

Dynamics in the Ultracold – Atomic gases far from equilibrium



Thomas Gasenzer
Institut für Theoretische Physik
Philosophenweg 16
69120 Heidelberg

email: t.gasenzer@thphys.uni-heidelberg.de

Overview

■ Preface

Ultracold gases out of equilibrium

■ Non-equilibrium quantum field theory

Functional approaches

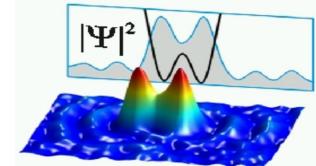
■ Equilibration of a 1D Bose gas

Initial value problems in QFT

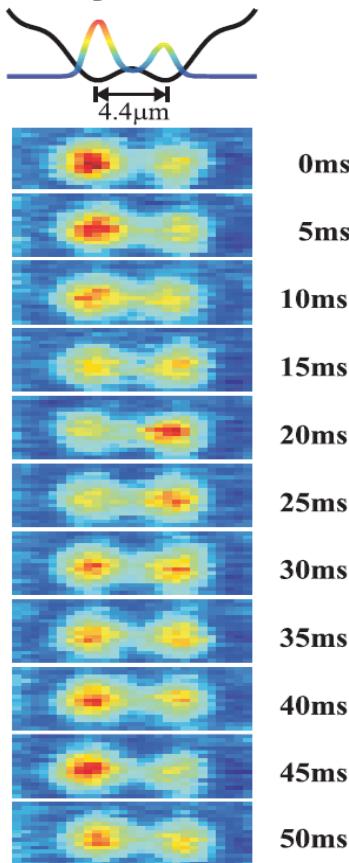


Preface

Cold-gases livestream on TV

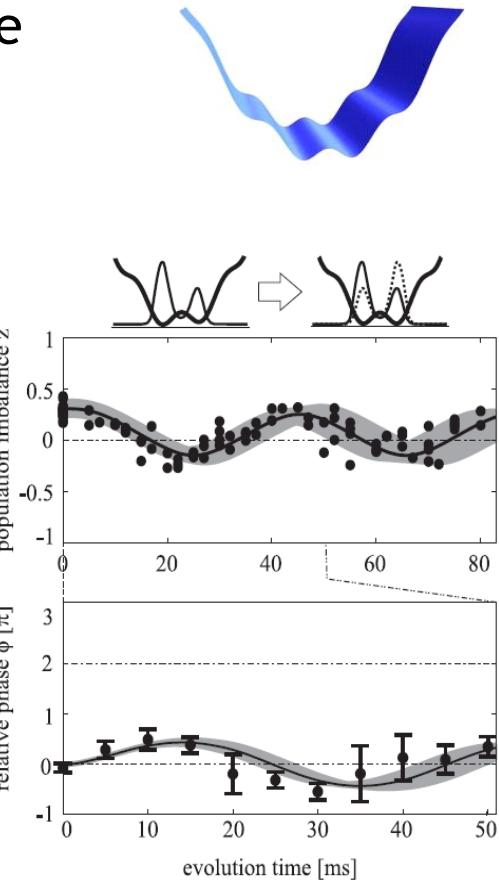


a Josephson oscillations



- ✓ Observe evolution in real time
- ✓ Model freely initial state
- ✓ Change boundary conditions
- ✓ Measure mean densities,
phases, fluctuations
- ✓ Reduce atom numbers to
a **few hundreds & less**

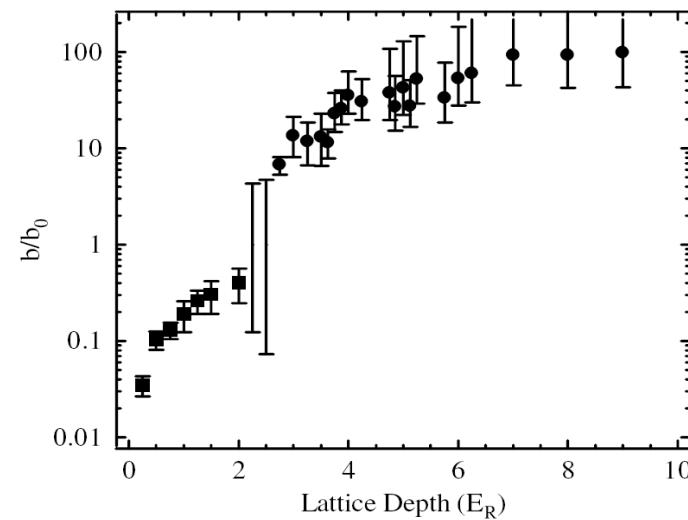
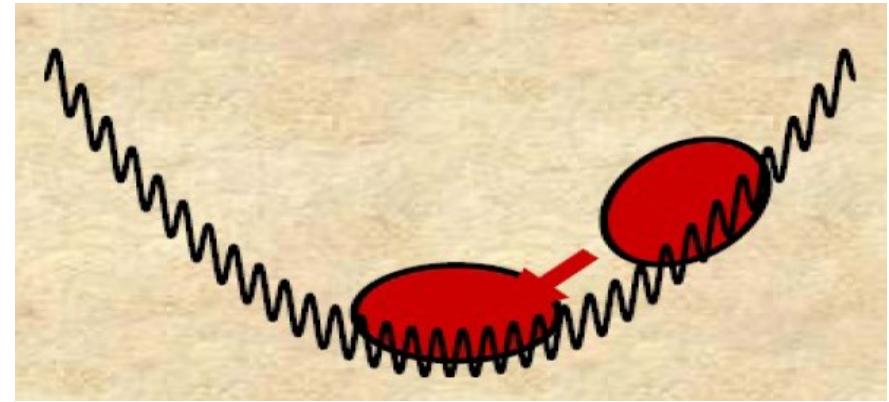
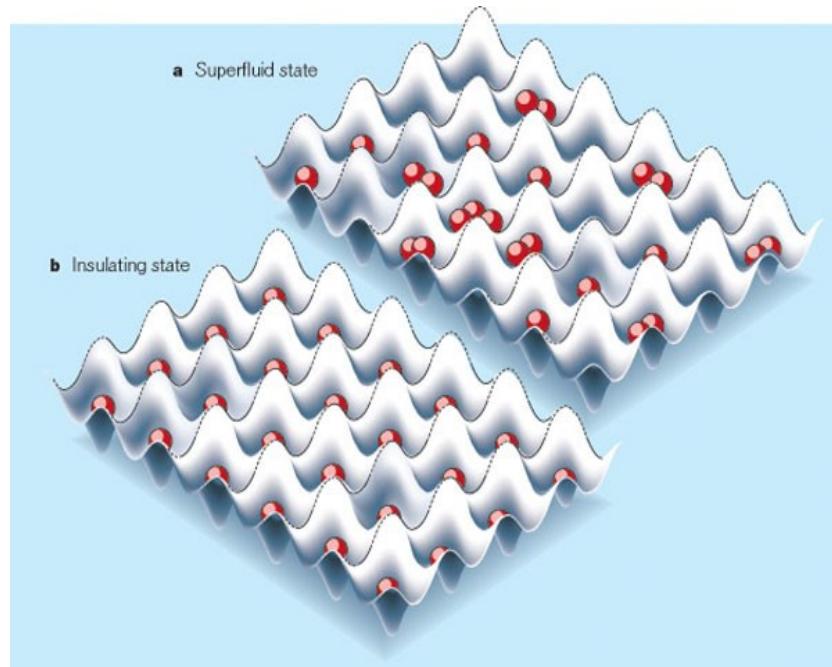
Oberthaler Labs
(Heidelberg)



“Strong” dynamics of ultracold gases

Dipole oscillations in lattices:
Damping rates?

[Expt. @ NIST: Fertig et al. PRL94 (05)]



Quantum Nonequilibrium Dynamics: Theory?

A few methods available, in particular for 1D.

- Exact methods
[Lieb & Liniger, Girardeau, McGuire, Gaudin, Minguzzi, Buljan, ...]
- DMRG, MPS/PEPS
[Vidal, Kollath, Schollwöck, White, Feiguin, Manmana, Muramatsu, Wolf, Cirac, ...]
- Quantum Monte Carlo
[Mak, Egger, Berges & Stamatescu, ...]

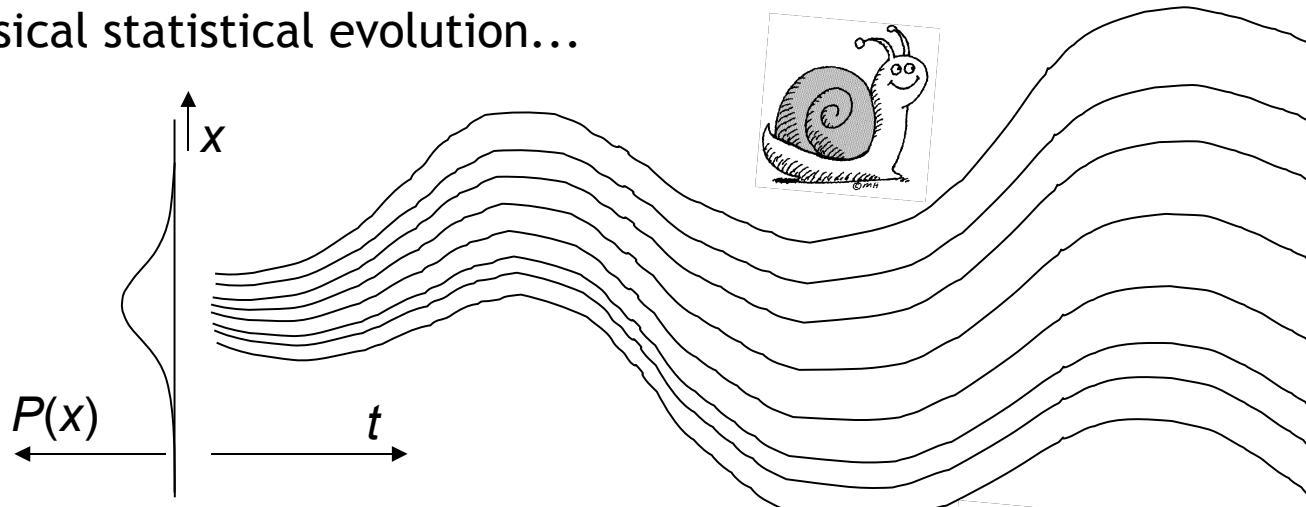
For higher dimensions...



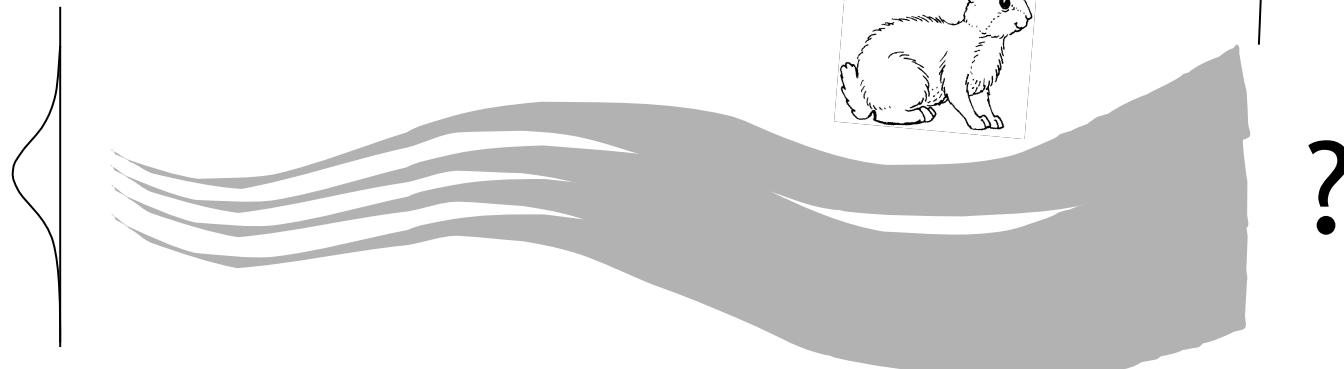
Evolving quantum fields...

...are difficult to describe due to quantum fluctuations.

Classical statistical evolution...

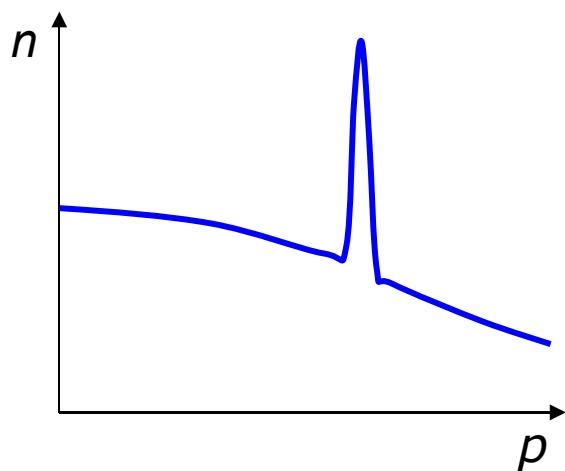


...vs quantum statistical evolution:



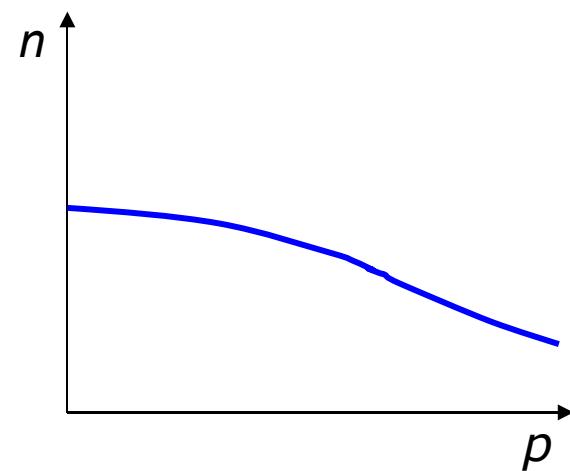
Non-equilibrium evolution in Quantum Field Theory

