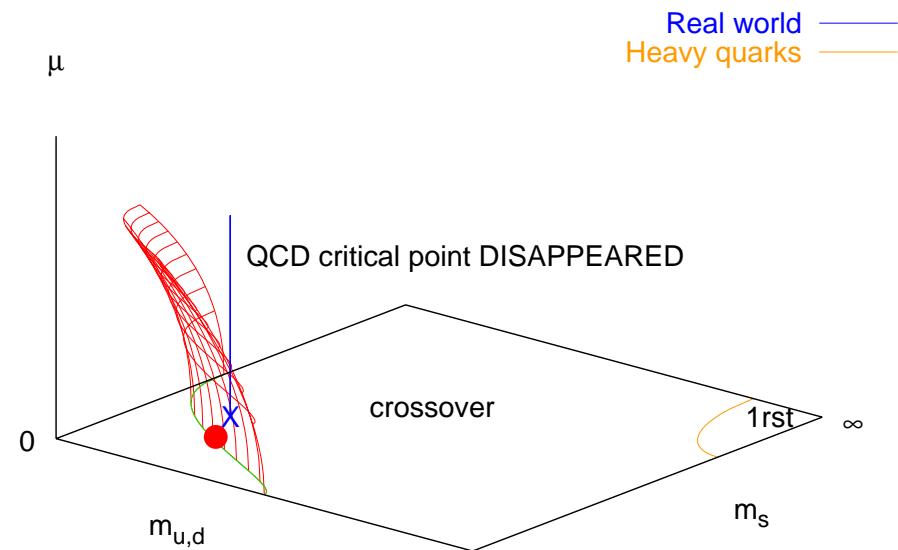
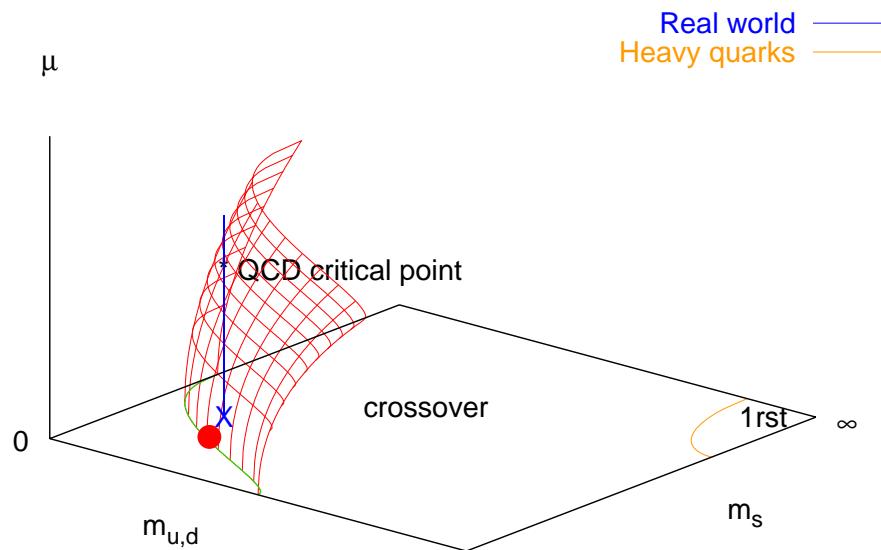


Strategy: tune m_q for 2nd-order P.T. at $\mu = 0$, then turn on infinitesimal μ

Does the transition become 1rst-order (**left**) or crossover (**right**)?

Answer: **little change** (\rightarrow surface almost **vertical**)

Lattice study with Owe Philipsen (hep-lat/0607017)



