Chiral Fermions on the Worldline

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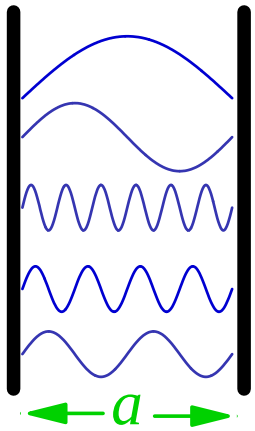
Functional Determinants

[COLLECTION BY G.V. DUNNE]

$$\det \mathcal{D}$$

- many applications in quantum field theory:
  - effective action, free energy, partition function
  - quantum forces, quantum equations of motion
  - tunneling rates, false vacuum decay,
  - unquenching in lattice gauge theory
- mathematical physics:
  - spectral properties of PDEs
- very few general techniques:
  - Laplacians in special backgrounds
  - zeta function
  - Gradient/heat-kernel expansion, WKB
  - lattice discretization
  - worldline numerics



For example: effective action Γ

▷ Gaußian fluctuations in a background A

(HEISENBERG&EULER'36)

(WEISSKOPF'36)

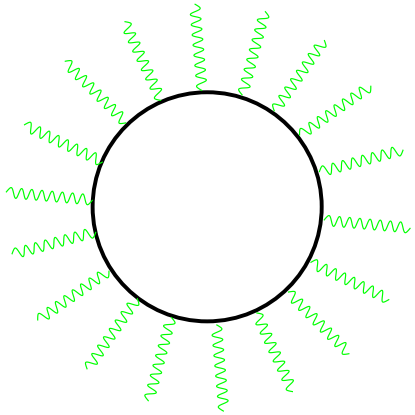
(CASIMIR'48)

(SCHWINGER'51)

$$\int \text{fluctuations} \quad \to \quad \Gamma[A]$$

$$\Gamma[A] = -\ln \int \mathcal{D}\phi \, e^{-\int -|D(A)\phi|^2 + m^2|\phi|^2} =$$

$$= \ln\det\Big(-D(A)^2 + m^2 \Big)$$



▷ spectrum of quantum fluctuations:

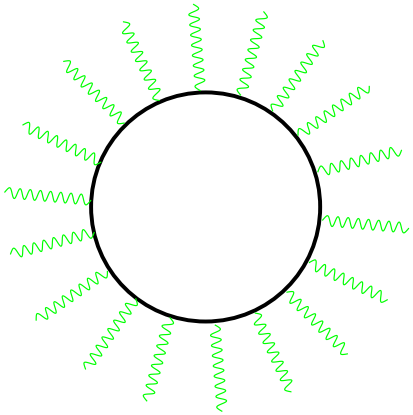
ScQED: $-D(A)^2 \; \phi = \lambda^2 \; \phi$

Scalar: $(-\partial^2 + A(x))\phi = \lambda^2 \; \phi$

Fermions: $i\not{D}[A(x)]\psi = \lambda \; \psi$

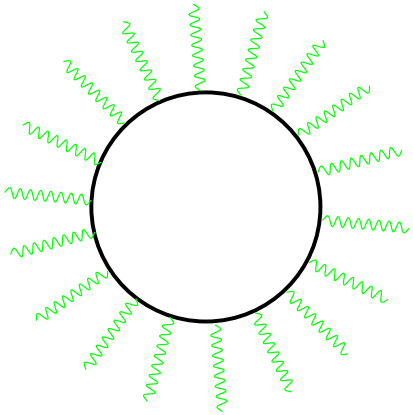
For example: effective action Γ

$$\Gamma[A] \quad = \quad \ln\det\big(-D(A)^2+m^2\big) \quad =$$



For example: effective action Γ

$$\Gamma[A] \quad = \quad \sum_{\lambda} \ln\left(\lambda^2 + m^2\right) \quad =$$



- ▷ Problem solved, "in principle"
  - find spectrum λ for a given background A
  - sum over spectrum

For example: effective action Γ

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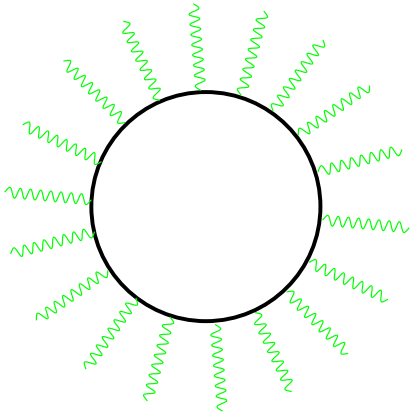
$$\Gamma[A] \quad = \quad \sum_{\lambda} \ln\left(\lambda^2 + m^2\right) \quad =$$



BUT:

▷ In general practice:

- spectrum $\{\lambda\}$ not known analytically
- spectrum $\{\lambda\}$ not bounded
- $\sum_\lambda \to \infty$ (regularization)
- renormalization

Worldline representation of $\det \mathcal{D}$

▷ pedestrian approach

$$\Gamma[A] = \ln\det\big(-D(A)^2+m^2\big) = \mathrm{Tr}\,\ln\Big[-(D(A))^2+m^2\Big]$$

$$= -\int_{1/\Lambda^2}^{\infty}\frac{dT}{T}\,e^{-m^2T}\,\underbrace{\mathrm{Tr}\,\exp\Big[D(A)^2\,T\Big]}_{=\langle x|e^{-HT}|x\rangle}$$

Worldline representation of $\det \mathcal{D}$

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$$= -\int_{1/\Lambda^2}^{\infty}\frac{dT}{T}\,e^{-m^2T}\,\mathcal{N}\int_{x(T)=x(0)}\mathcal{D}x(\tau)\,e^{-\int_0^T d\tau\left(\frac{\dot{x}^2}{4}+ie\,\dot{x}\cdot A(x(\tau))\right)}$$

Worldline representation of $\det \mathcal{D}$

$$\Gamma[A] \;=\; -\int_{1/\Lambda^2}^{\infty} \frac{dT}{T}\, e^{-m^2 T}\; \mathcal{N} \int_{x(T)=x(0)} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ie\,\dot{x}\cdot A(x\tau)\right)}$$

(FEYNMAN'50)

⋮

(HALPERN&SIEGEL'77)

(POLYAKOV'87)

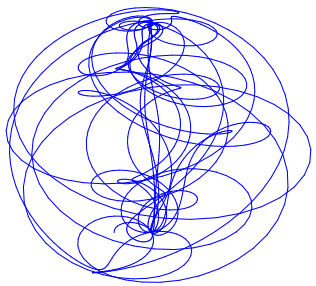
(FERNANDEZ,FRÖHLICH,SOKAL'92)

⋮

(BERN&KOSOWER'92; STRASSLER'92)

(SCHMIDT&SCHUBERT'93)

$x(T) \;=$



Worldline representation of $\det \mathcal{D}$

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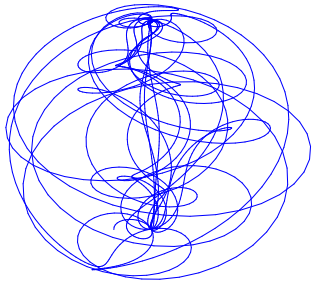
$x(T) \;=$ 

- ▷ Worldline approach:
  - effective action $\Gamma \sim \det \mathcal{D} \sim \int$ closed worldlines $x(\tau)$
  - worldline $\sim$ spacetime trajectory of ϕ fluctuations
  - gauge-field interaction $\sim$ "Wegner-Wilson loop"
  - finding $\{\lambda\}$ and $\sum_\lambda$ done in one finite (numerical) step

(HG&Langfeld'01)

Worldline Numerics

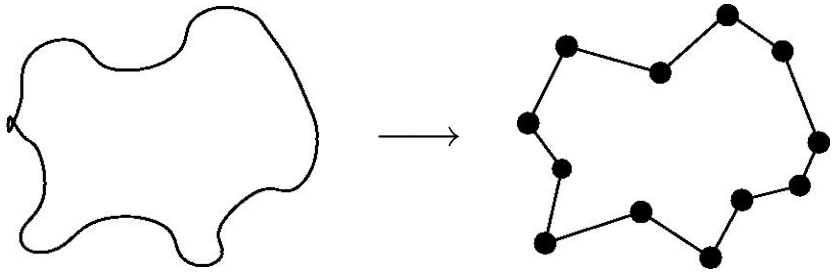
(HG&LANGFELD'01)

$$\int_{x(1)=x(0)} \mathcal{D}x(t) \quad \longrightarrow \quad \sum_{l=1}^{n_{\mathrm{L}}}, \qquad n_{\mathrm{L}} = \# \text{ of worldlines}$$