Near-equilibrium dynamics: Damping



Exponential damping

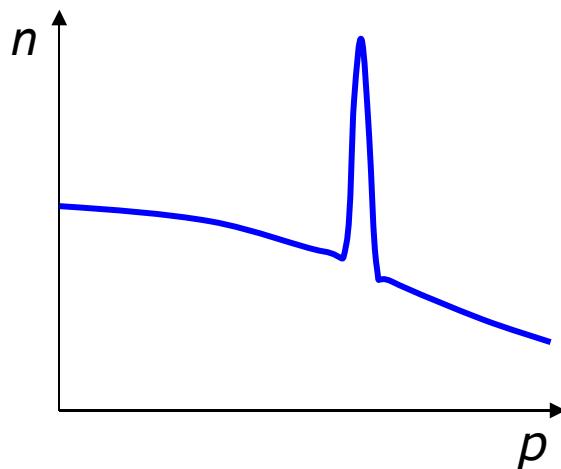
$$n - n_{\text{eq}} \sim \exp(-\Gamma t)$$



density info in \mathcal{F} = statistical correl. function

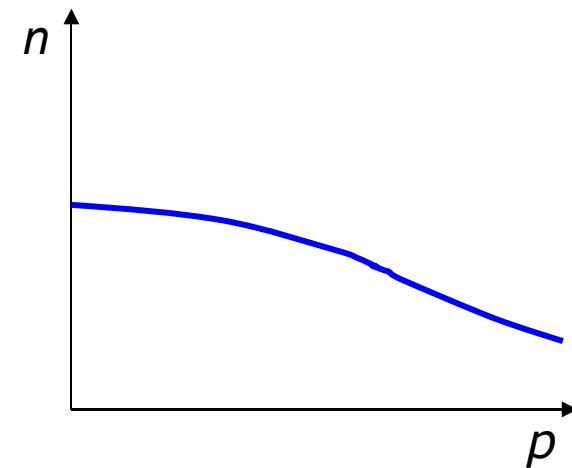


Near-equilibrium dynamics: Damping



Exponential damping

$$n - n_{\text{eq}} \sim \exp(-\Gamma t)$$

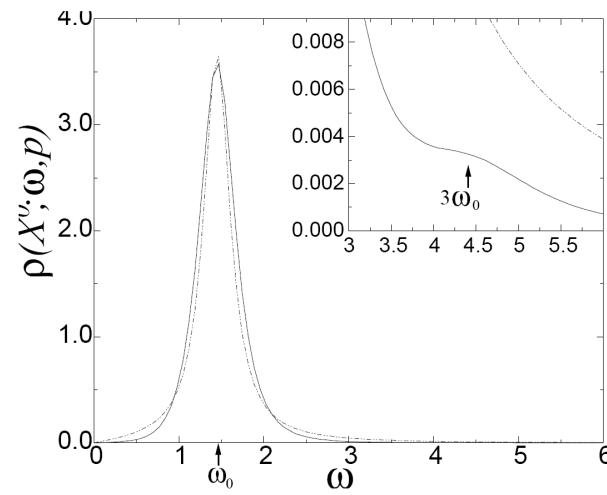


density info in \mathcal{F} = statistical correl. function

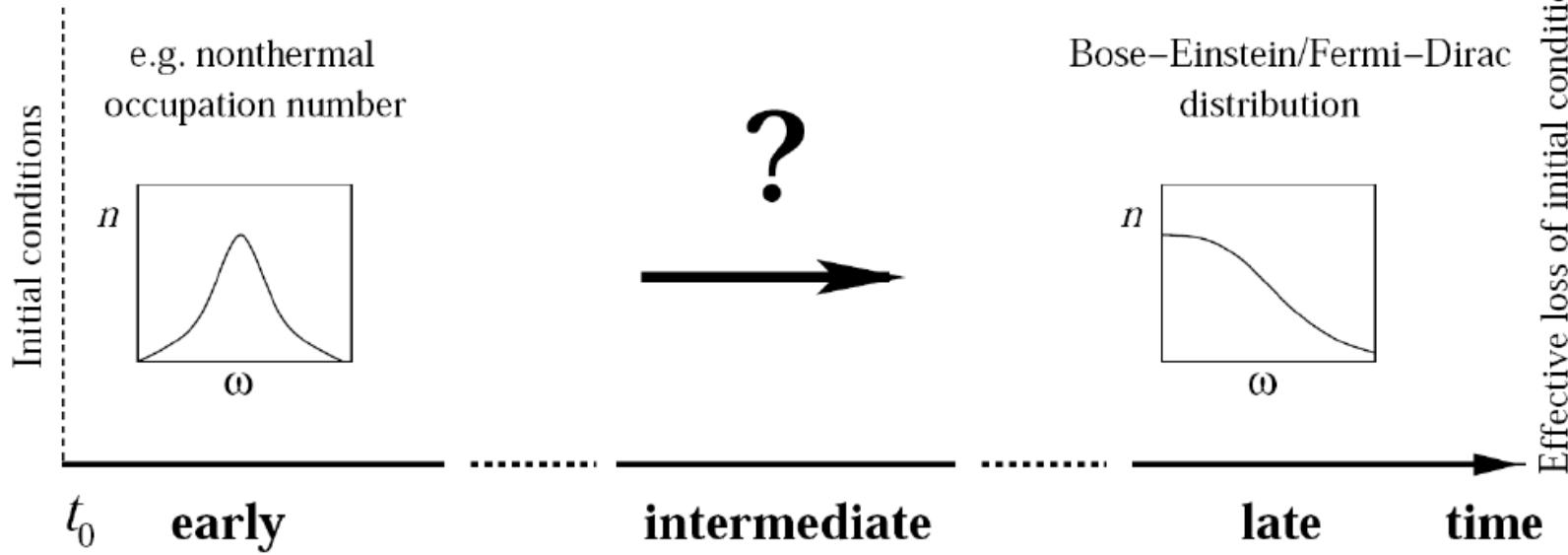
energy & decay width in ρ = spectral function

→ fulfill fluctuation-dissipation relation

$$\mathcal{F}_{\omega_p}^{(\text{eq})} = -i (n(\omega, T) + \frac{1}{2}) \rho_{\omega_p}^{(\text{eq})}$$



Far-from-equilibrium dynamics



Thermal equilibrium: **Loss of information** about prior evolution.

💡 Only a **few conserved quantities** persist.



Dynamical Field Theory



We will be interested, in particular, in the time dependence of the lowest-order correlation functions:

$$\phi_i(x) = \langle \Phi_i(x) \rangle$$

(mean field)

$$G_{ij}(x,y) = \langle \Phi_i(x) \Phi_j(y) \rangle$$

(density matrix)

$$x = (\mathbf{x}, t)$$



Path Integral Approach



e.g. transition amplitude:



$$\langle t_{\text{fin}} | t_{\text{ini}} \rangle = \int \mathcal{D}\varphi e^{i S[\varphi]/\hbar}$$

$$\mathcal{D}\varphi = \prod_{x=x_{\text{ini}}}^{x_{\text{fin}}} dx \varphi(x)$$

Classical dynamics of φ from $\delta S[\varphi] = 0$.



Path Integral Approach



Generating functional:



$$Z[J] = \int \mathcal{D}\varphi e^{iS[\varphi]/\hbar - i \int J \varphi}$$
$$\phi = i \frac{\delta \ln Z}{\delta J} \Big|_{J=0} = Z^{-1} \int \mathcal{D}\varphi \varphi e^{iS[\varphi]/\hbar}$$

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Effective Action



Generating functional:



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Classical dynamics of φ from $\delta S[\varphi] = 0$.

Quantum dynamics of ϕ from variation of an effective action, $\delta \Gamma[\phi]/\delta \phi = -J$:

$$Z[J] = \int \mathcal{D}\varphi \delta[\varphi - \phi] e^{i\Gamma[\varphi]/\hbar - i \int J \varphi}$$

$$\Gamma[\phi] = -i\hbar \ln Z[J] - \int J \phi$$



Effective Action



Generating functional:



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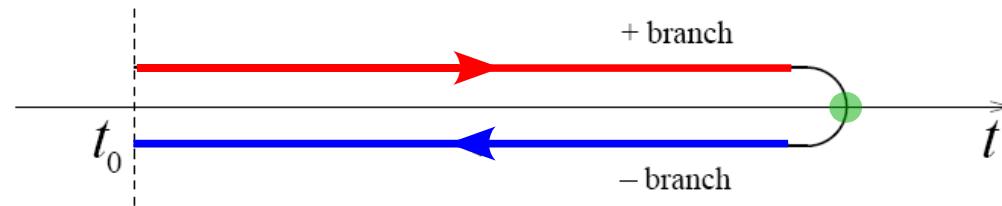
$$\begin{aligned} Z[J] &= \int \mathcal{D}\varphi \delta[\varphi - \phi] e^{i\Gamma[\varphi]/\hbar - i \int J \varphi} \\ \Gamma[\phi] &= -i\hbar \ln Z[J] - \int J \phi \\ &= S[\phi] - i/2 \operatorname{Tr} \ln G + \dots \end{aligned}$$



Initial value problems...

...require the Schwinger-Keldysh closed time path:

$$\begin{aligned}\langle t | O | t \rangle &= \langle t_0 | U^\dagger(t) O U(t) | t_0 \rangle \\ &= \int \mathcal{D}\varphi_0 \mathcal{D}\varphi_0 \rho[\varphi_0, \varphi_0] \int \mathcal{D}\varphi' \mathcal{D}\varphi' O e^{i(S[\varphi] - S[\varphi'])/\hbar}\end{aligned}$$



Functional approaches

2PI Effective Action (Φ -Functional)

[Luttinger, Ward (60); Baym (62); Cornwall, Jackiw, Tomboulis (74)]

- Double **Legendre transform**:

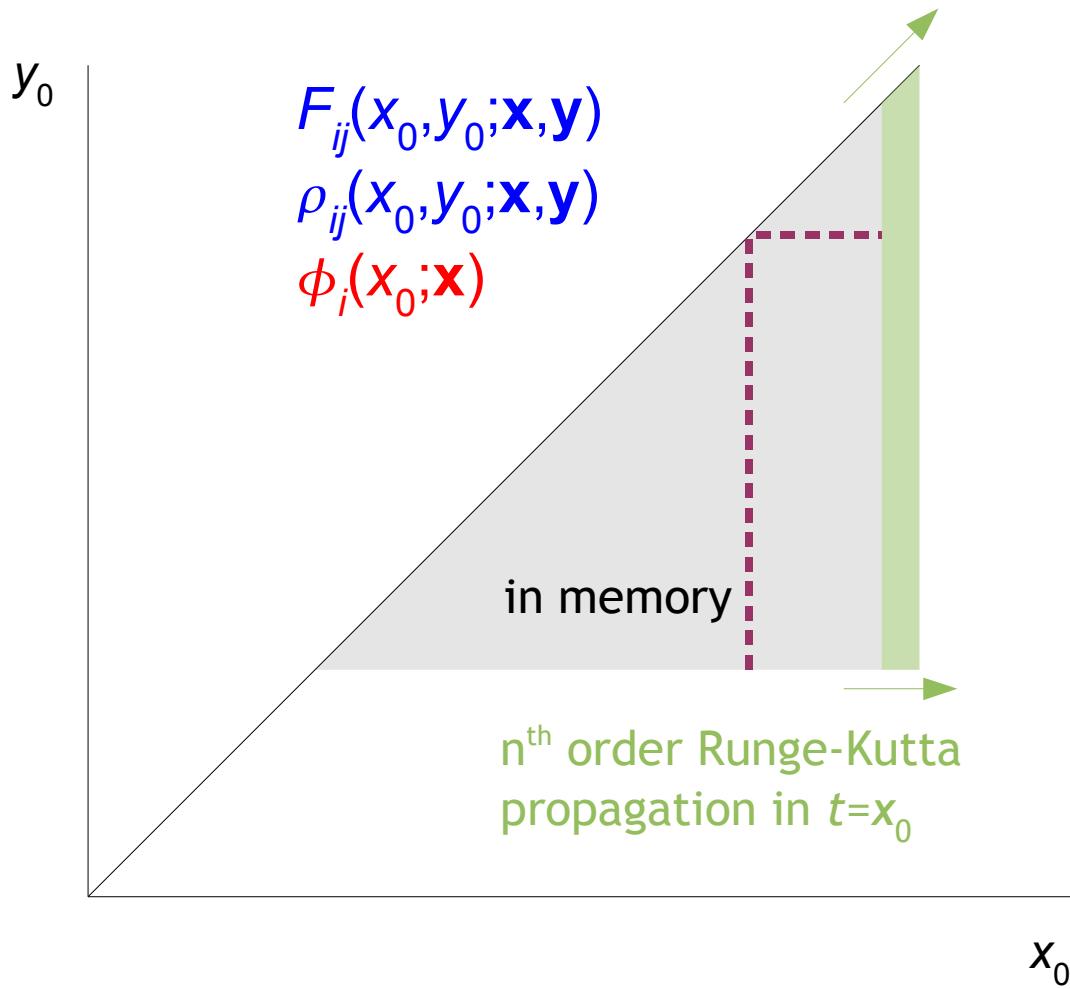
$$\Gamma[\phi, G] = -i \ln Z[J, K] - \phi_i J_i - \frac{1}{2}(\phi_i \phi_j + G_{ij}) K_{ij},$$
$$-i \frac{\delta \ln Z[J, K]}{\delta J_i} \Big|_{J=K \equiv 0} = \phi_i = \langle \hat{\Phi}_i \rangle,$$
$$-2i \frac{\delta \ln Z[J, K]}{\delta K_{ij}} \Big|_{J=K \equiv 0} = \phi_i \phi_j + G_{ij} = \langle T \hat{\Phi}_i \hat{\Phi}_j \rangle.$$

- Dynamic equations:

$$\frac{\delta \Gamma[\phi, G]}{\delta \phi_x} = 0, \quad \frac{\delta \Gamma[\phi, G]}{\delta G(x, y)} = 0$$



Numerical Demand



e.g.

- 16 x 16 spatial grid
- 1000 x 1000 temporal grid
- $\mathcal{N} \times \mathcal{N}$ index grid, $\mathcal{N}=2$
⇒ ~16 GB RAM

@ ITP:

10+ AMD DualCore
2 GB RAM/node

@ IWR Heidelberg:

8 x SPARC Ultra III 900 MHz
64 GB shared RAM



Conserved quantities

- **Energy** conservation à la Emmy Noether according to

$$\delta\Gamma[\phi, G] = \int \left\{ \frac{\delta\Gamma[\phi, G]}{\delta\phi} \delta\phi + \frac{\delta\Gamma[\phi, G]}{\delta G} \delta G \right\} = 0,$$

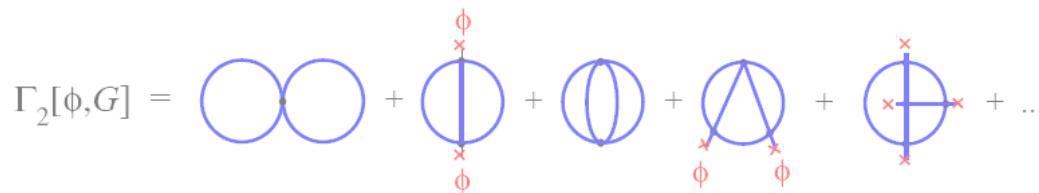
with time translations encoded into $\delta\phi$, δG .

- **Number** conservation at any diagrammatic truncation of $\Gamma[\phi, G]$ as a consequence of $O(\mathcal{N})$ invariance.