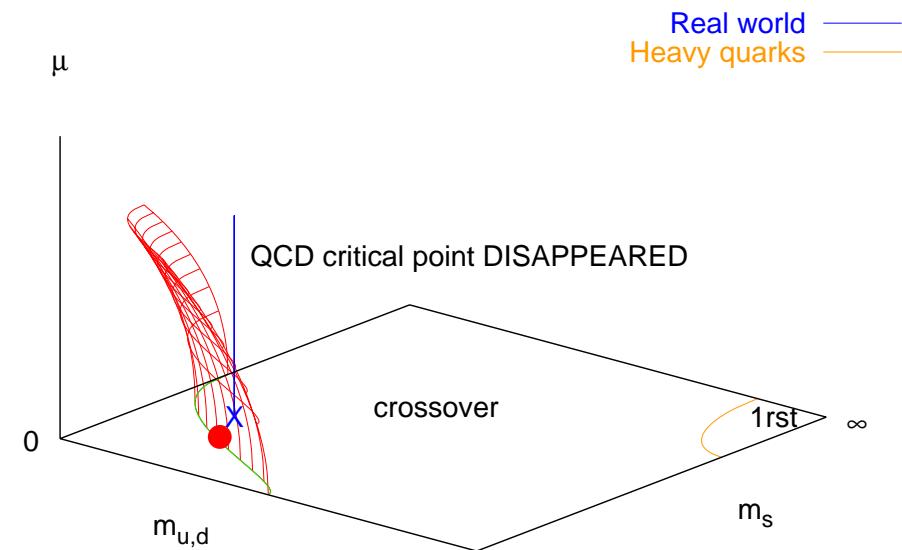
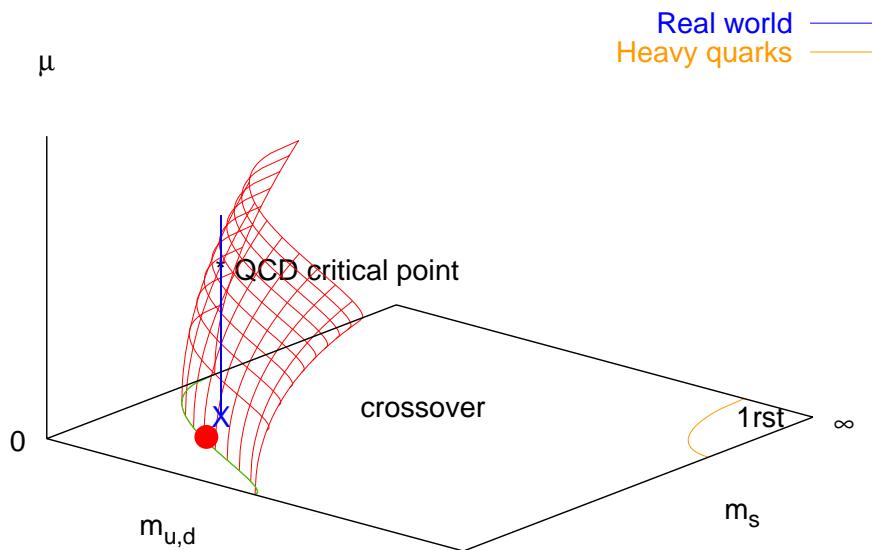
Strategy: tune m_q for 2nd-order P.T. at $\mu = 0$, then turn on infinitesimal μ

Does the transition become 1rst-order (**left**) or crossover (**right**)?

Answer: **little change** (\rightarrow surface almost **vertical**)

$$\begin{aligned}
 \text{2006: crossover favored (2}\sigma\text{)} \quad & \frac{m_c(\mu)}{m_c(0)} = 1 - 0.7(4) \left(\frac{\mu}{\pi T} \right)^2 (\mu^2 \text{ fit}) \\
 & = 1 - 2.6(1.2) \left(\frac{\mu}{\pi T} \right)^2 (\mu^2 + \mu^4 \text{ fit})
 \end{aligned}$$

Lattice study with Owe Philipsen (hep-lat/0607017)