$\longrightarrow$ statistical error

$$x(t) \quad \longrightarrow \quad x_i, \qquad i = 1, \ldots, N \text{ (ppl)}$$

$\longrightarrow$ systematical error

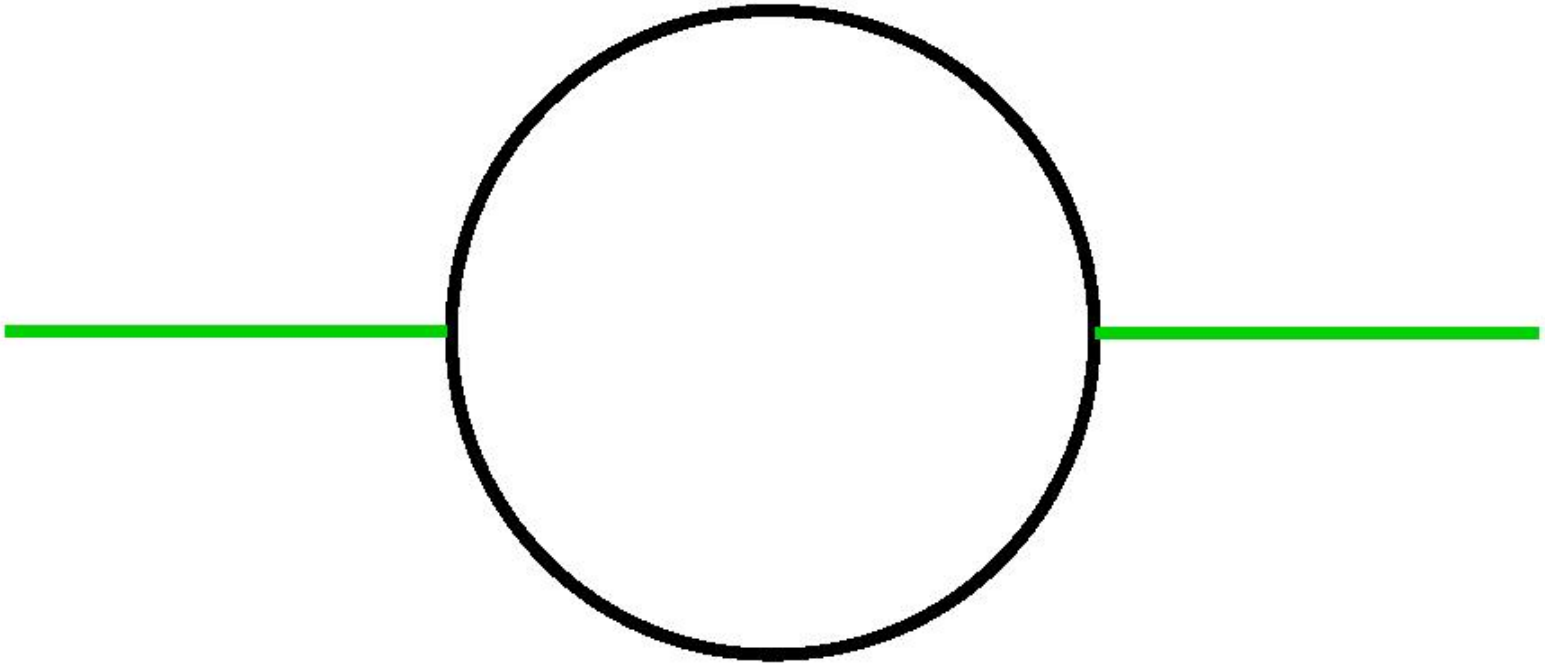


$\longrightarrow$ spacetime remains continuous

lattice version: (SCHMIDT&STAMATESCU'02)

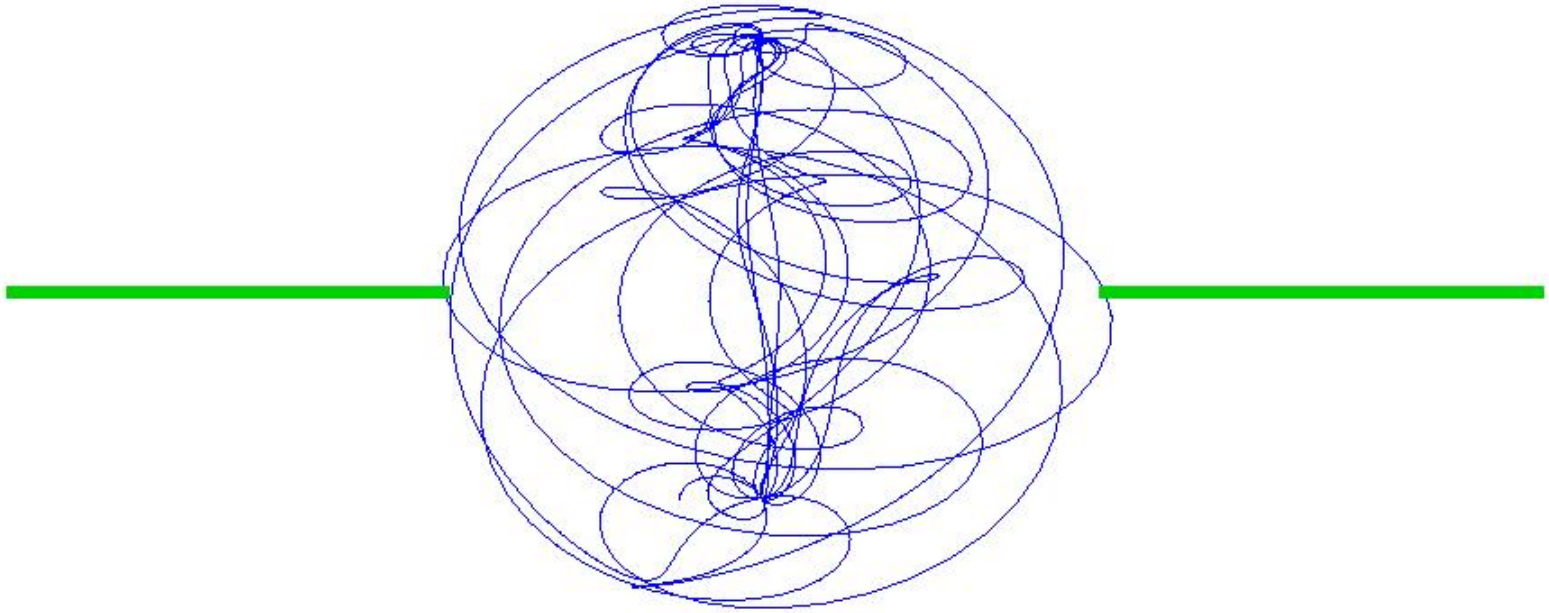
Trajectory of a Quantum Fluctuation

- Feynman diagram (conventionally in momentum space)



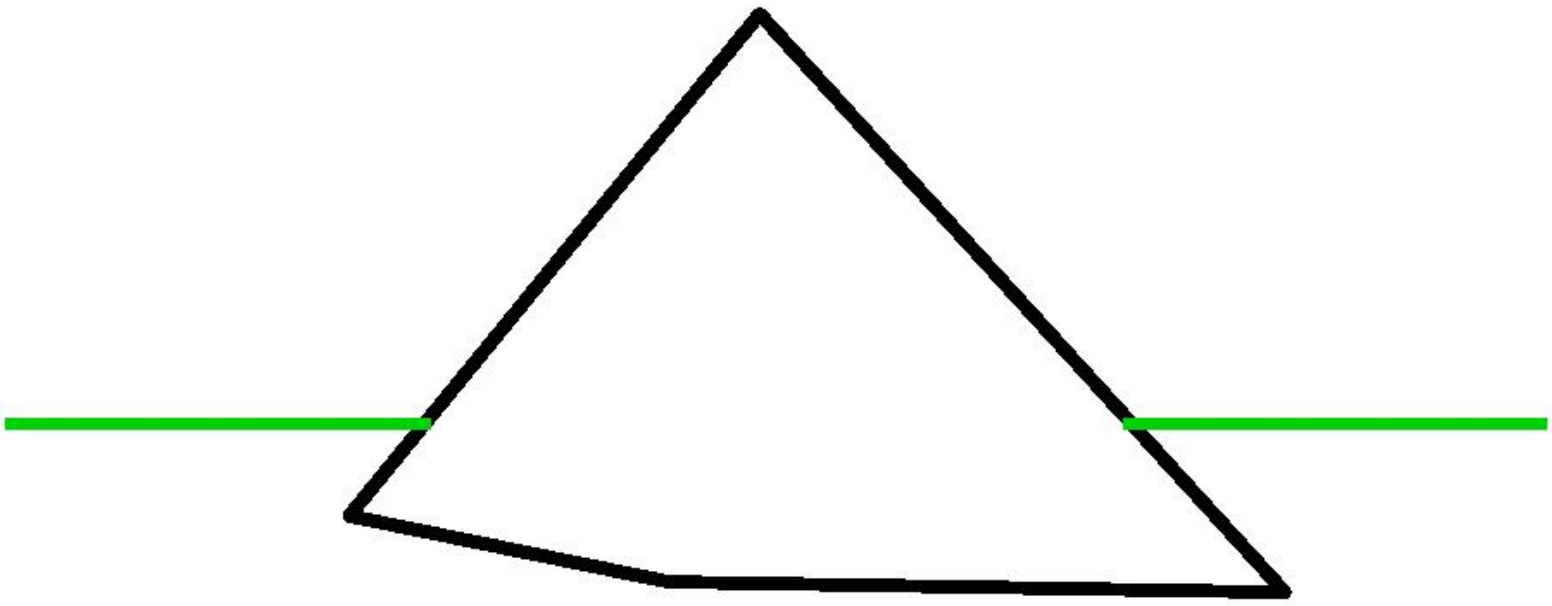
Trajectory of a Quantum Fluctuation

- worldline (artist's view)