2PI (2-particle irreducible) **Effective Action**:

$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2} \text{Tr} \ln(G^{-1} + G_0^{-1}(\phi)G) + \Gamma_2[\phi, G],$$

with $G_{0,ij}^{-1} = \delta^2 S[\phi]/(\delta\phi_i^\dagger \delta\phi_j)$



Conserved quantities

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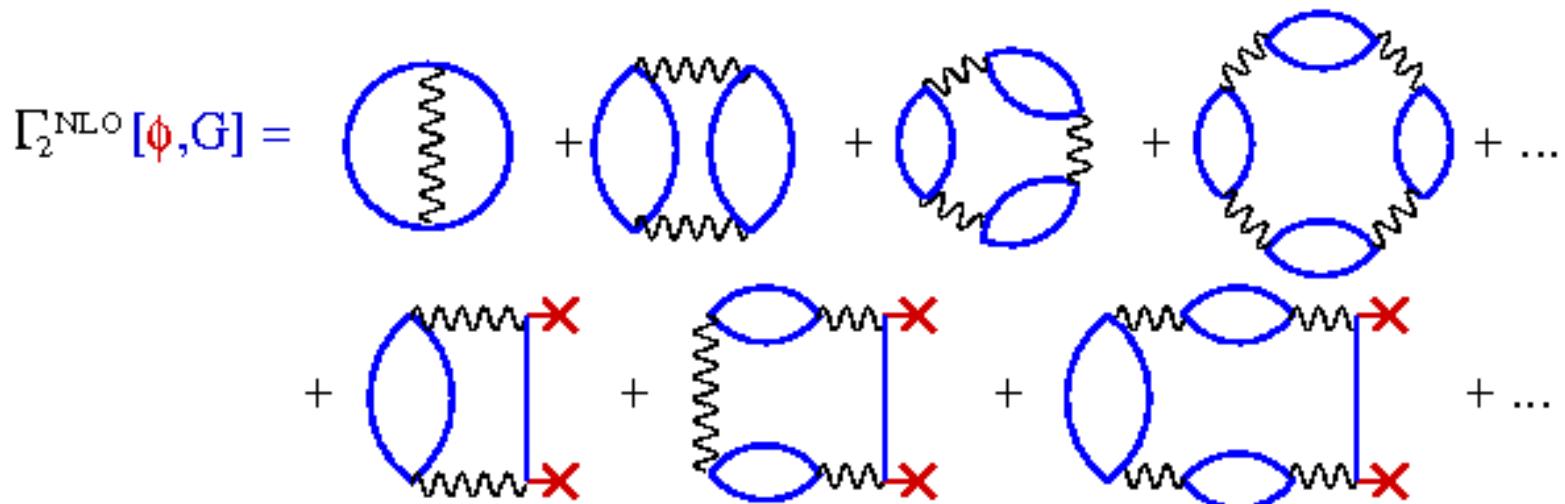
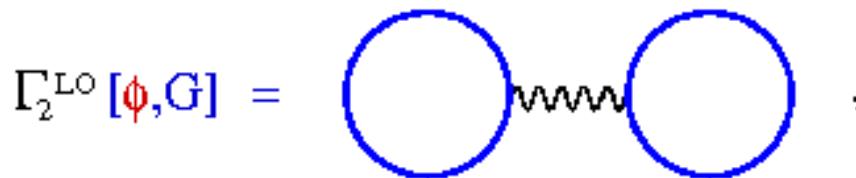
$$\Gamma_2[\phi, G] = \text{Diagram } 1 + \text{Diagram } 2 + \text{Diagram } 3 + \text{Diagram } 4 + \text{Diagram } 5 + \dots$$



2PI $1/\mathcal{N}$ Expansion

[Berges, Nucl. Phys. A (02); Aarts, Arensmaier, Baier, Berges, & Serreau, PRD (02)]

$$S[\Phi] = \int_{x,\textcolor{pink}{c}} \left[\Phi_i(x) i D_{ij}^{-1}(x) \Phi_j(x) - \frac{g}{\mathcal{N}} \Phi_i(x) \Phi_i(x) \Phi_j(x) \Phi_j(x) \right]$$

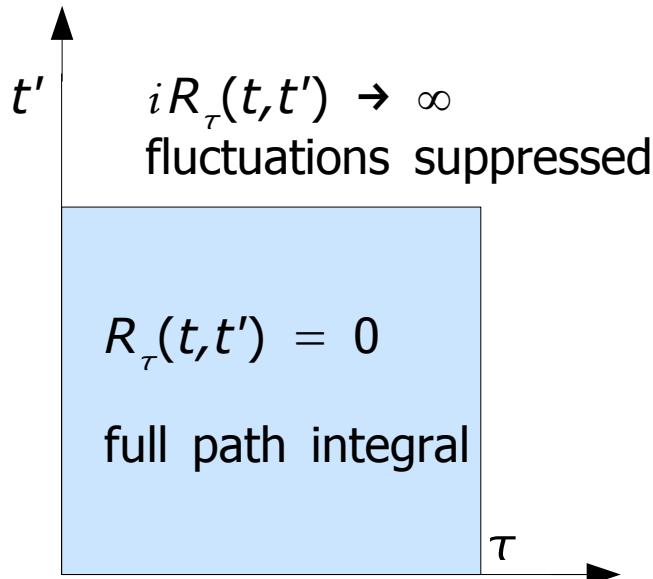


Renormalisation-group approach to far-from-equilibrium dynamics

[TG & J.M. Pawłowski, cond-mat/0710.4627]

- Regularise generating functional

$$Z_{\tau} = \exp \left\{ i \int_{x,y;\textcolor{red}{c}} \frac{\delta}{\delta \mathbf{J}_a(x)} \mathbf{R}_{\tau,ab}(x,y) \frac{\delta}{\delta \mathbf{J}_b(y)} \right\} Z$$
$$Z[\mathbf{J}; \rho_0] = \int \mathcal{D}\varphi \rho_0 \exp \left\{ iS[\varphi] + i \int_{x,\textcolor{red}{c}} \mathbf{J}_a \varphi_a \right\},$$

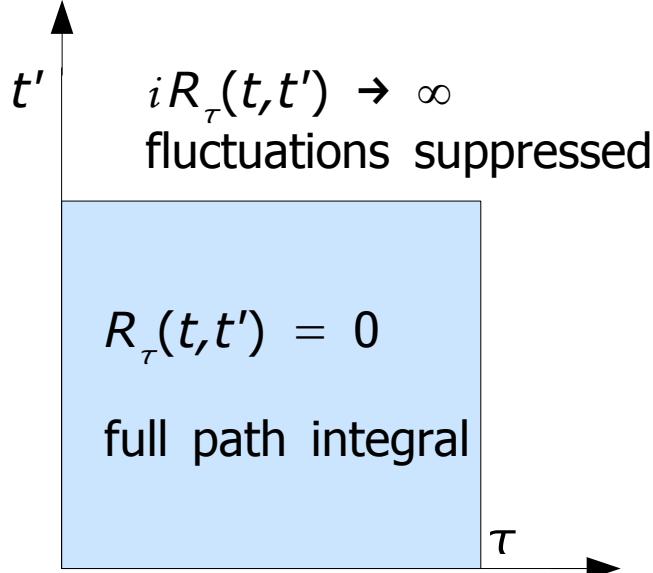


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- Exact flow equation [Wetterich (92)]

$$\partial_{\tau} \Gamma_{\tau} = \frac{i}{2} \int_{\textcolor{red}{c}} \left[\frac{1}{\Gamma_{\tau}^{(2)} + \mathbf{R}_{\tau}} \right]_{ab} \partial_{\tau} \mathbf{R}_{\tau,ab}$$

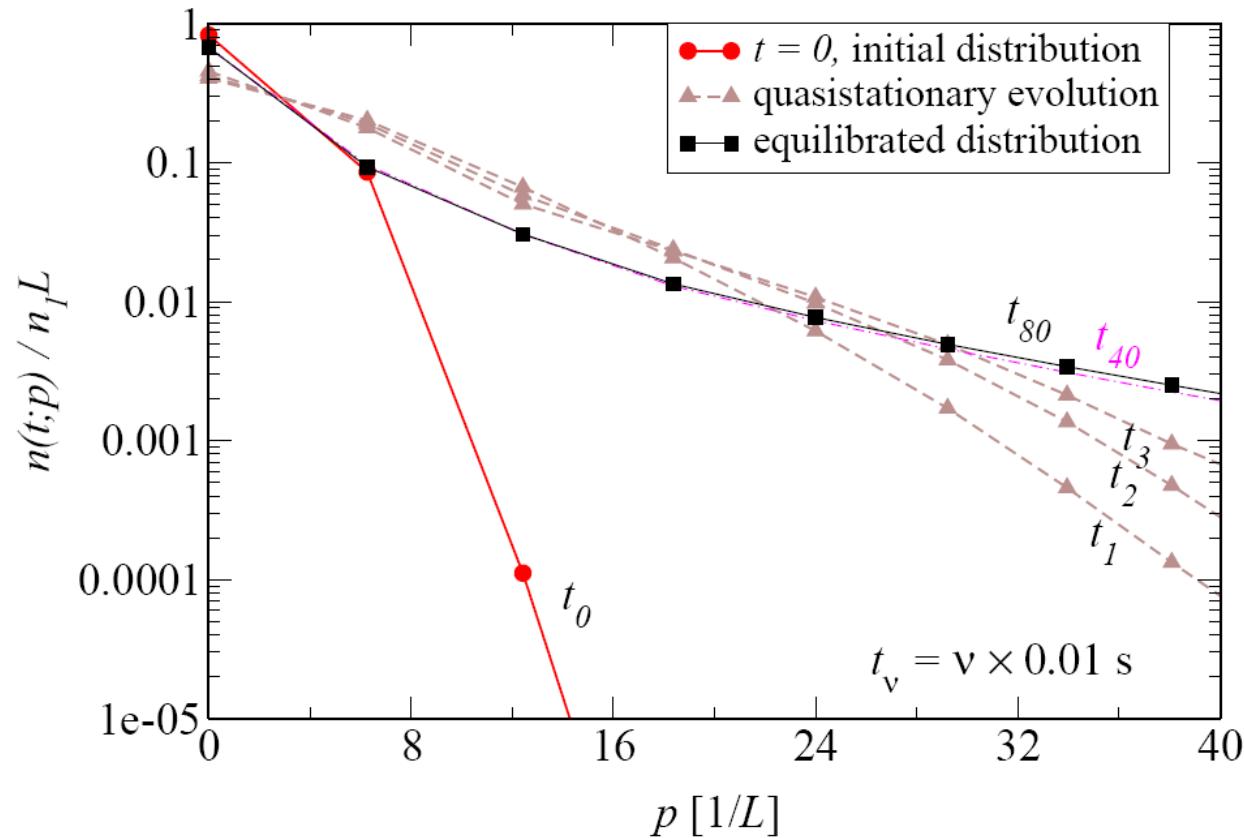


Equilibration of a 1D Bose gas

Equilibration of a 1D Bose gas

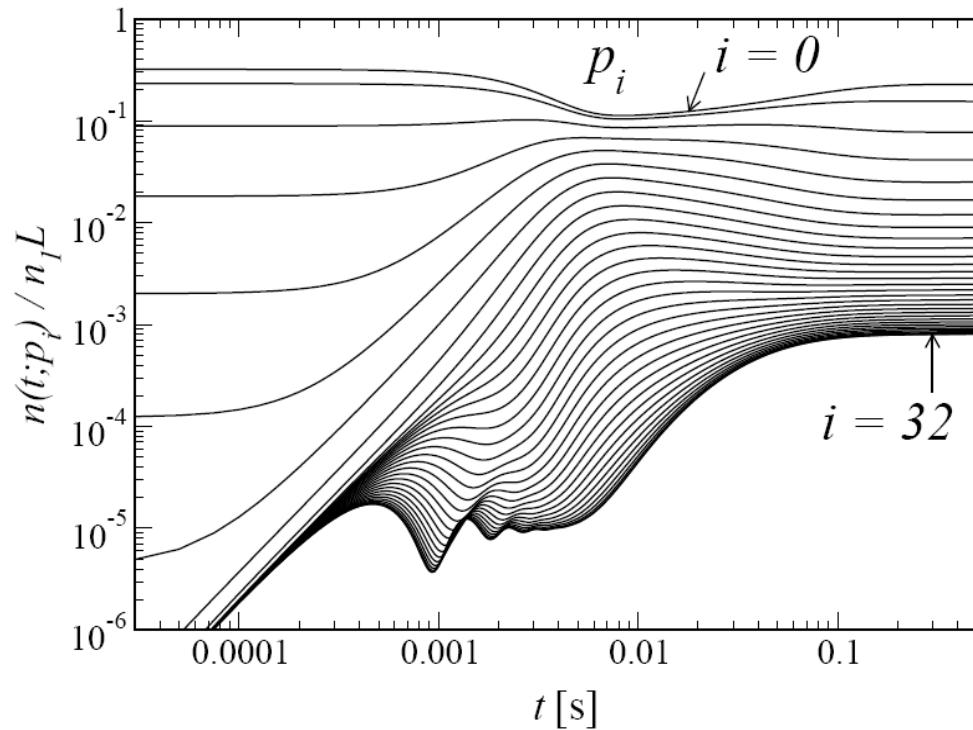
Momentum distribution for different times:

[TG, J. Berges, M. Seco & M.G.Schmidt, PRA (05); J. Berges & TG, PRA 76 (07)]



Far-from-equilibrium evolution

Time evolution of mode occupation no^s [J. Berges & TG, PRA 76 (07)]

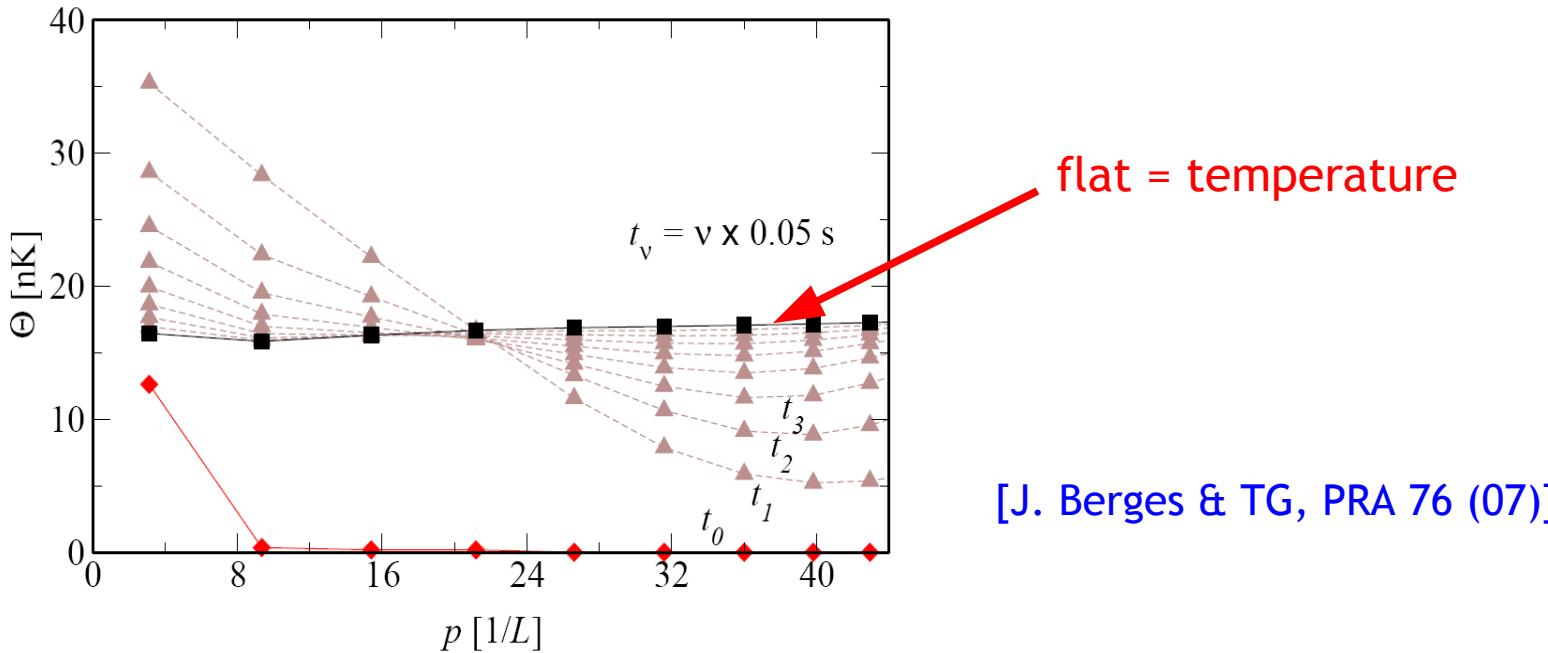


- initial state:
- ^{23}Na atoms in 1D, $n_1 = 10^7 \text{ m}^{-1}$
 - interaction parameter $\gamma = \lambda m / (\hbar^2 n_1) = 7.5 \cdot 10^{-4}$
 - Gaussian momentum distribution