Strategy: tune m_q for 2nd-order P.T. at $\mu = 0$, then turn on infinitesimal μ

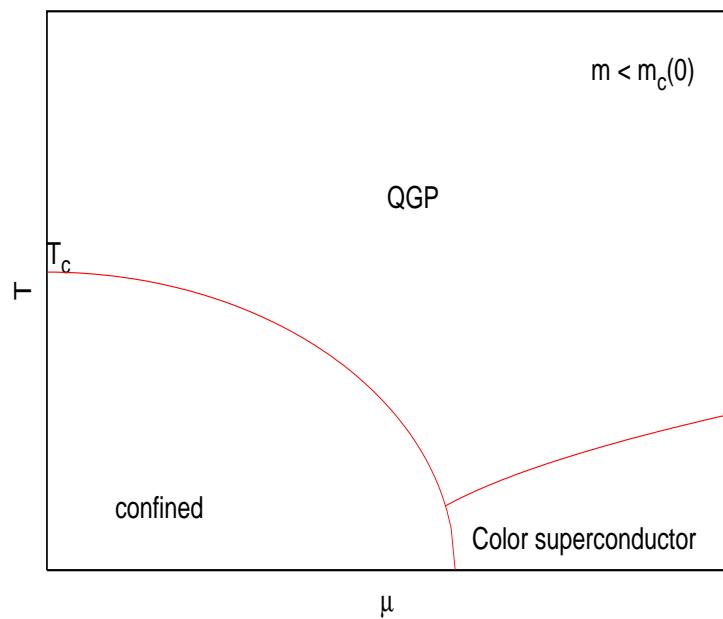
Does the transition become 1rst-order (**left**) or crossover (**right**)?

Answer: **little change** (\rightarrow surface almost **vertical**)

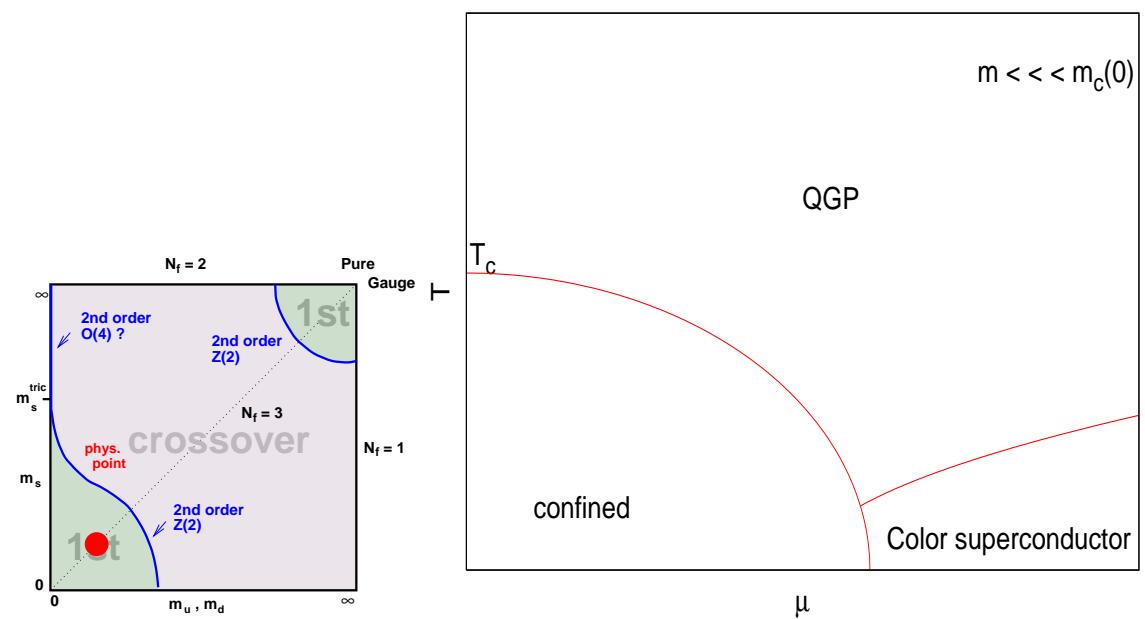
2007: measure δB_4 under $\delta \mu^2 \rightarrow$ **crossover:** $\frac{m_c(\mu)}{m_c(0)} = 1 - 3.3(5) \left(\frac{\mu}{\pi T} \right)^2$

Resulting phase diagram (simplest possibility)