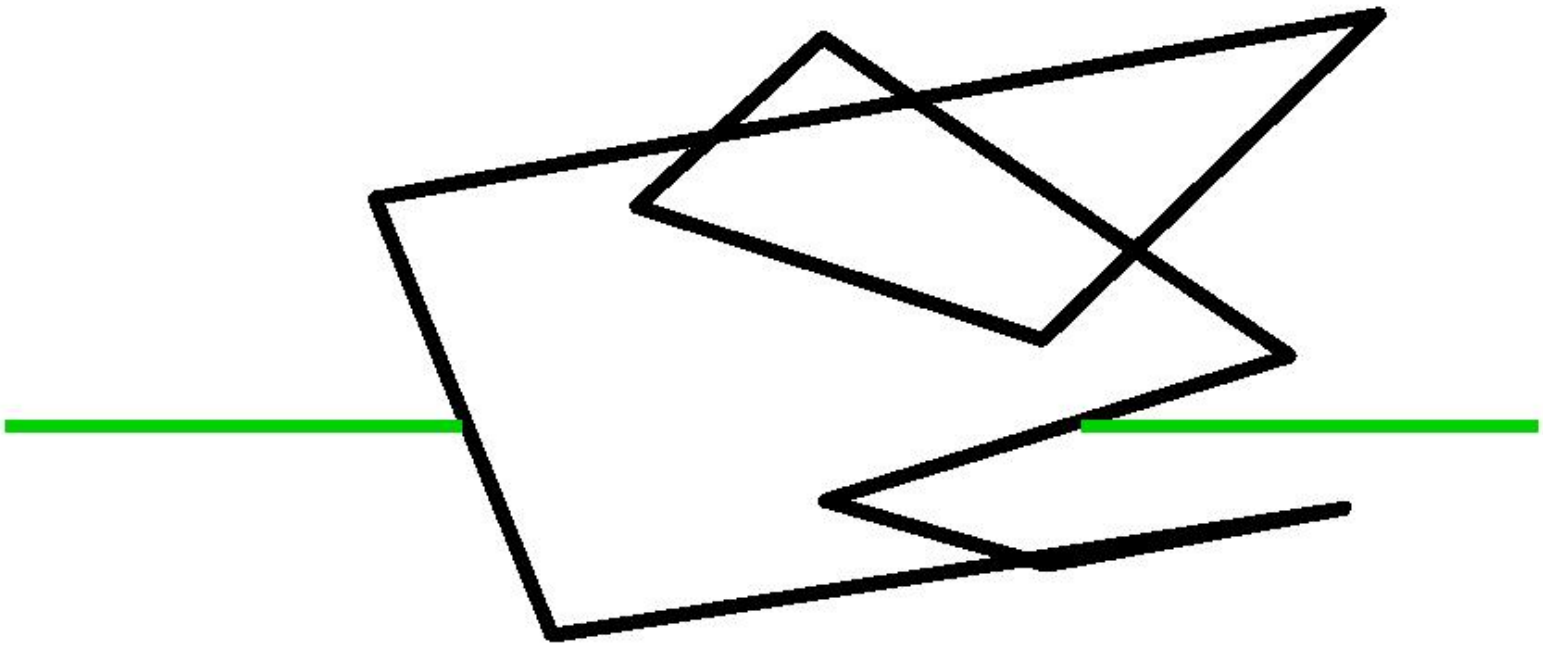
Trajectory of a Quantum Fluctuation

- worldline numerics: $N = 4$ points per loop (ppl)



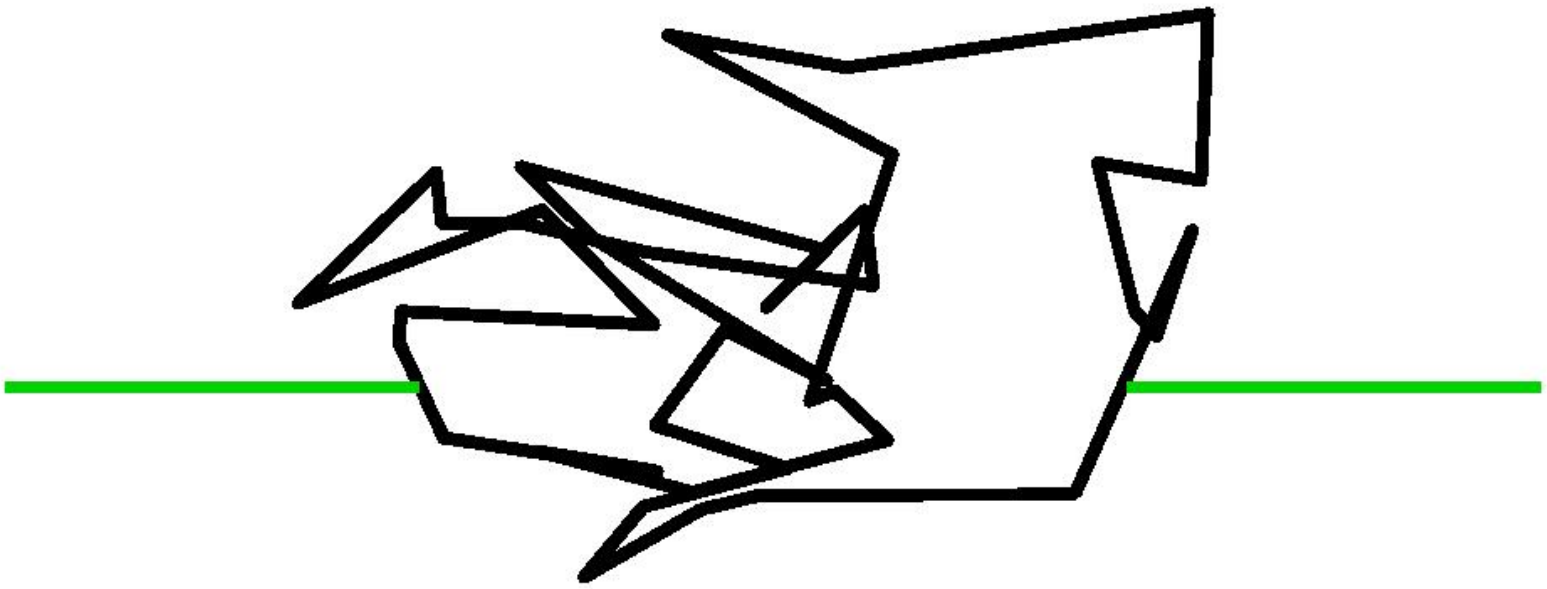
Trajectory of a Quantum Fluctuation

- worldline numerics: $N = 10$ points per loop (ppl)



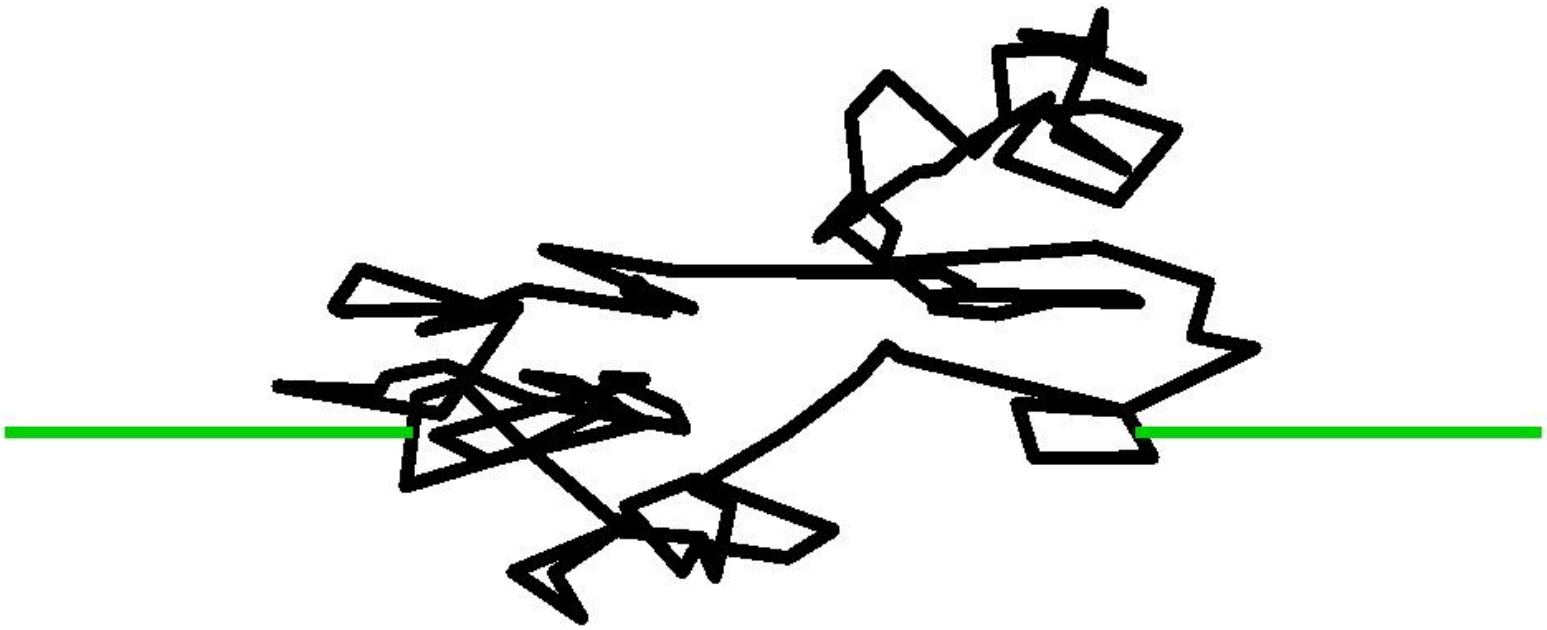
Trajectory of a Quantum Fluctuation

▷ worldline numerics: $N = 40$ points per loop (ppl)