Temperature appears

'Temperature' parameter $\Theta(p)$ at t_n



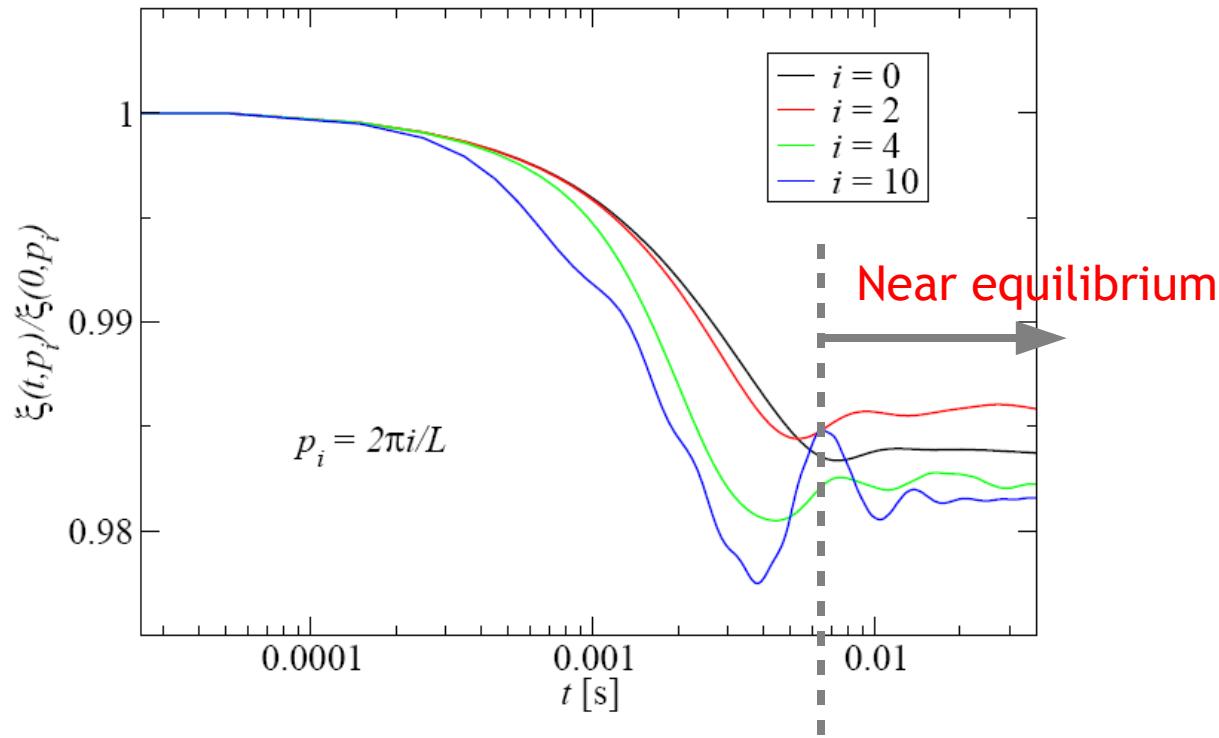
$$n(t; p) = \frac{1}{e^{\frac{1}{k_B \Theta(p)} (\frac{p^2}{2m} - \mu)} - 1}$$



Onset of near-equilibrium evolution

Time evolution of temporal correlations

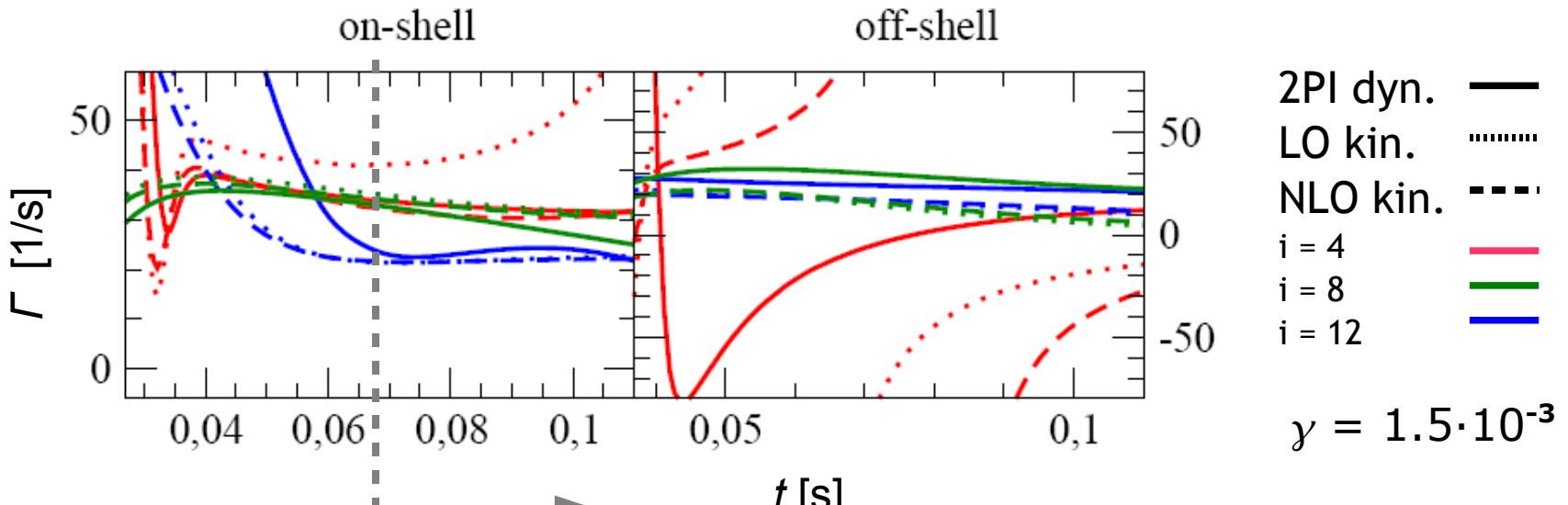
$$\xi(t, p) = F(t, 0; p)/\rho(t, 0; p):$$



$$(\text{Fluctuation-Dissipation rel.: } \mathbf{F}_{\omega_p}^{(\text{eq})} = -i (n(\omega, T) + \frac{1}{2}) \boldsymbol{\rho}_{\omega_p}^{(\text{eq})})$$

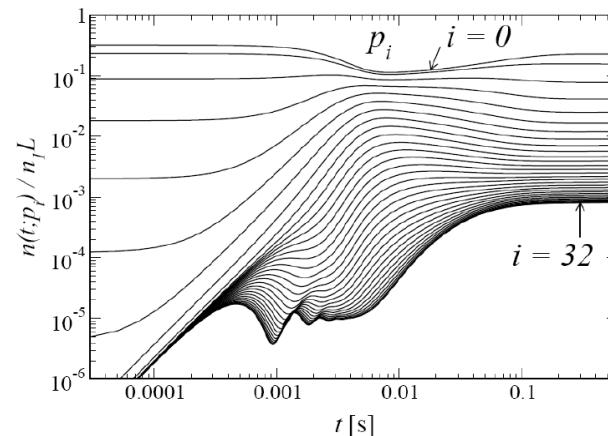


Comparison with kinetic theory



Time evol. of densities n :

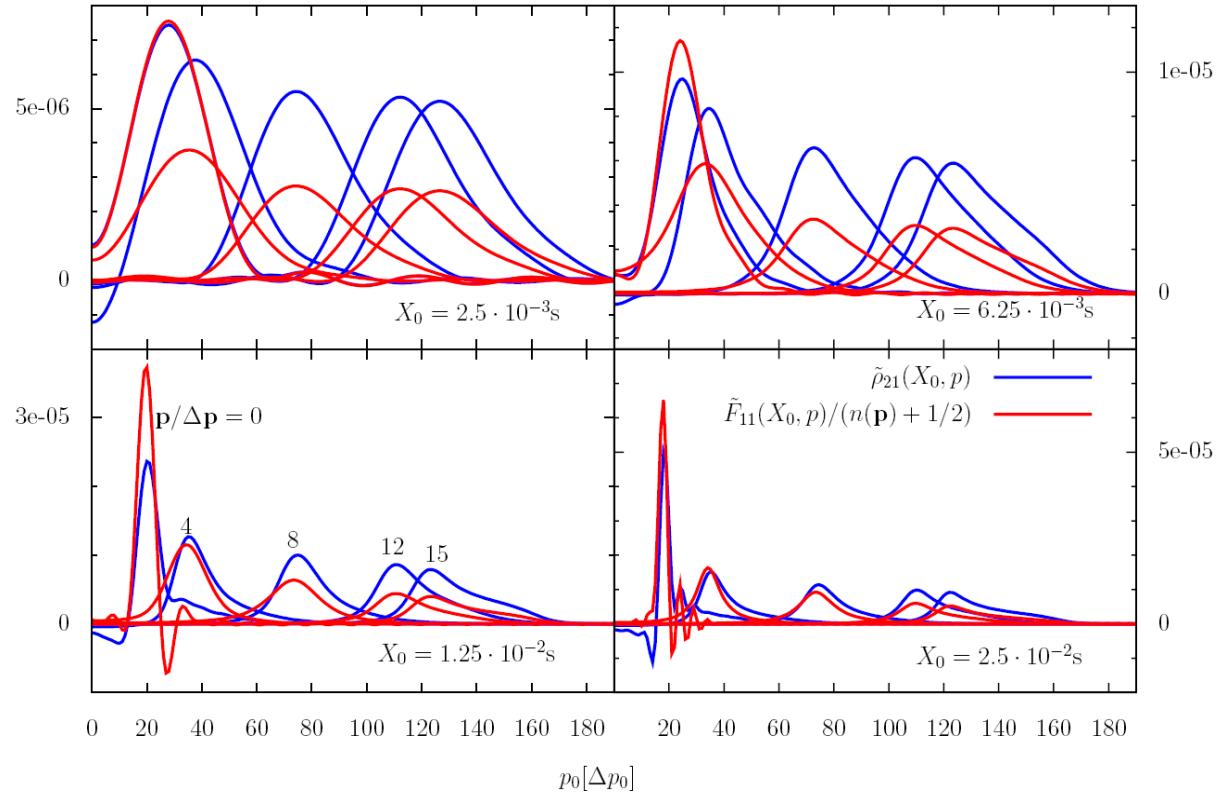
$$\Gamma = n''/n'$$



[A. Branschädel & TG, in prep. (07)]



Fluctuation-dissipation relation

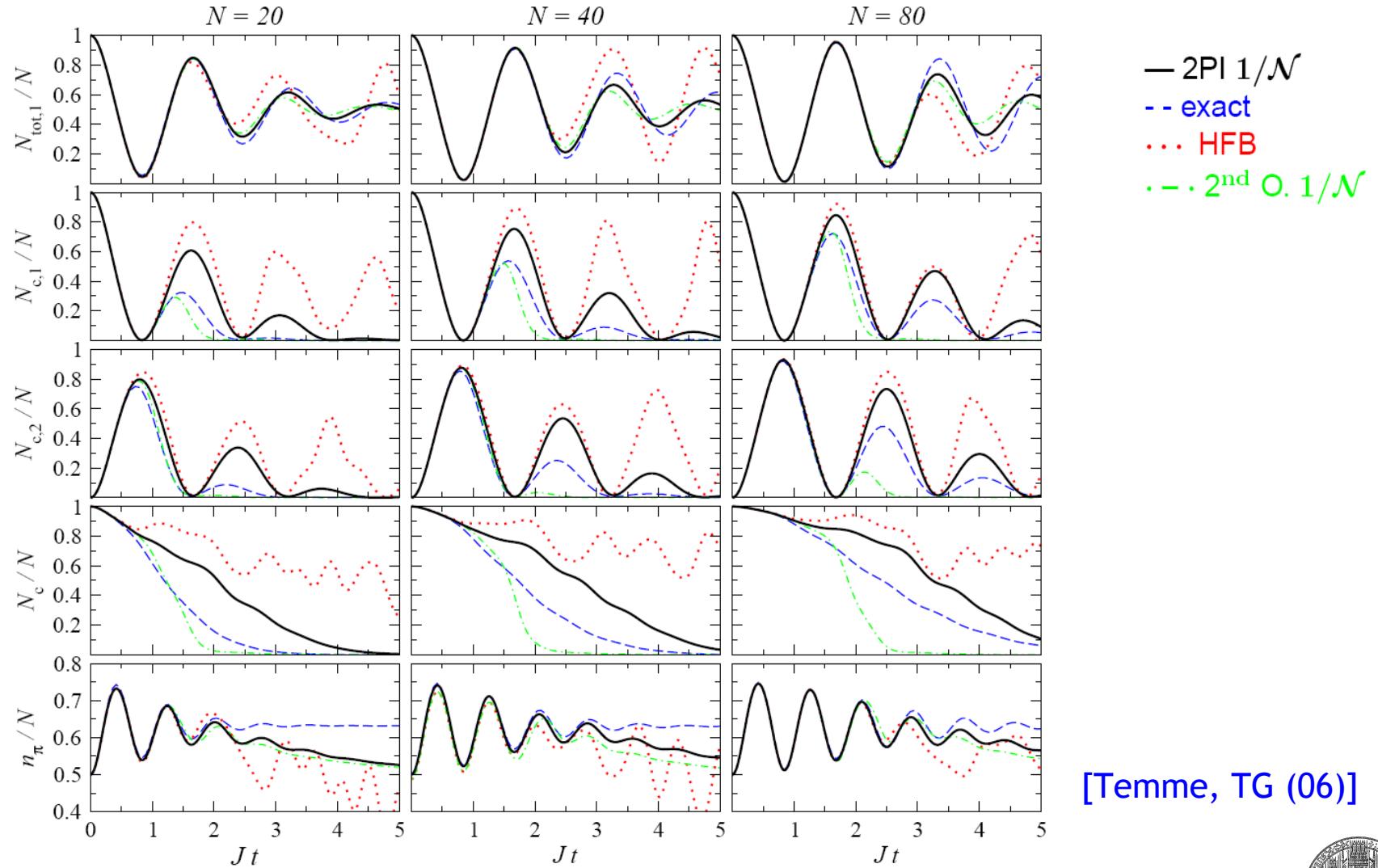


F = statistical corr. fct. – ρ = spectral fct.