Standard scenario

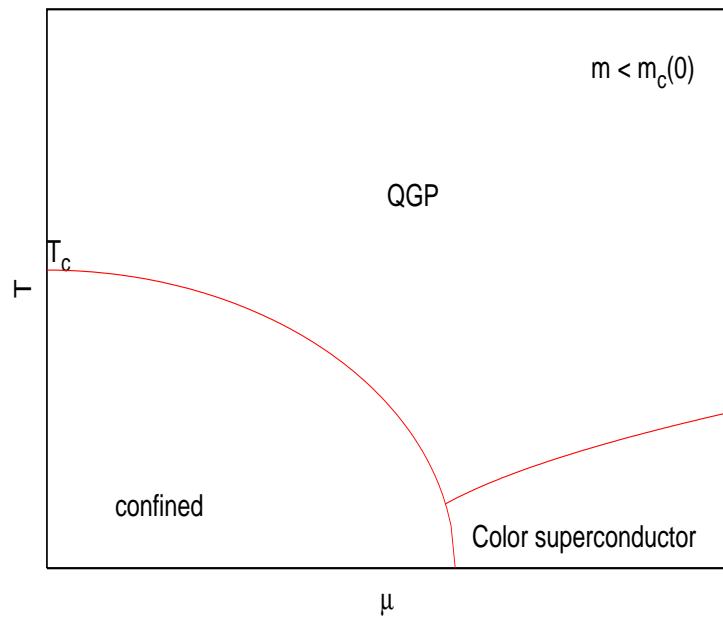


Exotic scenario

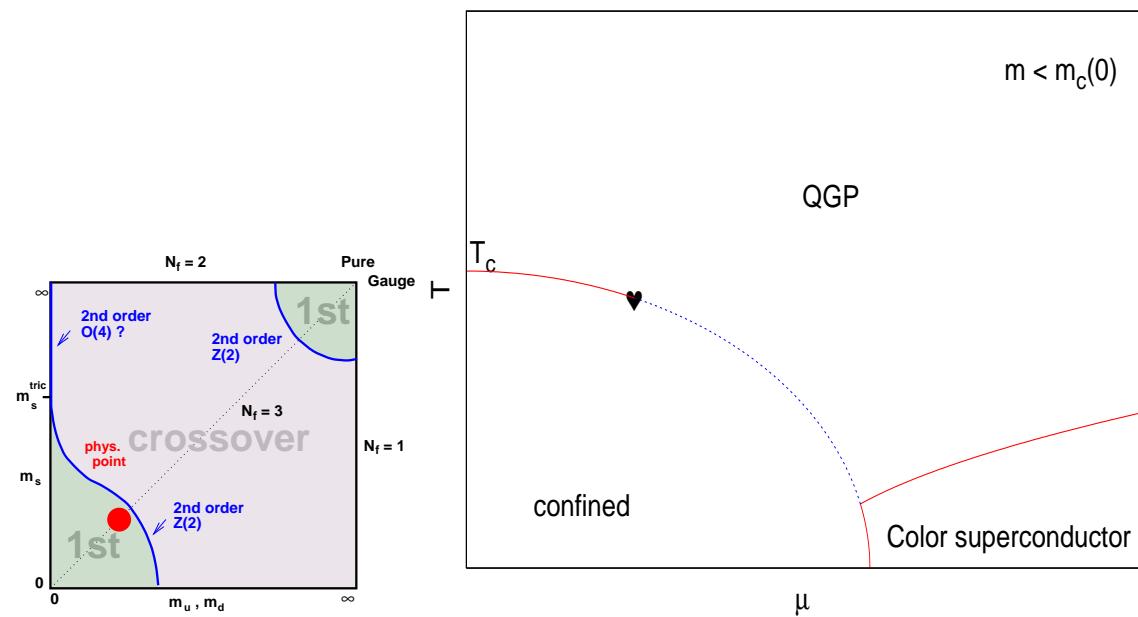


Resulting phase diagram (simplest possibility)