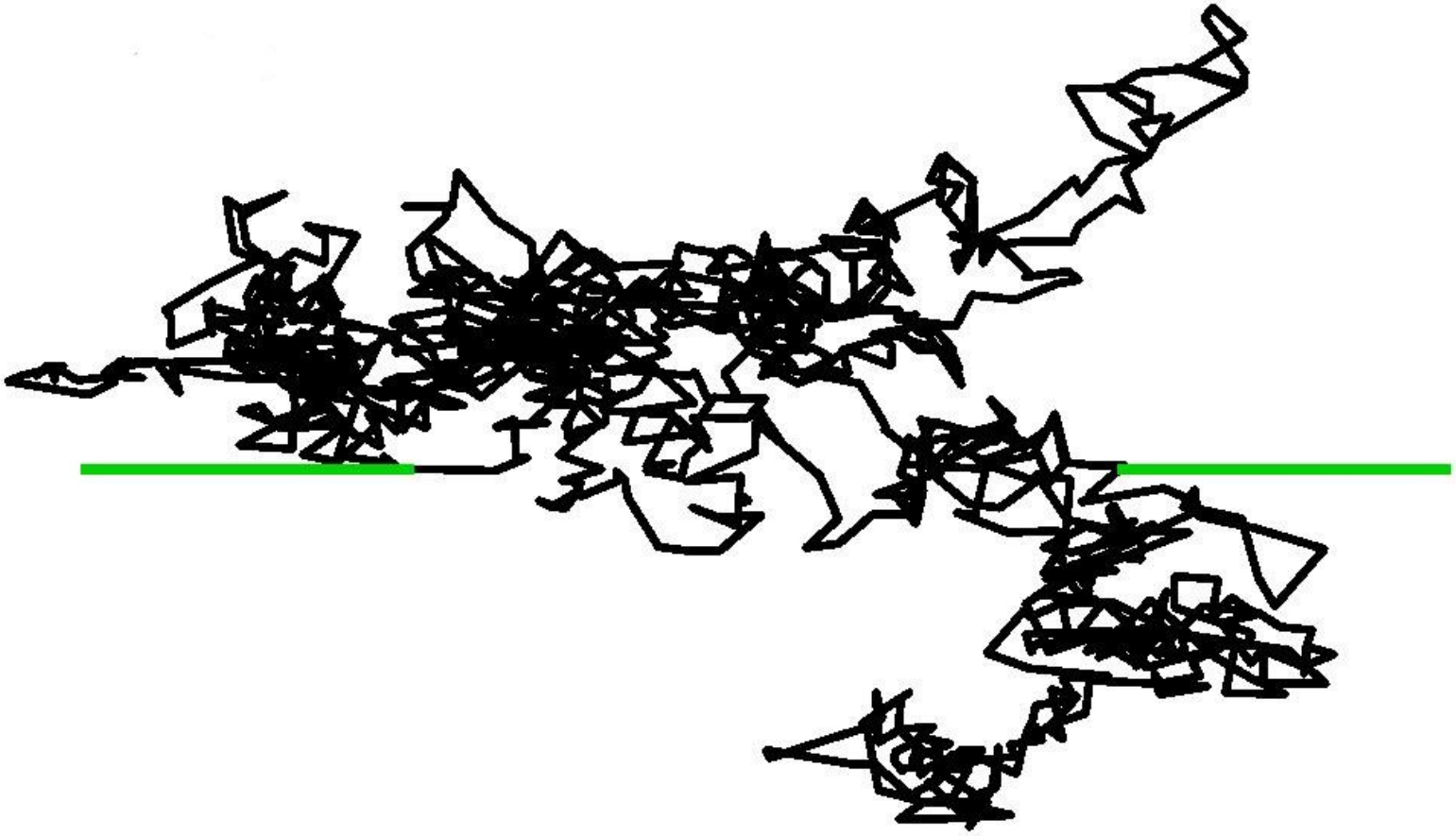
Trajectory of a Quantum Fluctuation

- worldline numerics: $N = 100$ points per loop (ppl)



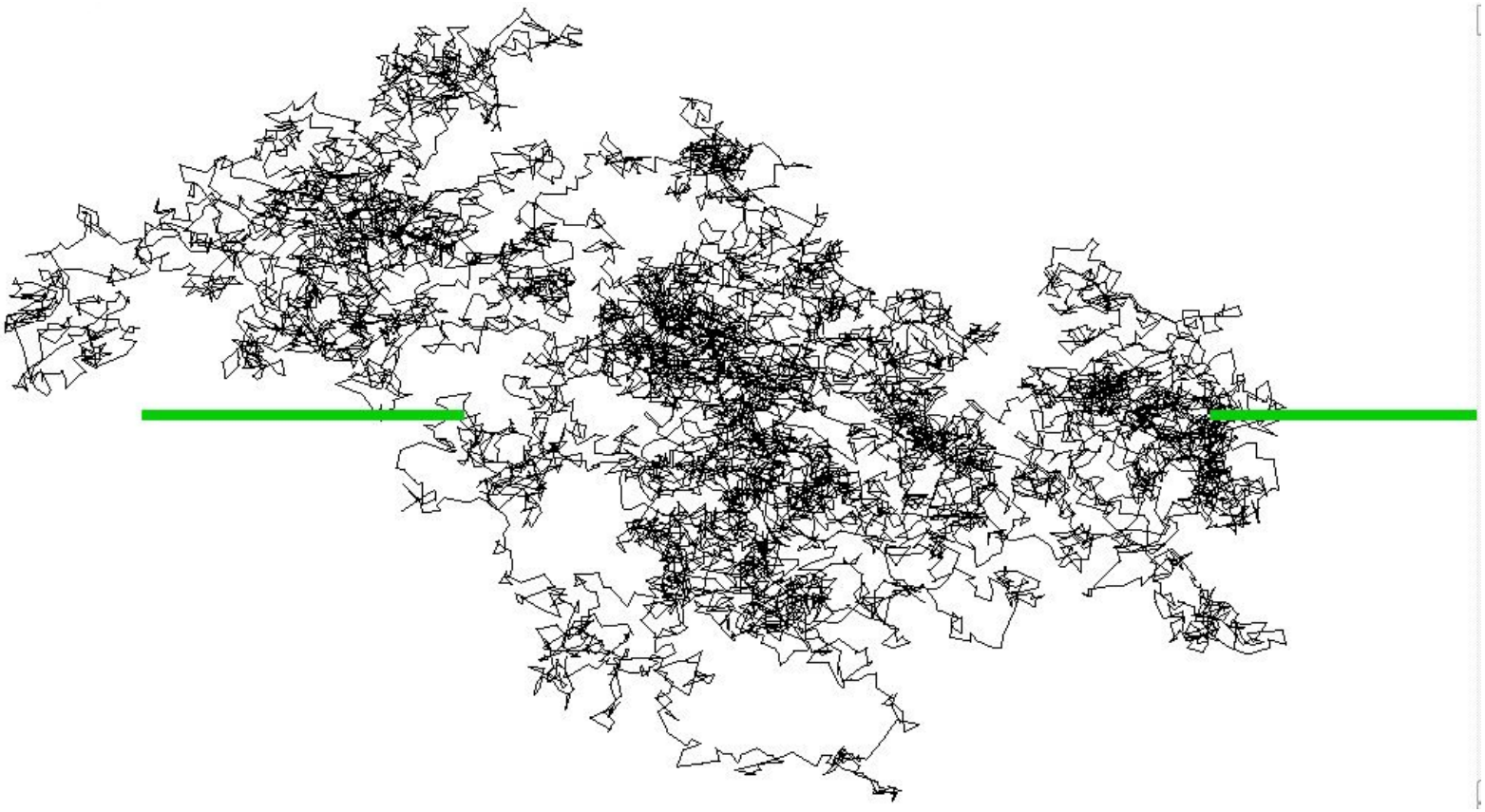
Trajectory of a Quantum Fluctuation

- worldline numerics: $N = 1000$ points per loop (ppl)



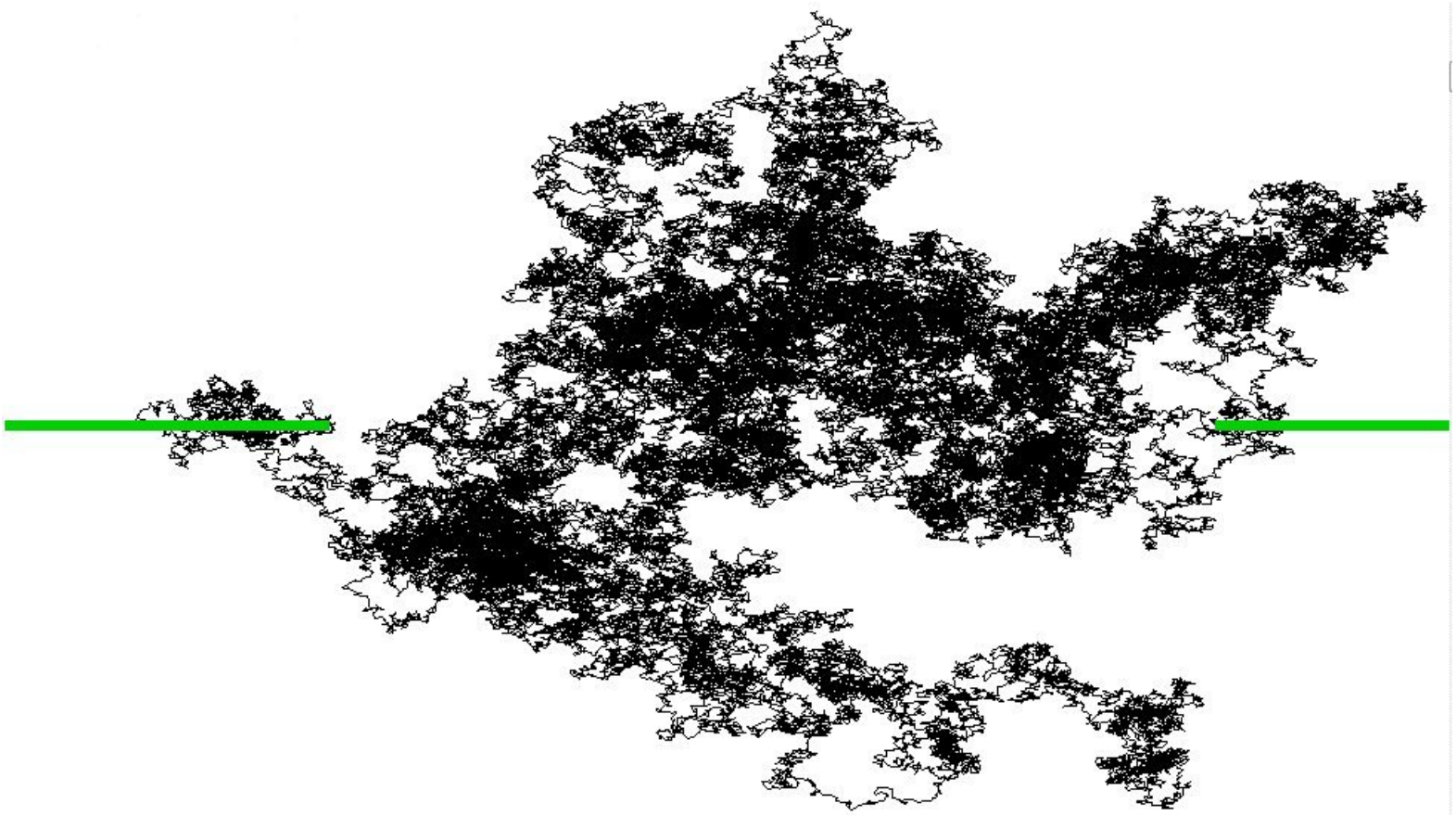
Trajectory of a Quantum Fluctuation

▷ worldline numerics: $N = 10000$ points per loop (ppl)