→ fulfill fluctuation-dissipation relation $F_{\omega_p}^{(\text{eq})} = -i(n(\omega, T) + \frac{1}{2}) \rho_{\omega_p}^{(\text{eq})}$



Bose-Einstein condensate in lattice potential



[Temme, TG (06)]



Thanks & credits to...

...my work group in Heidelberg:

Cédric Bodet
Alexander Branschädel
Stefan Keßler
Matthias Kronenwett
Philipp Struck
Kristan Temme (\rightarrow Vienna)



...my collaborators

Jürgen Berges • Darmstadt
Hrvoje Buljan • Zagreb
Jan M. Pawłowski • Heidelberg
Robert Pezer • Sisak
Michael G. Schmidt • Heidelberg
Marcos Seco • Santiago de Compostela

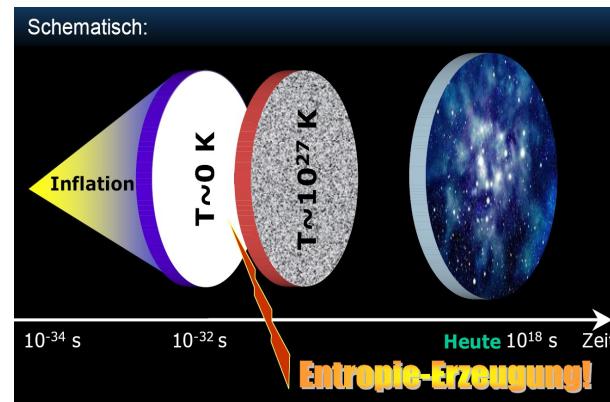
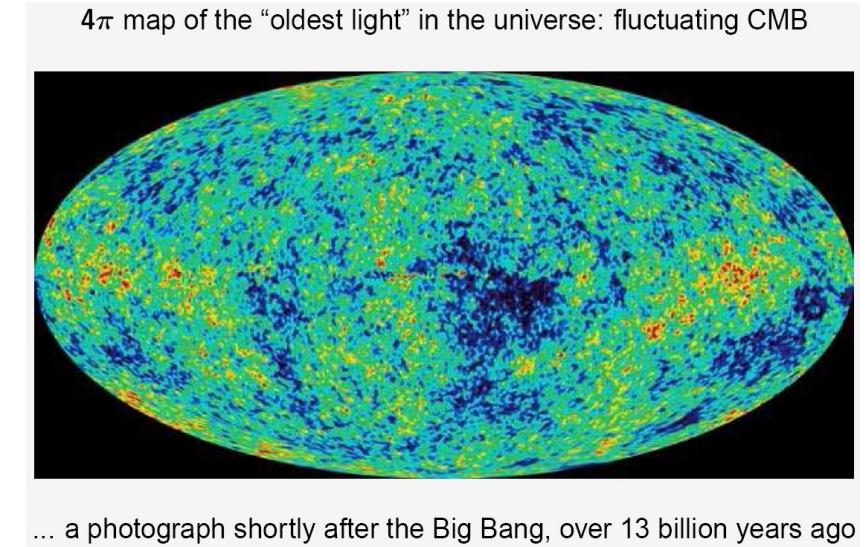
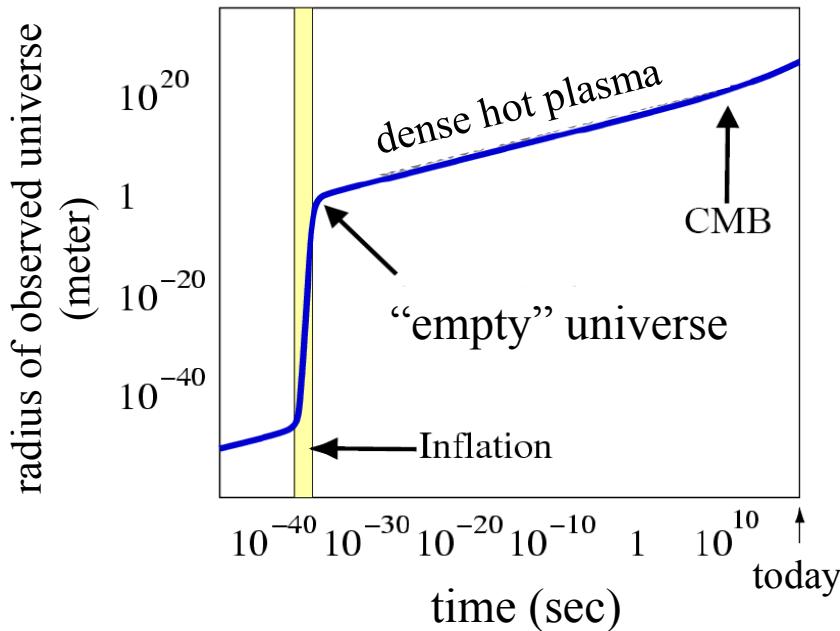
...also to

Keith Burnett • Oxford/Sheffield • David Hutchinson • Otago • Thorsten Köhler • UCL • Paul S. Julienne • NIST Gaithersburg • David Roberts • ENS Paris • Janne Ruostekoski • Southampton



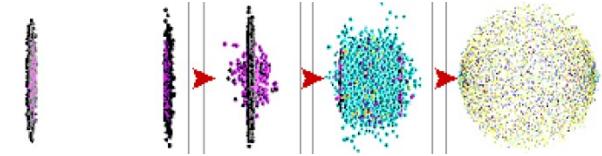
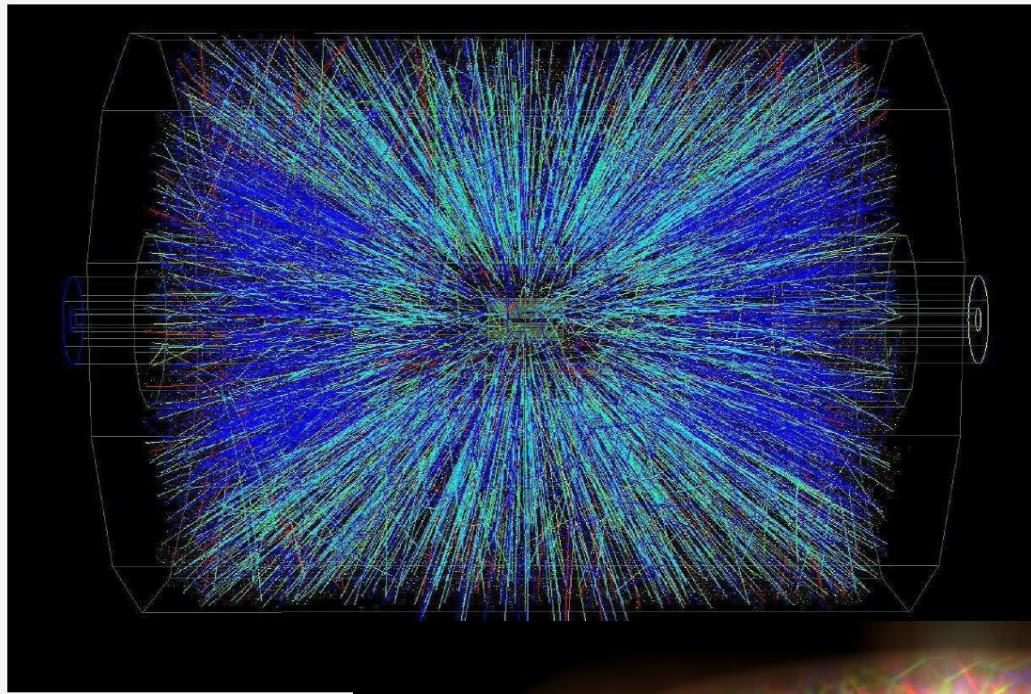
Outreach

Inflation dynamics



Heavy-Ion-Collisions

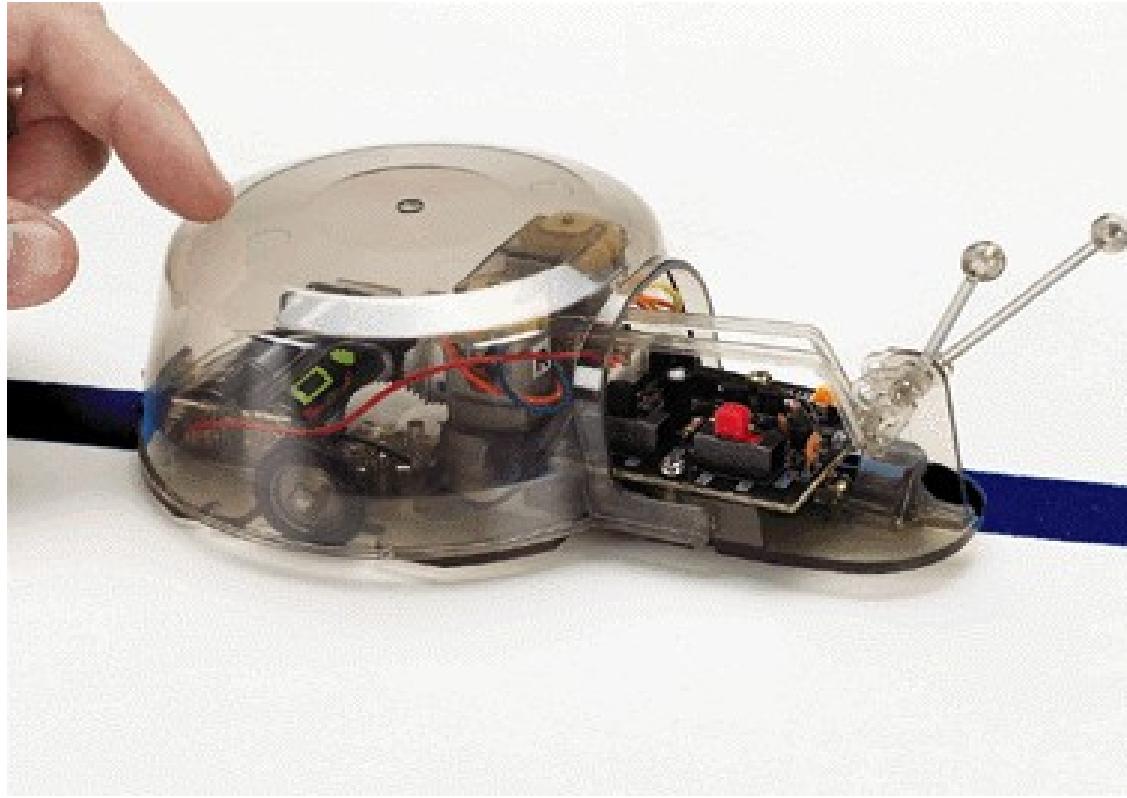
Result of colliding two Gold nuclei (Relativistic Heavy Ion Collider, BNL):



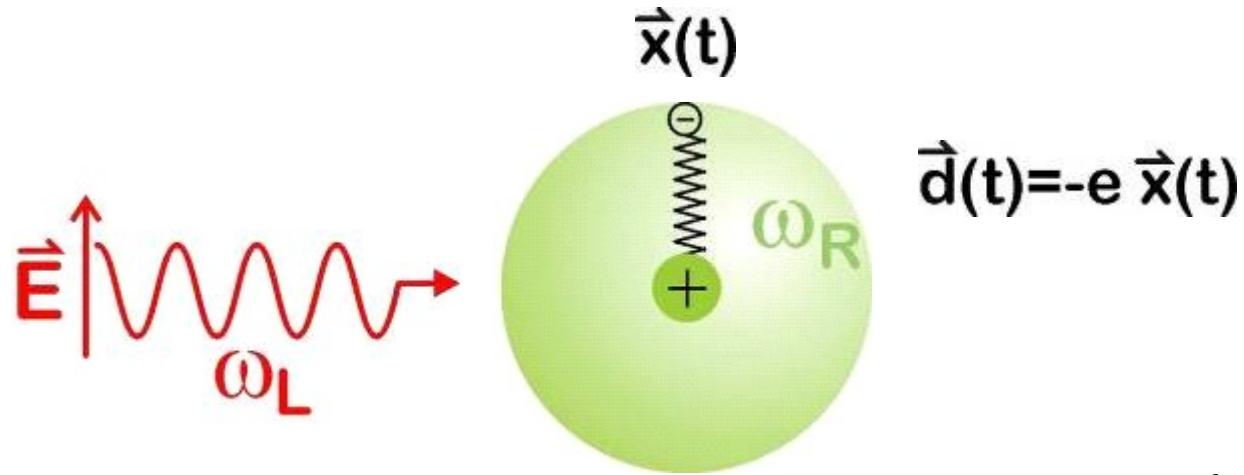
[Arrizabalaga, Tranberg,
Smit (05, 06),
Berges et al. (07)]



Classical Propagator



Atom-light-interactions - the way to produce almost arbitrary potentials



$$U = -\langle \vec{d} \cdot \vec{E} \rangle \propto \frac{\vec{E}^2}{\Delta} \propto \frac{I}{\Delta}$$

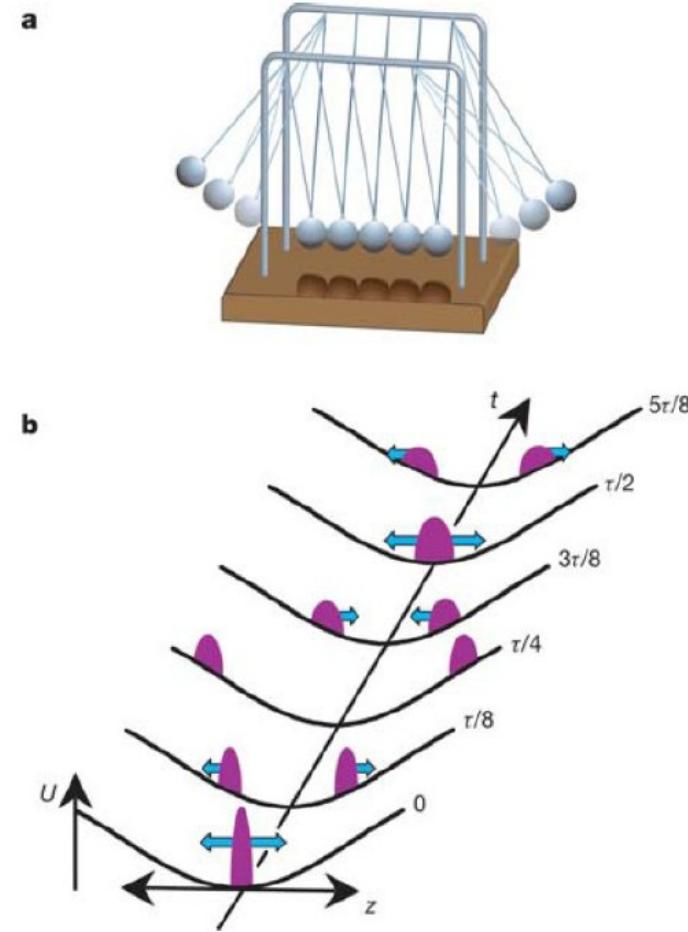
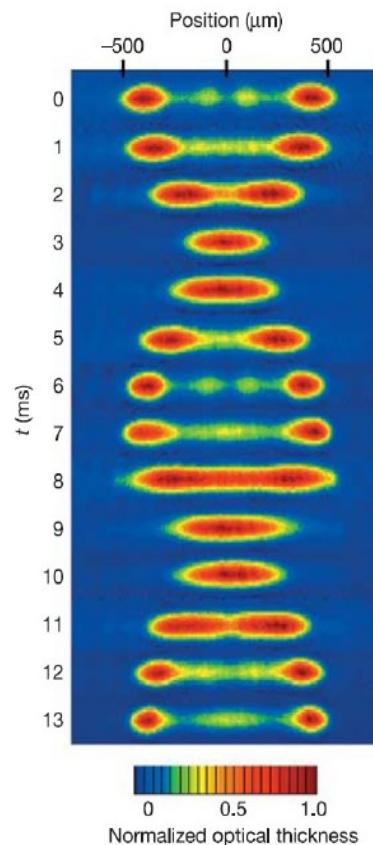
Simple example: Focused laser beam



Long-time dynamics of ultracold gases

A quantum Newton's cradle.