Standard scenario

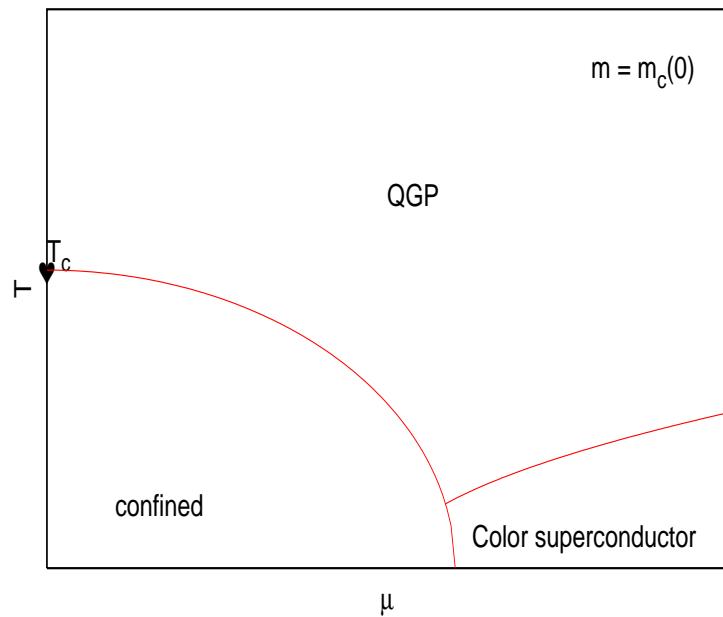


Exotic scenario

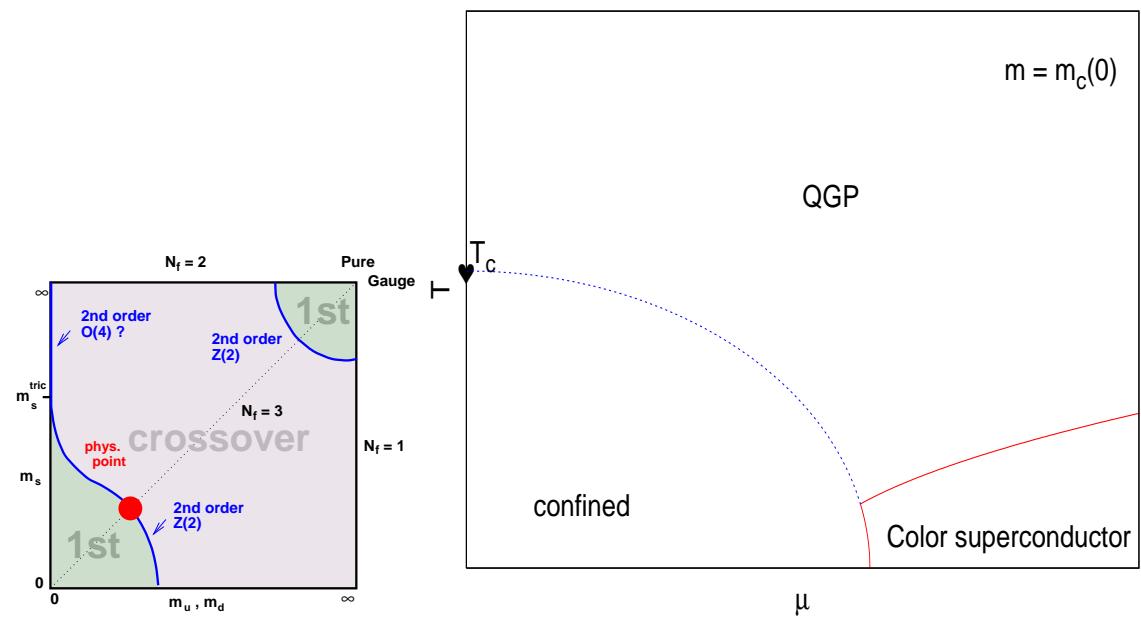


Resulting phase diagram (simplest possibility)