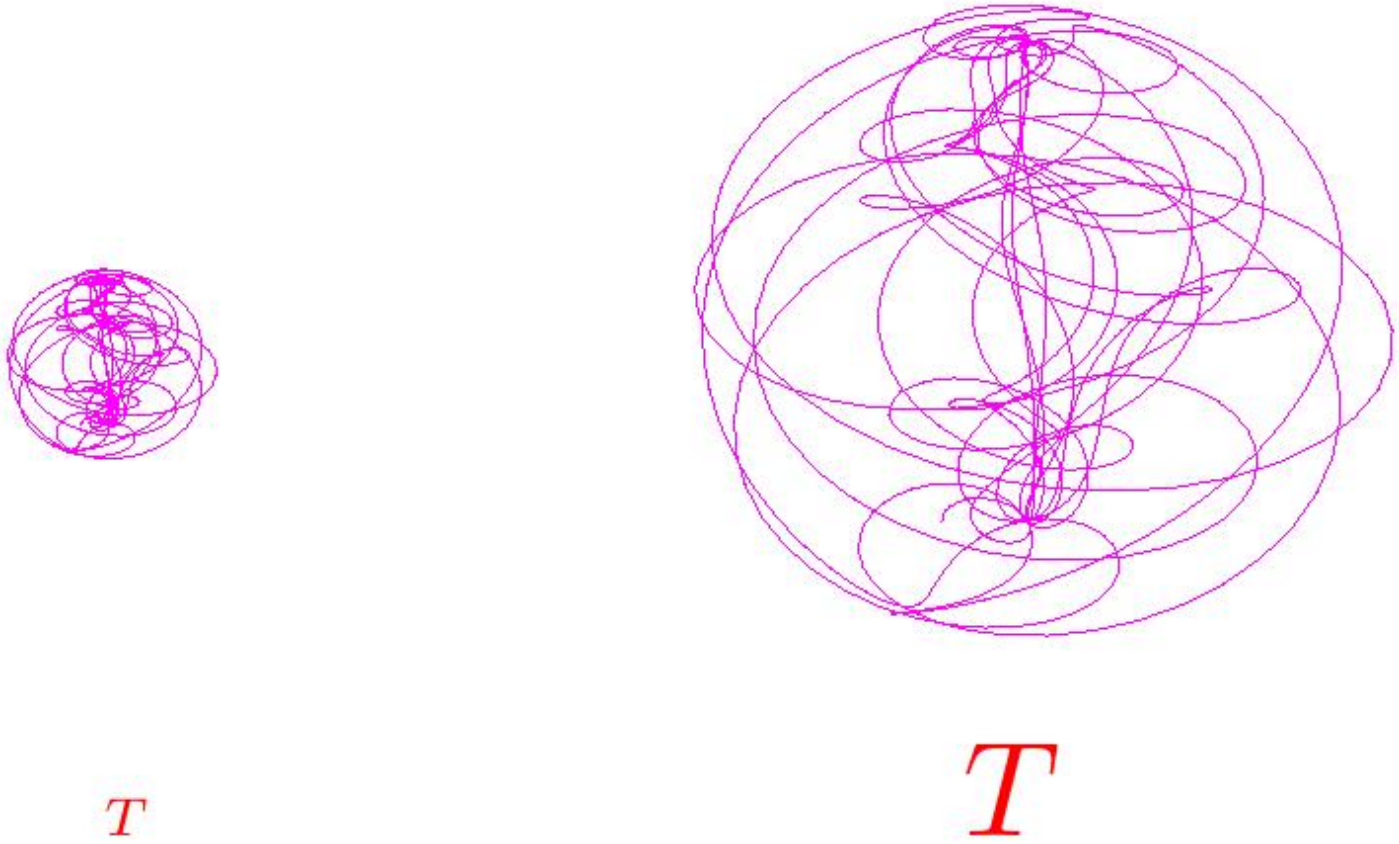


Trajectory of a Quantum Fluctuation

▷ worldline numerics: $N = 100000$ points per loop (ppl)