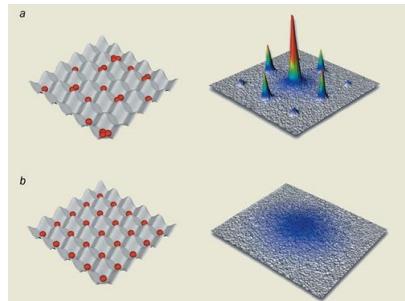
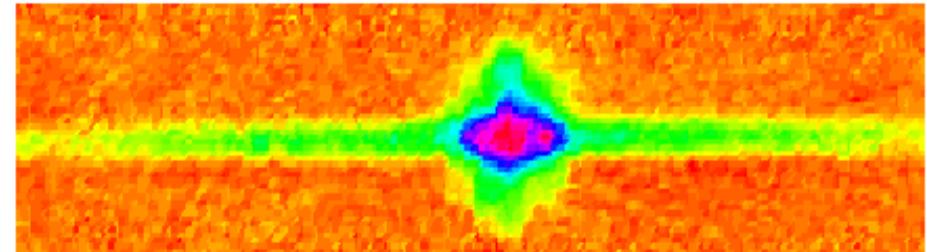
[T. Kinoshita et al. Nature 440 (06)]



“Strong” dynamics of ultracold gases

Feshbach resonances
allow fast changes of
collisional interaction strengths

”Bosenovae” [C. Wieman]



Superfluid - Mott-insulator
quantum phase transition

[I. Bloch]

Crossover from a Superfluid of Bosons
to a “Superconductor” of Fermion pairs

[JILA, Boulder]



Quantum Field Theory



Fields generally allow an **effective** description,
e.g. through

$$\rho(\mathbf{x},t), \mathbf{v}(\mathbf{x},t)$$

instead of **coordinates/velocities** of many particles;
similarly:

$$\phi_i(x) = \langle \Phi_i(x) \rangle \quad (\text{mean field})$$

$$G_{ij}(x,y) = \langle \Phi_i(x) \Phi_j(y) \rangle \quad (\text{density matrix})$$

$$x = (\mathbf{x},t)$$



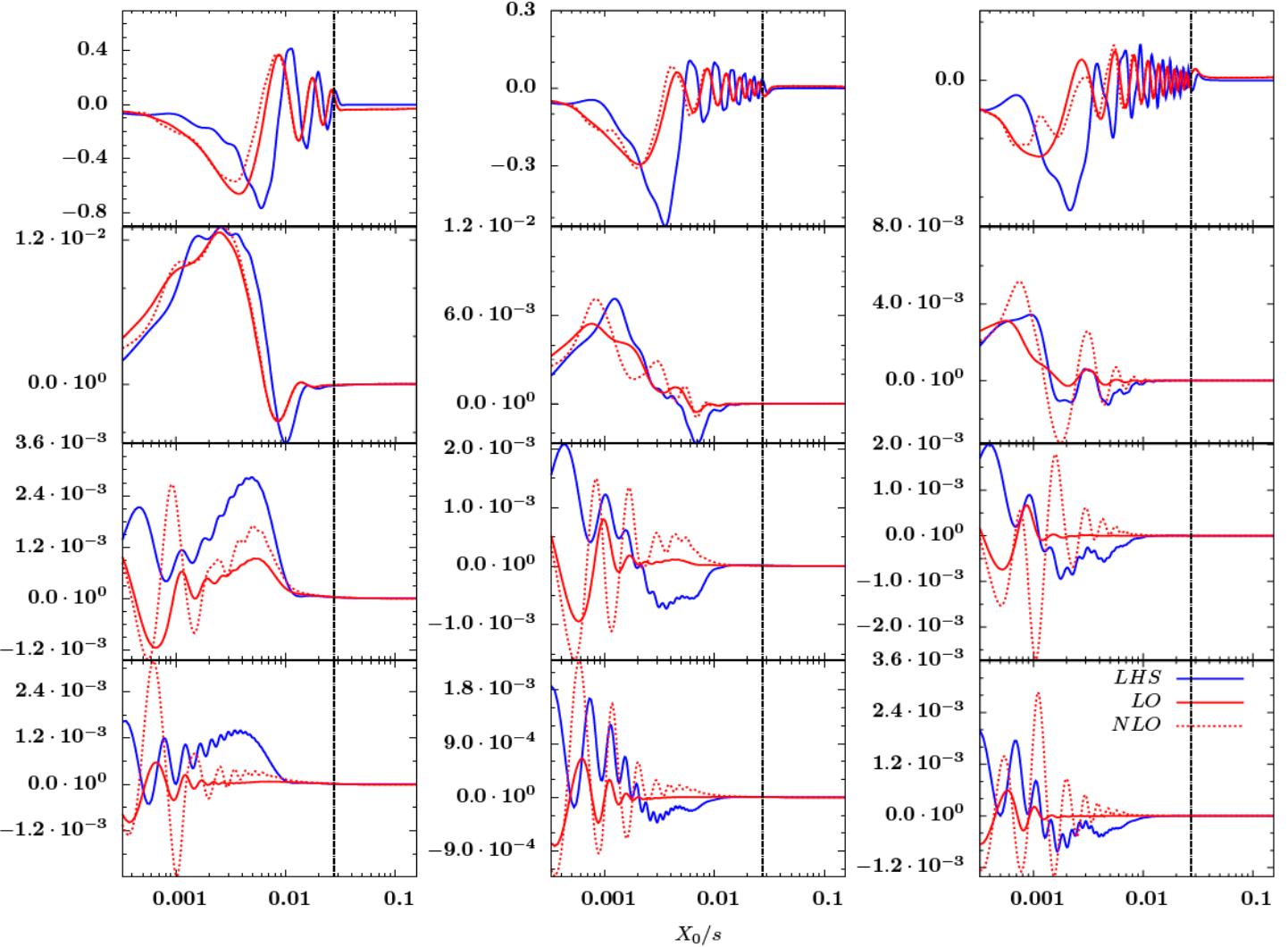
Comparison with kinetic theory

- Go to Wigner representation (central & relative times)
- Send initial time to minus infinity
- Fourier transform w.r.t. relative time
- Gradient expansion (Markov approx. & corrections)

[in our context cf.: [A.M. Rey *et al.*, PRA '05](#)]



Comparison with kinetic theory



$\gamma = 0.15$

(prelim.!)



Extensive work on

Kinetic Theories for Ultracold Quantum Gases

e.g.

- Semiclassical hydrodynamics
Griffin, Nikuni, Zaremba, ...
 - Quantum-Boltzmann equations, linear response theory:
Burnett, Giorgini, Proukakis, Rusch, Stoof, ...
 - Generalized master equations, quantum Boltzmann equations, non-Markovian extensions:
Bhongale, Cooper, Holland, Kokkelmans, Wachter, Walser, Williams, ...
 - Quantum stochastic master equations, quantum Boltzmann equations, classical simulations:
Ballagh, Burnett, Davis, Gardiner, Jaksch, Zoller, ...
 - Fokker-Planck equation, Langevin field equation, quantum Boltzmann equations:
Al Khawaja, Bijlsma, Proukakis, Stoof, ...
 - Greens-function approaches:
Boyanovsky, Griffin, Imamovic-Tomasovic, Clark, Rey, Hu, ...
- + many more on (finite-T) stationary properties
(Burnett, Castin, Clark, Fedichev, Griffin, Hutchinson, Morgan, Shlyapnikov, Stoof, Stringari, Williams, Zaremba, ...)



Observables

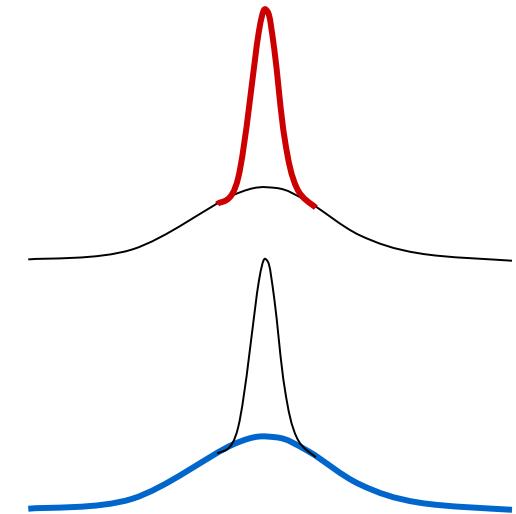
For **bosons**: $[\hat{\Phi}_{t,x}, \hat{\Phi}_{t,x'}^\dagger] = \delta(x - x')$

- Matter wave **mean field** [$x = (x_0, x) = (t, x)$]

$$\phi_x = \langle \hat{\Phi}_x \rangle, \quad |\phi_x|^2 = n_c(x) = \text{condensate density},$$

- Density of **non-condensed atoms** ($\hat{\Phi} = \phi + \tilde{\Phi}$, $\phi = \langle \hat{\Phi} \rangle$)

$$\langle \tilde{\Phi}_x^\dagger \tilde{\Phi}_x \rangle = n_{nc}(x) \equiv n(x) - n_c(x),$$



- Total one-body **density matrix**

$$G_{11}(x, y) = \langle \tilde{\Phi}_x^\dagger \tilde{\Phi}_y \rangle \Rightarrow \text{spatial Fourier transform: momentum distribution } n(\mathbf{p}, t) \\ \Rightarrow 1^{\text{st}}\text{-order phase coherence}$$

- **Anomalous one-body density** matrix

$$G_{12}(x, y) = \langle \tilde{\Phi}_x \tilde{\Phi}_y \rangle \Rightarrow \text{e.g., number of Bose-condensed bound pairs (molecules)}$$



Dynamic Equations

- **Exact** dynamic equations for **mean field** and **2-point correlators** \mathbf{F}, ρ :

$$\begin{aligned} & \left[-i\sigma_2 \partial_{x_0} - \frac{1}{2} \int_z V_{x-z} \phi_z \phi_z \right] \phi_x - \int_y M_{xy}[0, \mathbf{F}] \phi_y = \int_0^{x_0} dy \bar{\Sigma}_{xy}^\rho[0, \mathbf{G}] \phi_y, \\ & \left[-i\sigma_2 \delta_{xz} \partial_{z_0} - \int_z M_{xz}[\phi, \mathbf{F}] \right] \begin{pmatrix} \mathbf{F}_{zy} \\ \rho_{zy} \end{pmatrix} = \begin{pmatrix} \int_0^{x_0} dz \bar{\Sigma}_{xz}^\rho[\phi, \mathbf{G}] & -\int_0^{y_0} dz \bar{\Sigma}_{xz}^F[\phi, \mathbf{G}] \\ 0 & \int_{y_0}^{x_0} dz \bar{\Sigma}_{xz}^\rho[\phi, \mathbf{G}] \end{pmatrix} \begin{pmatrix} \mathbf{F}_{zy} \\ \rho_{zy} \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} M_{xy}[\phi, \mathbf{F}] &= \delta_{xy} \left[H_{1B}(x) + \frac{1}{2} \int_z V_{x-z} (\phi_z \phi_z + \mathbf{F}_{zz}) \right] + V_{x-y} (\phi_x \phi_y + \mathbf{F}_{xy}), \\ H_{1B}(x) &= -\frac{\Delta}{2m} + V_{\text{ext}}(x). \end{aligned}$$

Statistical and spectral correlation functions:

$$\begin{aligned} \langle \mathcal{T} \tilde{\Phi}_x \tilde{\Phi}_y \rangle &= \mathbf{G}_{xy} = \mathbf{F}_{xy} - \frac{i}{2} \text{sign}_{\mathcal{C}}(x_0 - y_0) \rho_{xy} \\ \implies \quad \mathbf{F}_{xy} &= \langle \{\tilde{\Phi}_x, \tilde{\Phi}_y\} \rangle, \\ \rho_{xy} &= \langle [\tilde{\Phi}_x, \tilde{\Phi}_y] \rangle, \end{aligned}$$



Classicality condition

[J. Berges, TG, PRA (to appear)]

Under the rescaling $\varphi_a(x) \rightarrow \varphi'_a(x) = \sqrt{g} \varphi_a(x),$
 $\tilde{\varphi}_a(x) \rightarrow \tilde{\varphi}'_a(x) = (1/\sqrt{g}) \tilde{\varphi}_a(x)$