Standard scenario

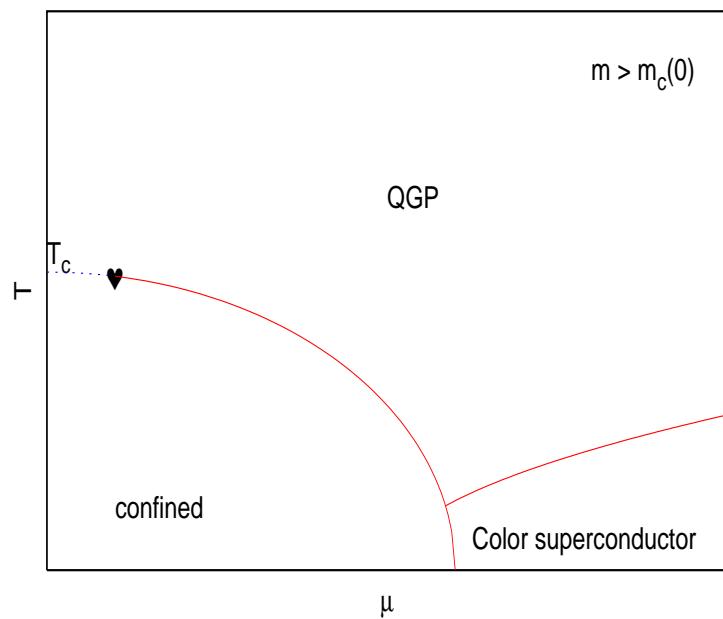


Exotic scenario

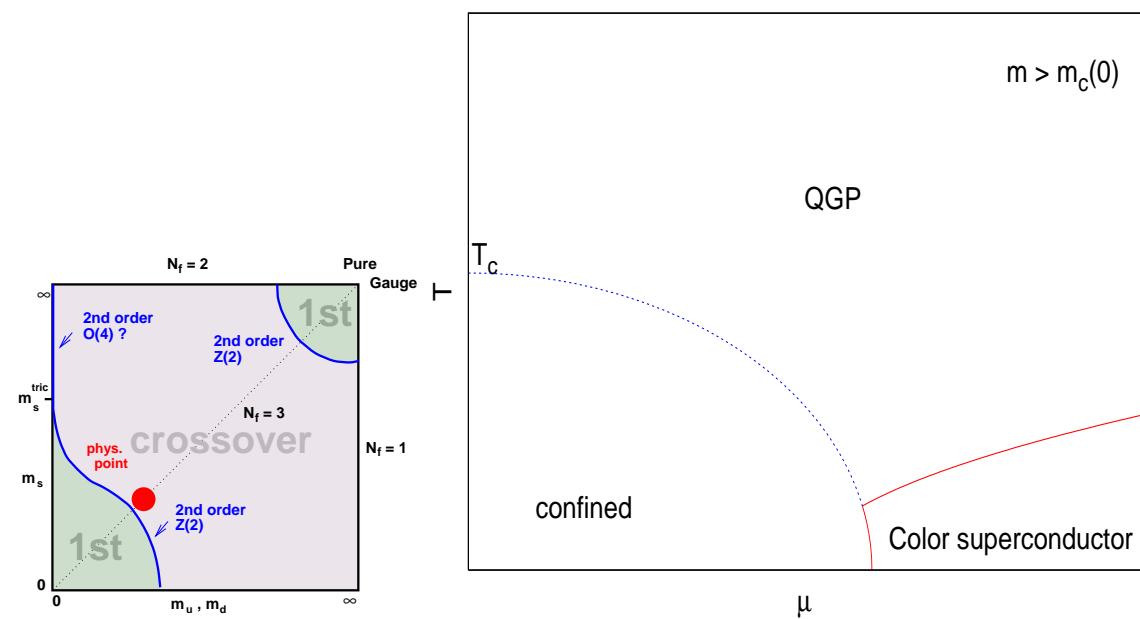


Resulting phase diagram (simplest possibility)

Standard scenario



Exotic scenario



Conclusions

- $\mu = 0$ transition is broad **crossover**, “ T_c ” settled soon
- “Phase” boundary $\frac{T_c(\mu)}{T_c(\mu=0)} = 1 - c \left(\frac{\mu}{\pi T}\right)^2$:
 - good description up to $\frac{\mu}{T} \sim 1$
 - **can be determined in continuum QCD**
 - presently, $c \sim \frac{1}{3}$ freeze-out curve ?
- QCD **critical point**:
 - very large cutoff errors $\mathcal{O}(100\%)$
 - $\frac{m_c(\mu)}{m_c(\mu=0)} = 1 + \gamma \left(\frac{\mu}{\pi T}\right)^2$; **γ can be determined**
 - keep open mind: does μ strengthen or weaken the transition?
 - **small μ_E unlikely** (sign of γ + cutoff effects)
- Need finer lattices ($N_t \geq 6$) even for **qualitative picture**
- Another talk: w/Misha Stephanov & Urs Wenger
 fabulously interesting physics with **isospin** chem. pot. $\mu_u = -\mu_d$
 confinement vs Bose condensation of π^+ , BEC-BCS crossover,...