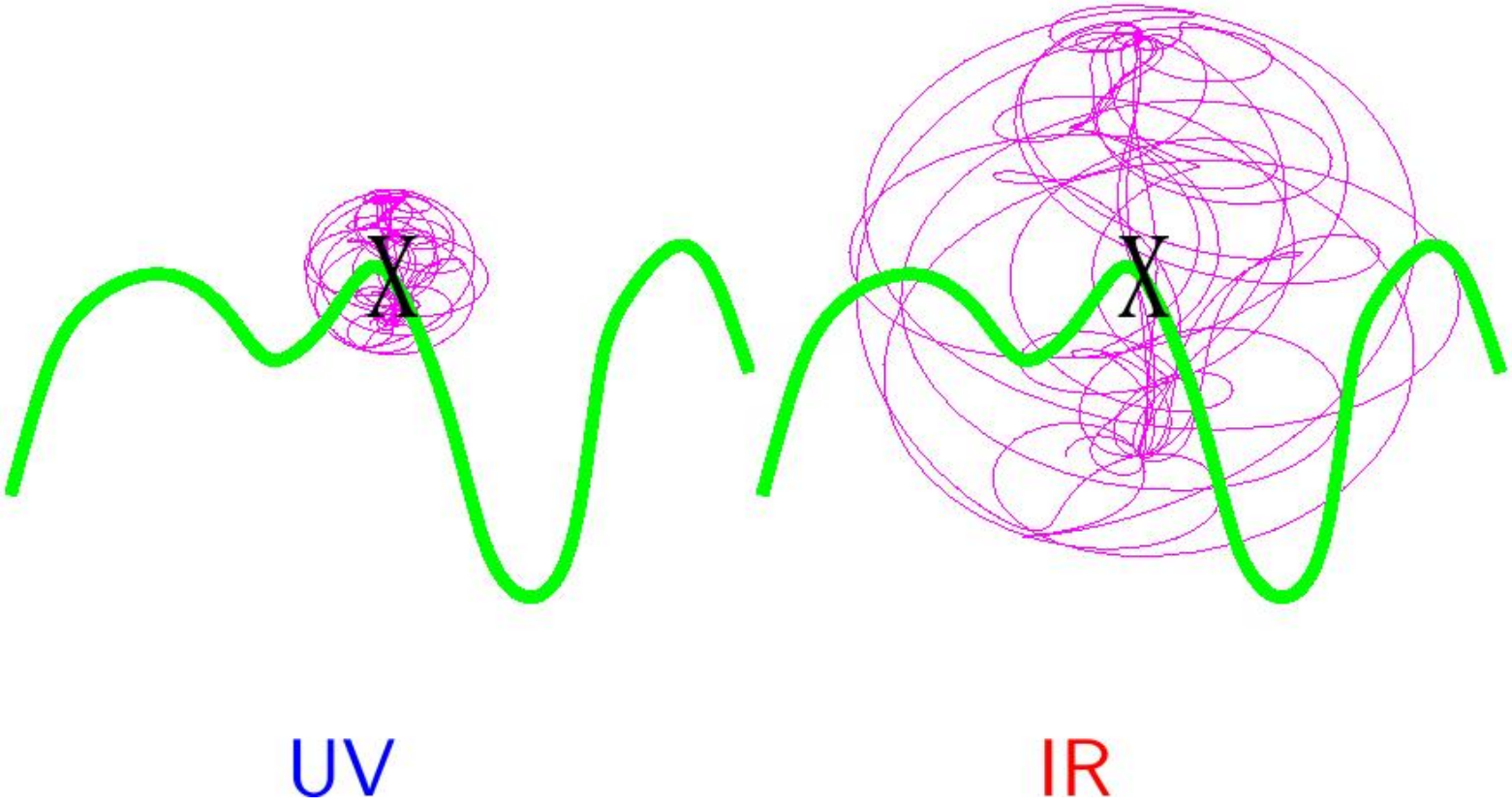
Propertime T



$T \quad \sim$ regulator scale of smeared momentum shells

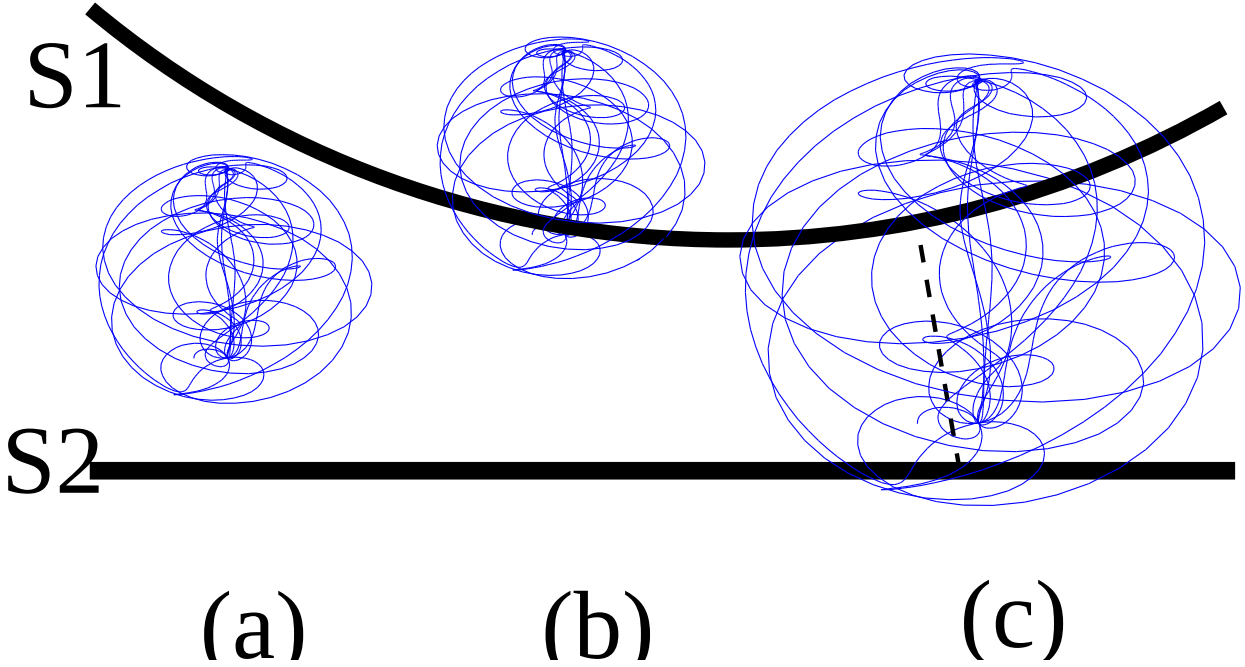
Propertime T

▷ "Measuring" the Wegner-Wilson loop $\exp\left(-\mathrm{i}e\oint dx\cdot A\right)$



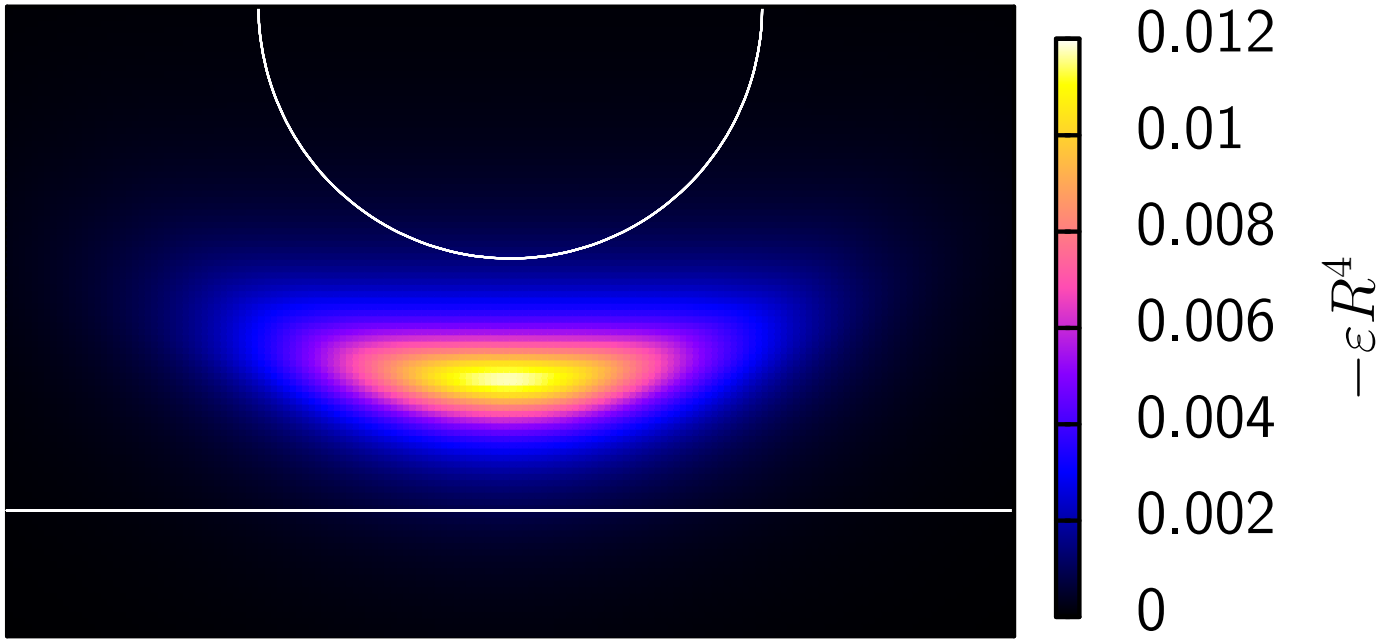
For instance: Casimir Geometries

(HG, LANGFELD, MOYAERTS'03)



For instance: Casimir Geometries

(HG,KLINGMULLER'06)



Fermions on the worldline I

▷ Grassmann loops

(. . . ,Polyakov'87;Bern&Kosower'92)

(Strassler'92; Schmidt&Schubert'93)

$$\Gamma^1_{\text{spin}} = \ln\det\left[\gamma^\mu\partial_\mu + \mathrm{i}e\gamma^\mu A_\mu + m\right]$$

$$= -\frac{1}{2(4\pi)^{D/2}}\int_{1/\Lambda^2}^{\infty}\frac{dT}{T^{1+D/2}}\,\mathrm{e}^{-m^2T}\int_P\mathcal{D}x\int_A\mathcal{D}\psi\;\mathrm{e}^{-\int_0^T d\tau L_{\text{spin}}}$$

$$L_{\text{spin}} = \frac{1}{4}\dot{x}^2 + \mathrm{i}e\dot{x}^\mu A_\mu + \frac{1}{2}\psi_\mu\dot{\psi}^\mu - \mathrm{i}e\psi^\mu F_{\mu\nu}\psi^\nu$$

Fermions on the worldline II

▷ geometric spin factor

(STROMINGER'80, POLYAKOV'88)

(HG&HAMMERLING'05)

$$\Gamma[A] = \frac{1}{2}\frac{1}{(4\pi)^{D/2}}\int_0^\infty \frac{dT}{T^{(1+D/2)}}\, e^{-m^2 T}\, \left\langle\, e^{-ie\oint dx A(x)}\, \Phi[x]\right\rangle_x$$

$$\Phi[x] := \mathrm{tr}_\gamma \mathcal{P} : e^{\frac{i}{2}\int_0^T d\tau\, \sigma\omega(\tau)} :$$

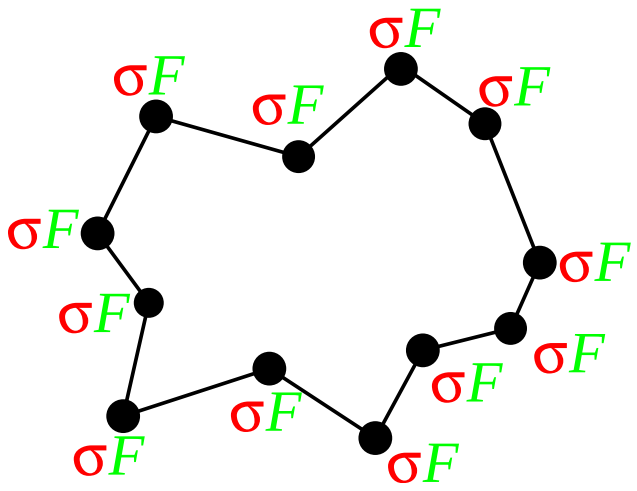
$$\omega_{\mu\nu}(\tau) = \frac{1}{4}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\, \ddot{x}_\mu(\tau+\frac{\rho}{2})\ddot{x}_\nu(\tau-\frac{\rho}{2})$$

Fermions on the worldline III

▷ standard Pauli term:

$$\Gamma \;=\; \frac{-\frac{1}{2}}{(4\pi)^{\frac{D}{2}}} \int d^D x_{\rm CM} \int\limits_{1/\Lambda^2}^{\infty} \frac{dT}{T^{\frac{D}{2}+1}}\, {\rm e}^{-m^2 T} \;\left\langle W_{\rm spin}[A] \right\rangle_x$$

$$W_{\rm spin}[A] = W[A] \;\times\; P_T \exp\left(\frac{{\rm i}e}{2} \int\limits_0^T d\tau\, \sigma_{\mu\nu} F^{\mu\nu} \right)$$



Specific Example: Gross-Neveu/NJL model at large N

- fermionic action (GN: $\zeta = 0$; NJL: $\zeta = 1$)

$$S_{\rm F} = \int d^D x \left(-\bar\psi \partial\!\!\!/ \psi + \frac{g^2}{2N} \left[(\bar\psi\psi)^2 - \zeta (\bar\psi\gamma_5\psi)^2 \right] \right)$$

- partially bosonized action a la Hubbard-Stratonovich

$$S_{\rm FB} = \int d^D x \left[\frac{N}{2g^2} \left(\sigma^2 + \frac{1}{\zeta}\pi^2 \right) - \bar\psi \partial\!\!\!/ \psi - i\sigma\bar\psi\psi + \pi\bar\psi\gamma_5\psi \right]$$