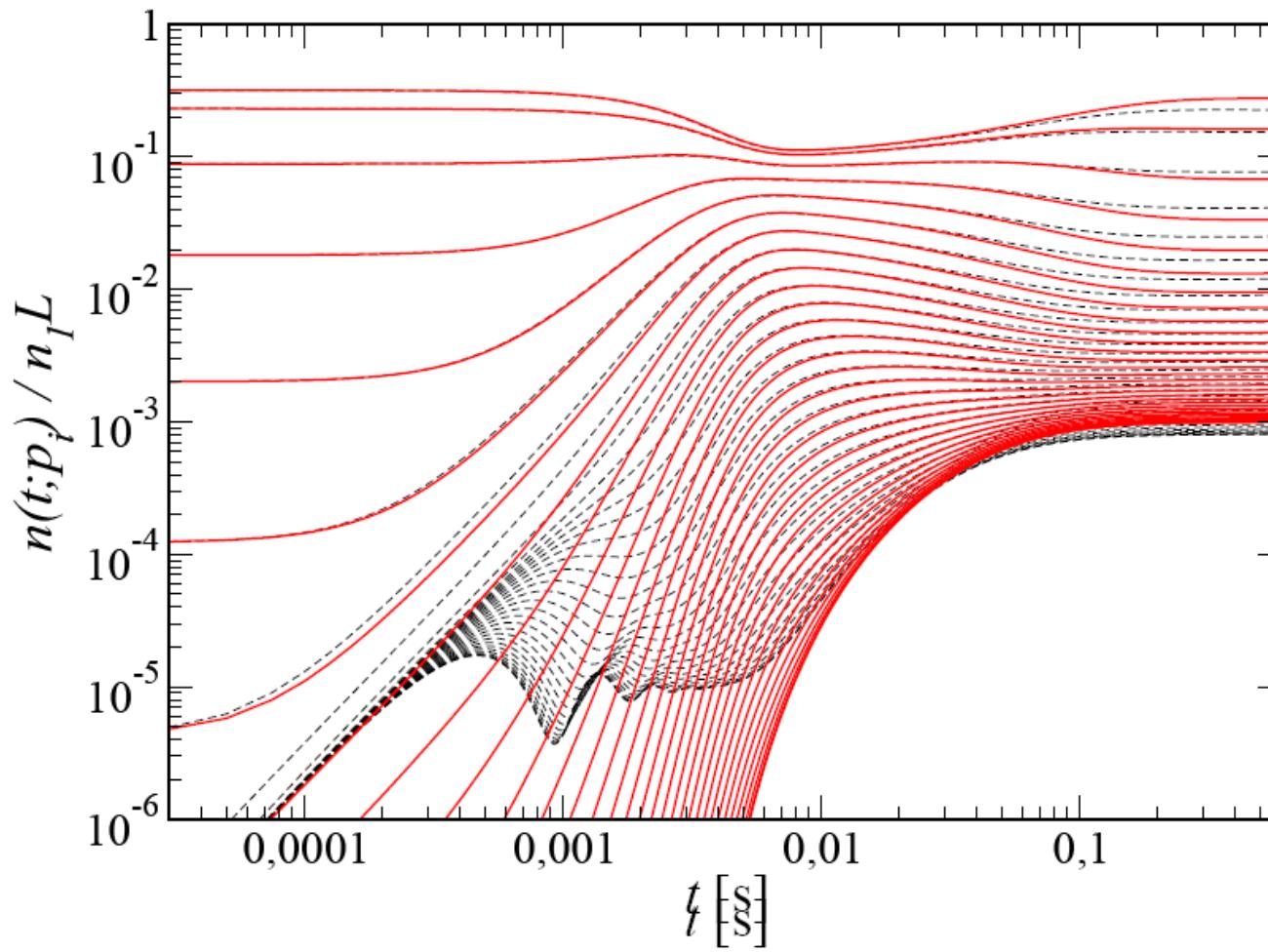
the interaction part becomes $\tilde{\varphi}' \varphi'^3 + g \tilde{\varphi}'^3 \varphi'$

Classicality condition:

$$|F'_{ab}(x, y) F'_{cd}(z, w)| \gg \frac{3}{4} g^2 |\rho_{ab}(x, y) \rho_{cd}(z, w)|$$



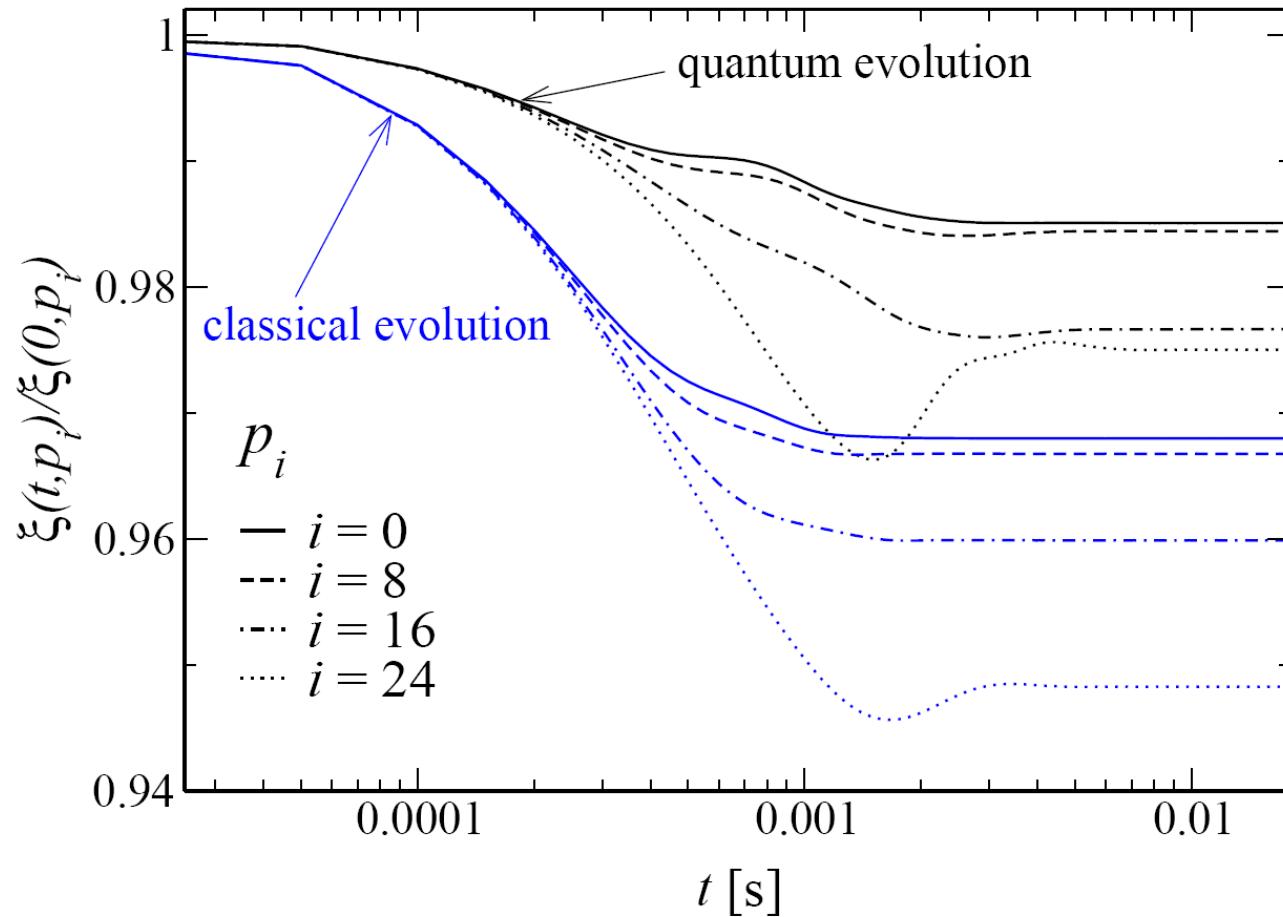
Far-from-equilibrium evolution



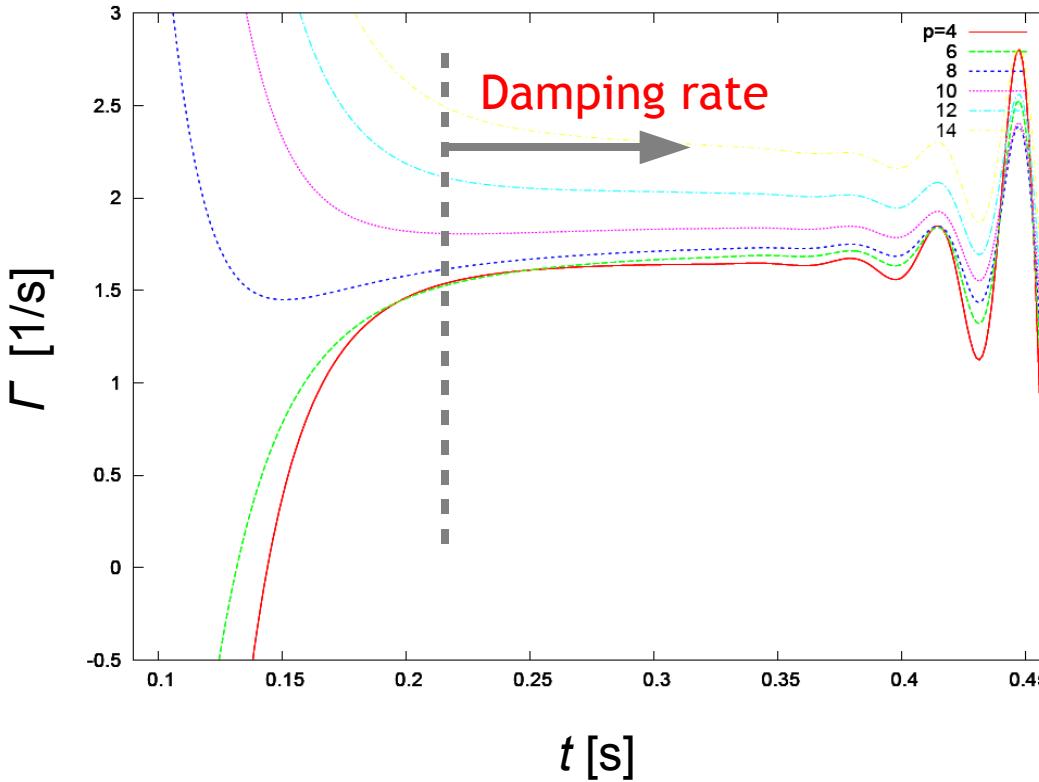
Strong coupling

Time evolution of temporal correlations

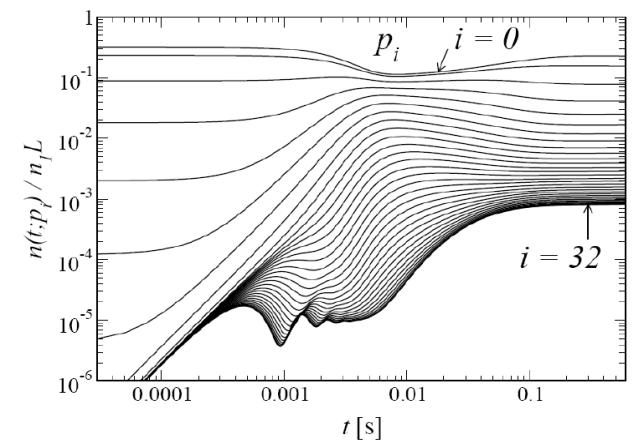
$$\xi(t, p) = F(t, 0; p)/\rho(t, 0; p):$$



Comparison with kinetic theory



Time evol. of densities n :
 $\Gamma = n''/n'$



[A. Branschädel & TG, in prep. (07)]

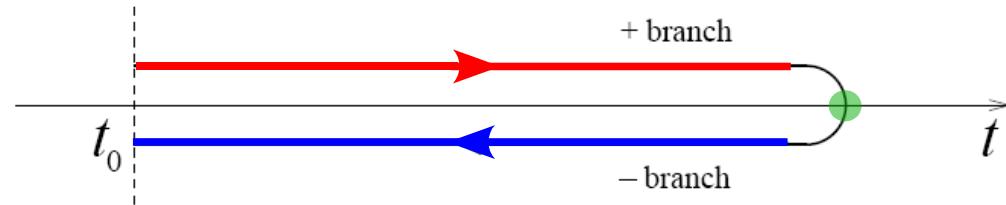
$$\gamma = 1.5 \cdot 10^{-3}$$



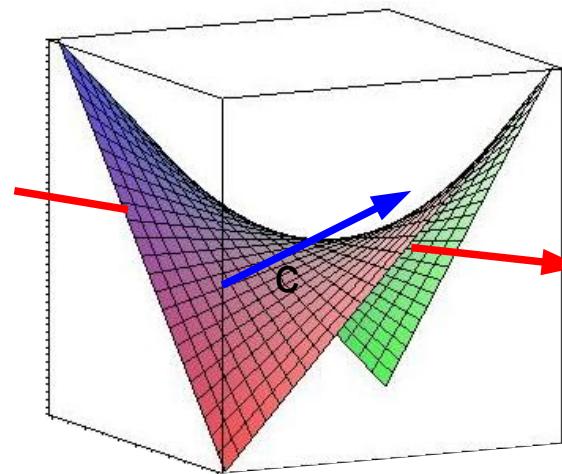
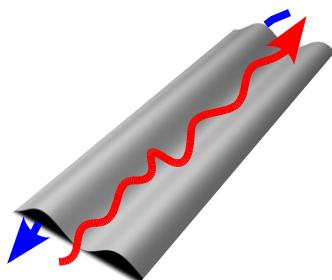
Initial value problem

$$\langle t | O | t \rangle = \langle t_0 | U^\dagger(t) O U(t) | t_0 \rangle = Z^{-1} \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} O e^{i(S[\varphi] - S[\bar{\varphi}])/\hbar}$$

Schwinger-Keldysh
closed time path:



Quadratic action (QM Harm. Osc.): $S[\varphi] \sim \int dt \{ (\partial_t \varphi)^2 - \varphi^2 \}$:



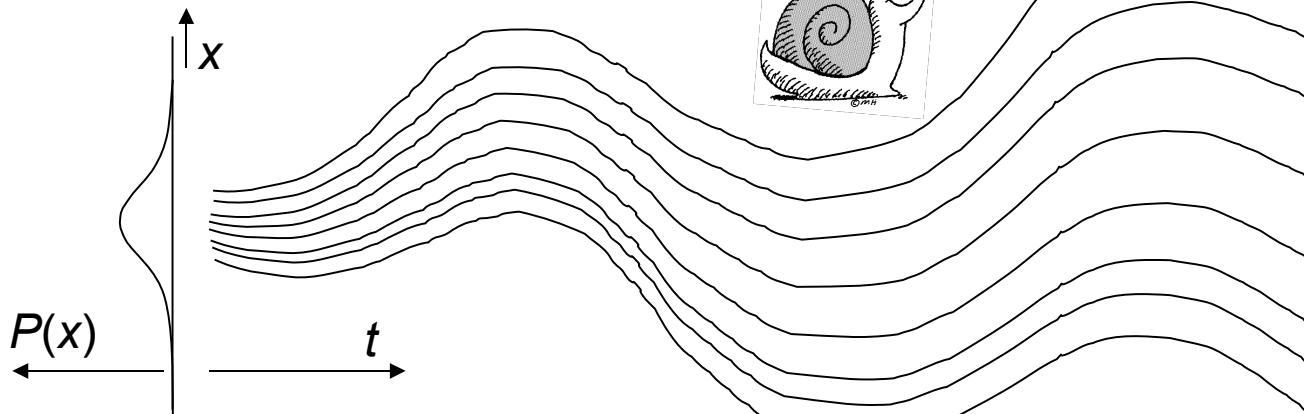
$$\begin{aligned} \varphi^2 - \bar{\varphi}^2 &= (\varphi - \bar{\varphi})(\varphi + \bar{\varphi}) \\ &=: \tilde{\varphi}\varphi \end{aligned}$$



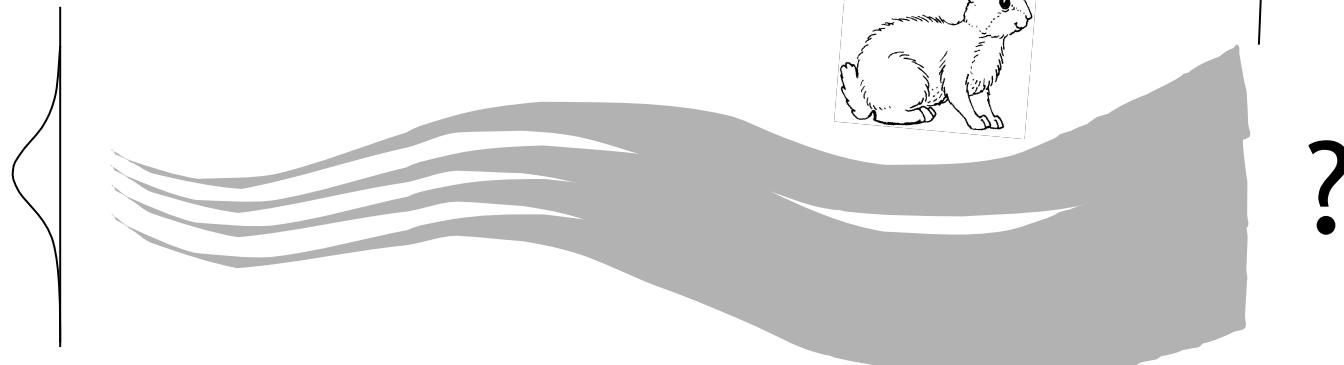
Evolving quantum fields...