- large-N effective action

$$\Gamma[\sigma,\pi] = \int d^D x \frac{N}{2g^2} \left(\sigma^2 + \frac{1}{\zeta}\pi^2 \right) - \ln\det(-\partial\!\!\!/ - i\sigma + \gamma_5\pi), \quad N \to \infty$$

Specific Example: Gross-Neveu/NJL model at large N

▷ (real part of the) effective action ($\mathcal{D} = -\partial\!\!\!/ - i\sigma + \gamma_5\pi$)

ImΓ on the worldline: (D'Hoker&Gagné'95)

$$
\begin{aligned}
\Delta\Gamma[\sigma,\pi] &= -\frac{1}{2}\ln\det\mathcal{D}\mathcal{D}^\dagger \\
&= -\frac{1}{2}\mathrm{Tr}\ln(-\partial^2+\sigma^2+\pi^2-i\partial\!\!\!/\sigma+\gamma_5\partial\!\!\!/\pi) \\
&= -\frac{1}{2}\frac{1}{(4\pi)^{D/2}}\int\limits_{1/\Lambda^2}^{\infty}\frac{dT}{T^{D/2+1}} \\
&\qquad\times\int\limits_{x(0)=x(T)}\mathcal{D}x\, e^{-\frac{1}{4}\int_0^T d\tau\,\dot{x}^2(\tau)}\, e^{-\int_0^T d\tau(\sigma^2+\pi^2)}\Phi[\sigma,\pi]
\end{aligned}
$$

▷ spin factor

$$
\Phi[\sigma,\pi] = \mathrm{tr}_\gamma\,\mathcal{P}\exp\left(i\int_0^T d\tau\;(\partial\!\!\!/\sigma\;+\;i\,\gamma_5\,\partial\!\!\!/\pi)\right)
$$

Chiral/Axial Symmetry

- fermions

$$\psi \to e^{-i\gamma_5\alpha}\psi, \quad \bar\psi \to \bar\psi e^{-i\gamma_5\alpha}$$

- bosons

$$\begin{pmatrix}\sigma\\ \pi\end{pmatrix} = \begin{pmatrix}\cos 2\alpha & -\sin 2\alpha\\ \sin 2\alpha & \cos 2\alpha\end{pmatrix}\cdot\begin{pmatrix}\sigma\\ \pi\end{pmatrix}$$

$$\Longrightarrow \qquad (\partial\!\!\!/\sigma + i\,\gamma_5\,\partial\!\!\!/\pi) \quad \to \quad U^\dagger\,(\partial\!\!\!/\sigma + i\,\gamma_5\,\partial\!\!\!/\pi)\,U, \quad U = e^{i\gamma_5\alpha}$$

$\Longrightarrow$ invariance of the spin factor

$$\Phi[\sigma,\pi] = \mathrm{tr}_\gamma\,\mathcal{P}\exp\left(i\int_0^T d\tau\;(\partial\!\!\!/\sigma + i\,\gamma_5\,\partial\!\!\!/\pi)\right)$$

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$\Longrightarrow$ invariance of the discretized spin factor

$$\Phi[\sigma,\pi] = \mathrm{tr}_\gamma\,\mathcal{P}\exp\left(i\,\frac{T}{N}\sum_{i=1}^{N}\,(\partial\!\!\!/\sigma(x_i) + i\,\gamma_5\,\partial\!\!\!/\pi(x_i))\right)$$

Chiral/Axial Symmetry

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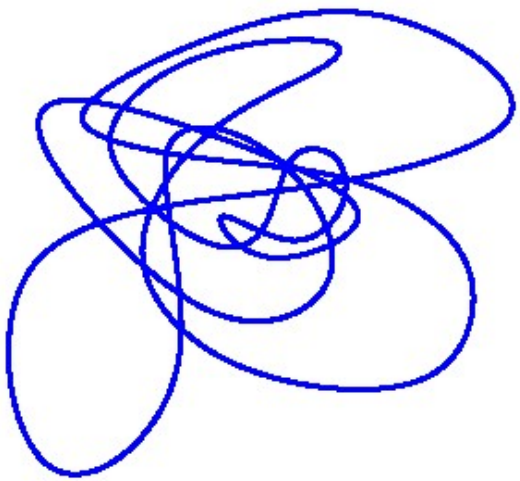

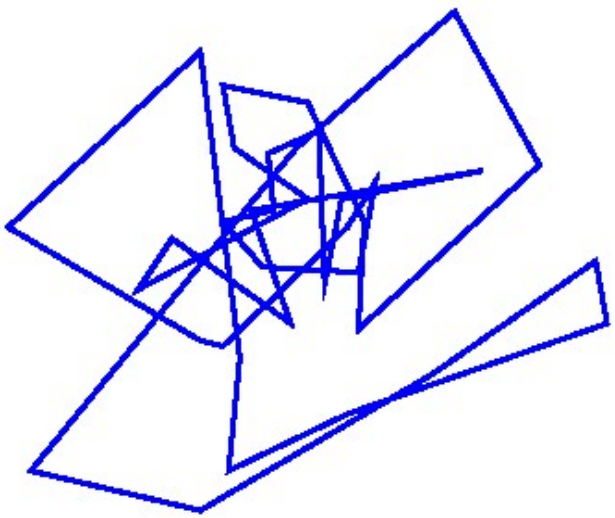

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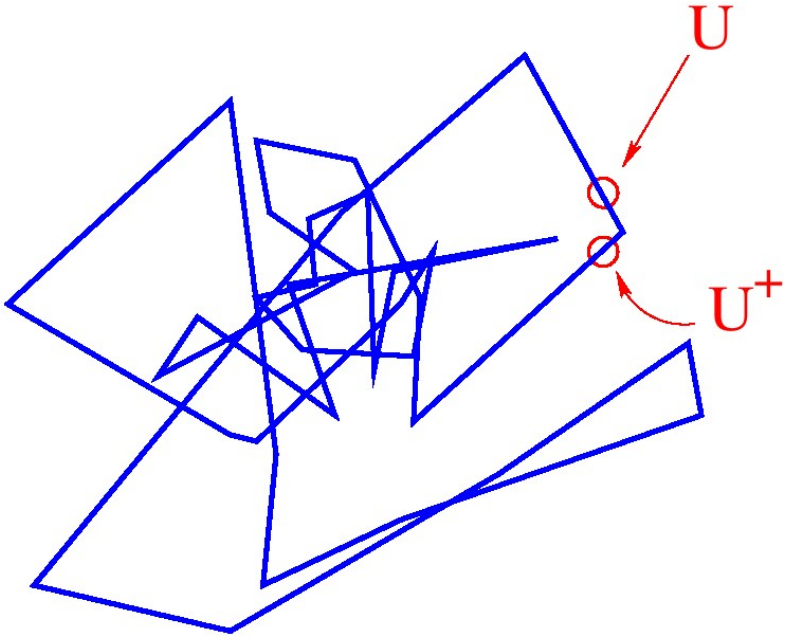


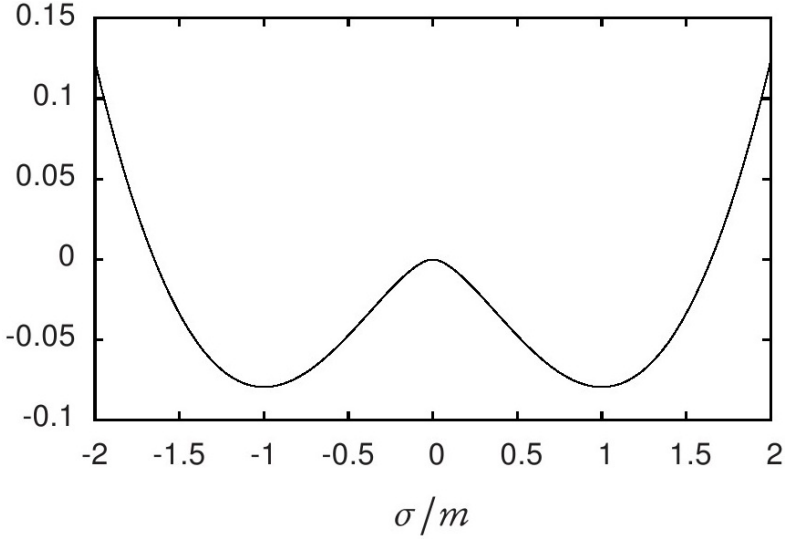
GN Model Studies in $D = 1 + 1$

▷ GN in $D = 1 + 1$:

$$\pi = 0, \quad \sigma(t,x) \to \sigma(x), \quad (\sigma^2 + i\partial\!\!\!/\sigma) \;\to\; (\sigma^2 \pm \sigma')$$

▷ GN vacuum:
spontaneous breaking of Z_2 symmetry

$$\Gamma[\sigma = \text{const.}] = \int d^2x \frac{\sigma^2}{4\pi}\left(\ln\frac{\sigma}{m} - 1\right)$$