...are difficult to describe due to quantum fluctuations.

Classical statistical evolution...



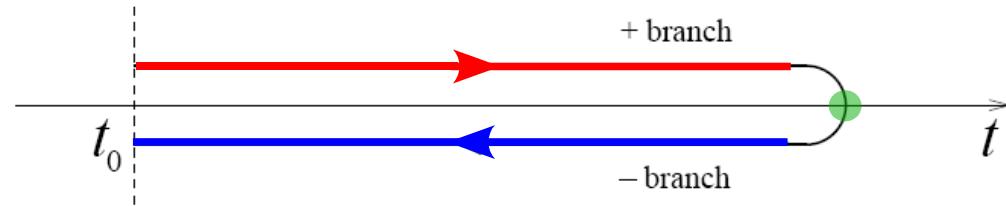
...vs quantum statistical evolution:



Initial value problem

$$\langle t | O | t \rangle = \langle t_0 | U^\dagger(t) O U(t) | t_0 \rangle = Z^{-1} \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} O e^{i(S[\varphi] - S[\bar{\varphi}])/\hbar}$$

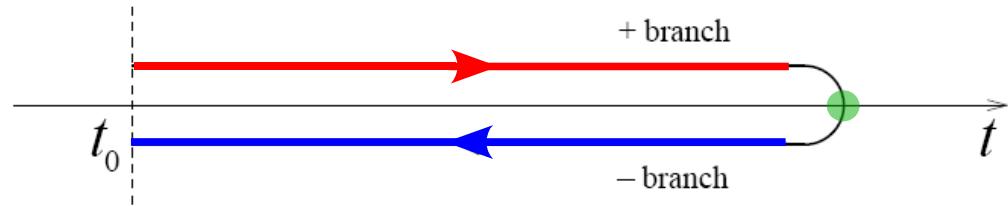
Schwinger-Keldysh
closed time path:



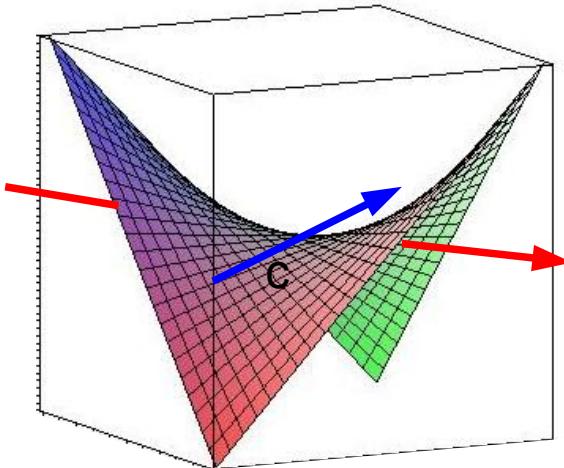
Initial value problem

$$\langle t | O | t \rangle = \langle t_0 | U^\dagger(t) O U(t) | t_0 \rangle = Z^{-1} \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} O e^{i(S[\varphi] - S[\bar{\varphi}])/\hbar}$$

Schwinger-Keldysh
closed time path:



Quadratic action (QM Harm. Osc.): $S[\varphi] \sim \int dt \{ (\partial_t \varphi)^2 - \varphi^2 \}$:



$$\begin{aligned} \varphi^2 - \varphi^2 &= (\varphi - \varphi)(\varphi + \varphi) \\ &=: \tilde{\varphi} \varphi \end{aligned}$$



Classical Path Integral

[J. Berges, TG, PRA (07)]

Consider QM Harm. Osc.:

$$\begin{aligned} S[\varphi] - S[\phi] &\sim -\int dt \{ \varphi (\partial_t^2 + \omega^2) \varphi^2 - \phi (\partial_t^2 - \omega^2) \phi^2 \} \\ &\sim -\int dt \tilde{\varphi} (\partial_t^2 + \omega^2) \varphi \end{aligned}$$



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Path integral evaluates to classical solution:

$$\begin{aligned} &\int \mathcal{D}\varphi \mathcal{D}\phi O \rho[\varphi_0, \phi_0] e^{i(S[\varphi] - S[\phi])/\hbar} \\ &\sim \int \mathcal{D}\tilde{\varphi} \mathcal{D}\varphi O \rho[\tilde{\varphi}_0, \varphi_0] \exp[-\int dt \tilde{\varphi}(\partial_t^2 + \omega^2)\varphi/\hbar] \end{aligned}$$



Classical Path Integral

[J. Berges, TG, PRA (07)]

Consider QM Harm. Osc.:

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Path integral evaluates to classical solution:

$$\begin{aligned} &\int \mathcal{D}\varphi \mathcal{D}\phi O \rho[\varphi_0, \phi_0] e^{i(S[\varphi] - S[\phi])/\hbar} \\ &\sim \int \mathcal{D}\tilde{\varphi} \mathcal{D}\varphi O \rho[\tilde{\varphi}_0, \varphi_0] \exp[-\int dt \tilde{\varphi}(\partial_t^2 + \omega^2)\varphi/\hbar] \\ &\sim \int \mathcal{D}\varphi O \mathcal{W}[\pi_0, \varphi_0] \delta[(\partial_t^2 + \omega^2)\varphi] \end{aligned}$$



Classical Path Integral

[J. Berges, TG, PRA (07)]

Consider QM Harm. Osc.:

$$S[\varphi] - S[\phi] \sim -\int dt \{ \varphi(\partial_t^2 + \omega^2)\varphi^2 - \phi(\partial_t^2 - \omega^2)\phi^2 \}$$

$$\sim -\int dt \tilde{\varphi}(\partial_t^2 + \omega^2)\varphi$$

Path integral evaluates to classical solution:

Not with interactions!
 $\text{g } (\varphi^4 - \phi^4) = \text{g } (\tilde{\varphi}\varphi^3 + \tilde{\varphi}^3\varphi)$

$$\int \mathcal{D}\varphi \mathcal{D}\phi O \rho[\varphi_0, \phi_0] e^{i(S[\varphi] - S[\phi])/\hbar}$$

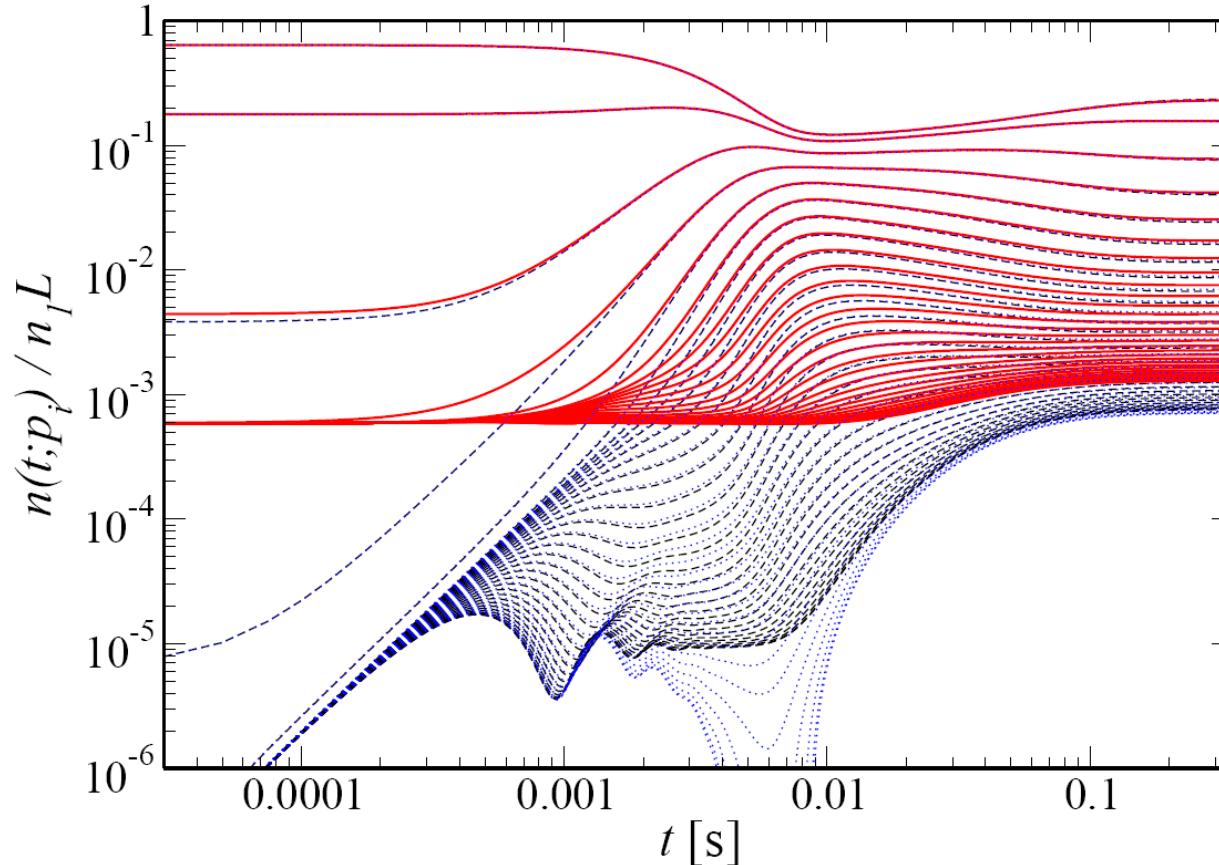
$$\sim \int \mathcal{D}\tilde{\varphi} \mathcal{D}\varphi O \rho[\tilde{\varphi}_0, \varphi_0] \exp[-\int dt \tilde{\varphi}(\partial_t^2 + \omega^2)\varphi/\hbar]$$

$$\sim \int \mathcal{D}\varphi O \mathcal{W}[\pi_0, \varphi_0] \delta[(\partial_t^2 + \omega^2)\varphi]$$



Classical vs. quantum evolution

Occupation numbers according to quantum dynamic equations...

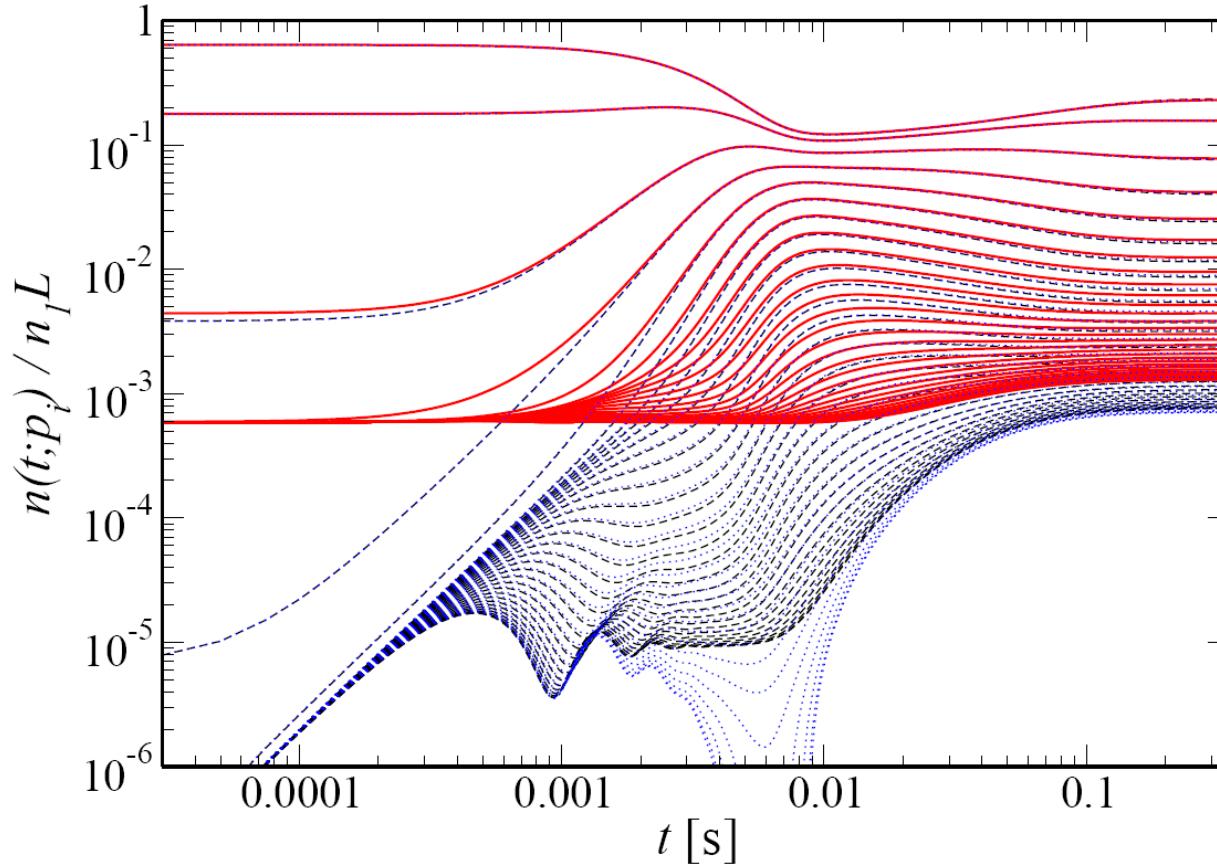


...vs. classical evolution for 'quantum' initial conditions.



Classical vs. quantum evolution

Occupation numbers according to quantum dynamic equations...



Compare to eq. **fluctuation-dissipation rel.:** $\overline{F^2(t, t'; p)} \simeq (n(t, p) + \frac{1}{2})^2 \overline{\rho^2(t, t'; p)}$.