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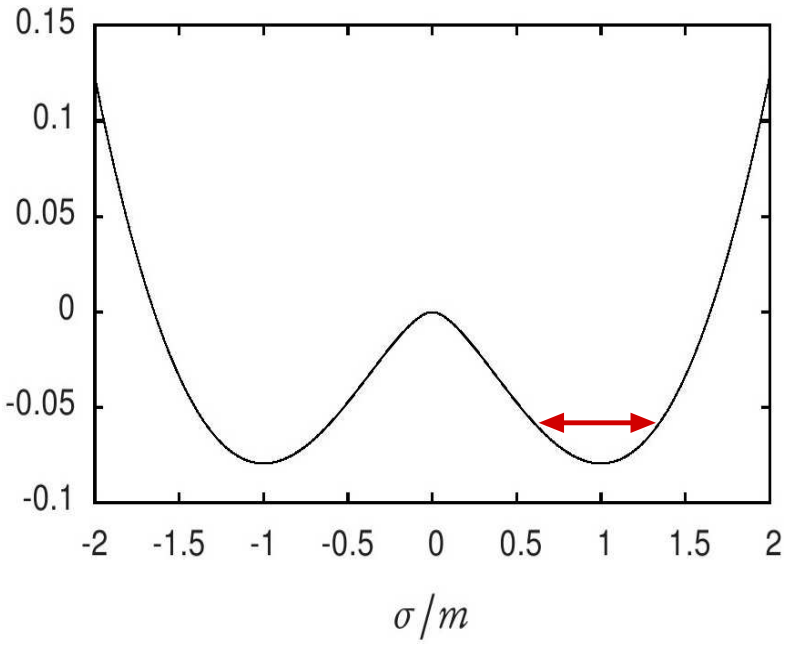
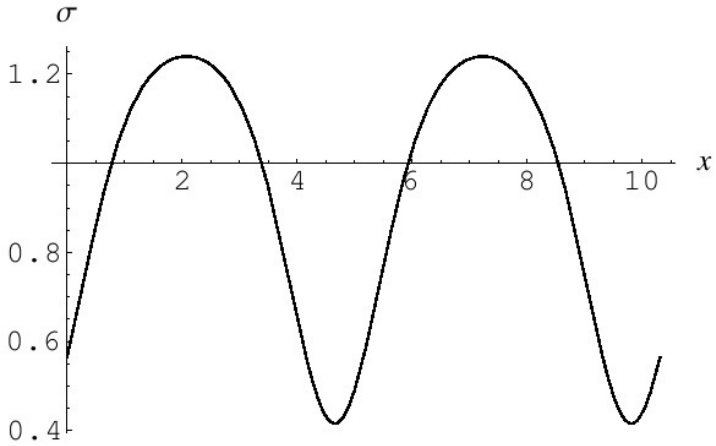
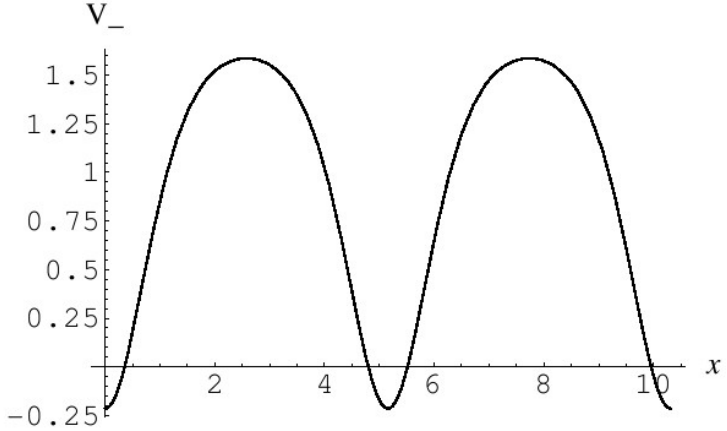


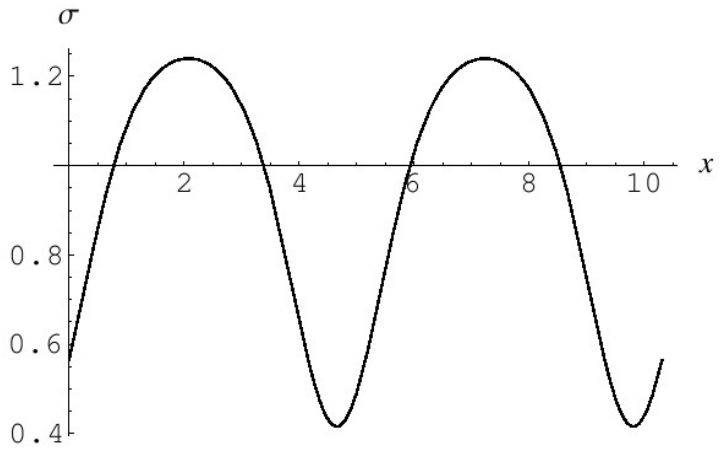
▷ periodic potentials ($\sigma^2 - \sigma = V_-$):

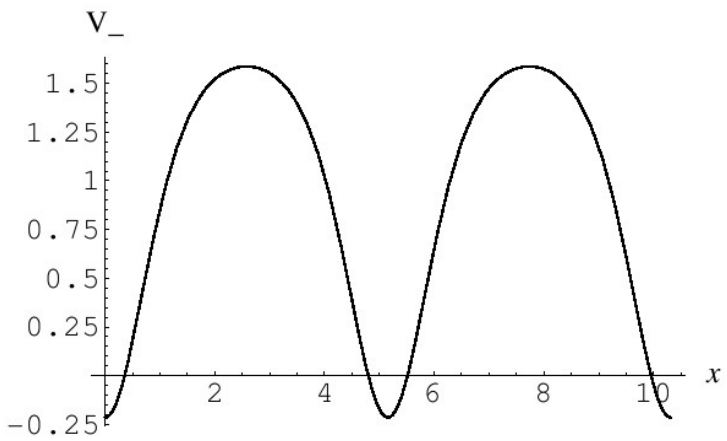




GN Model Studies in $D = 1 + 1$

▷ periodic potentials ($\sigma^2 - \sigma = V_-$):



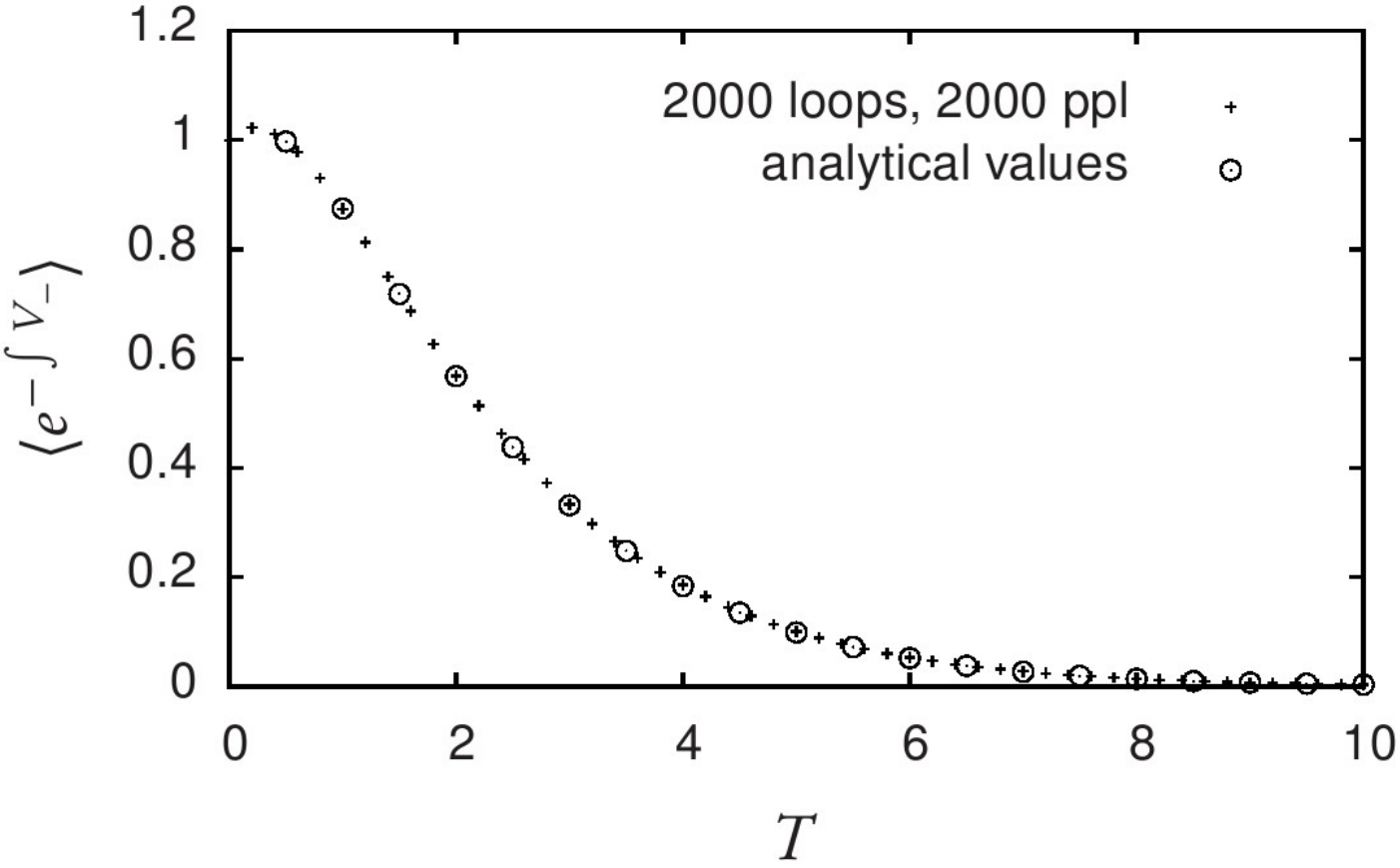


$$\sigma(x) = \varkappa^2 \left(\frac{\mathrm{cn}(b;\nu)\mathrm{dn}(b;\nu)}{\mathrm{sn}(b;\nu)} + \nu\, \mathrm{sn}(b;\nu)\mathrm{sn}(\varkappa x;\nu)\mathrm{sn}(\varkappa x + b;\nu) \right)$$

(MACHIDA&NAKANISHI'84)

GN Model Studies in $D = 1 + 1$

▷ worldline numerics vs. exact result, e.g., $b = 1, \nu = 0.9$

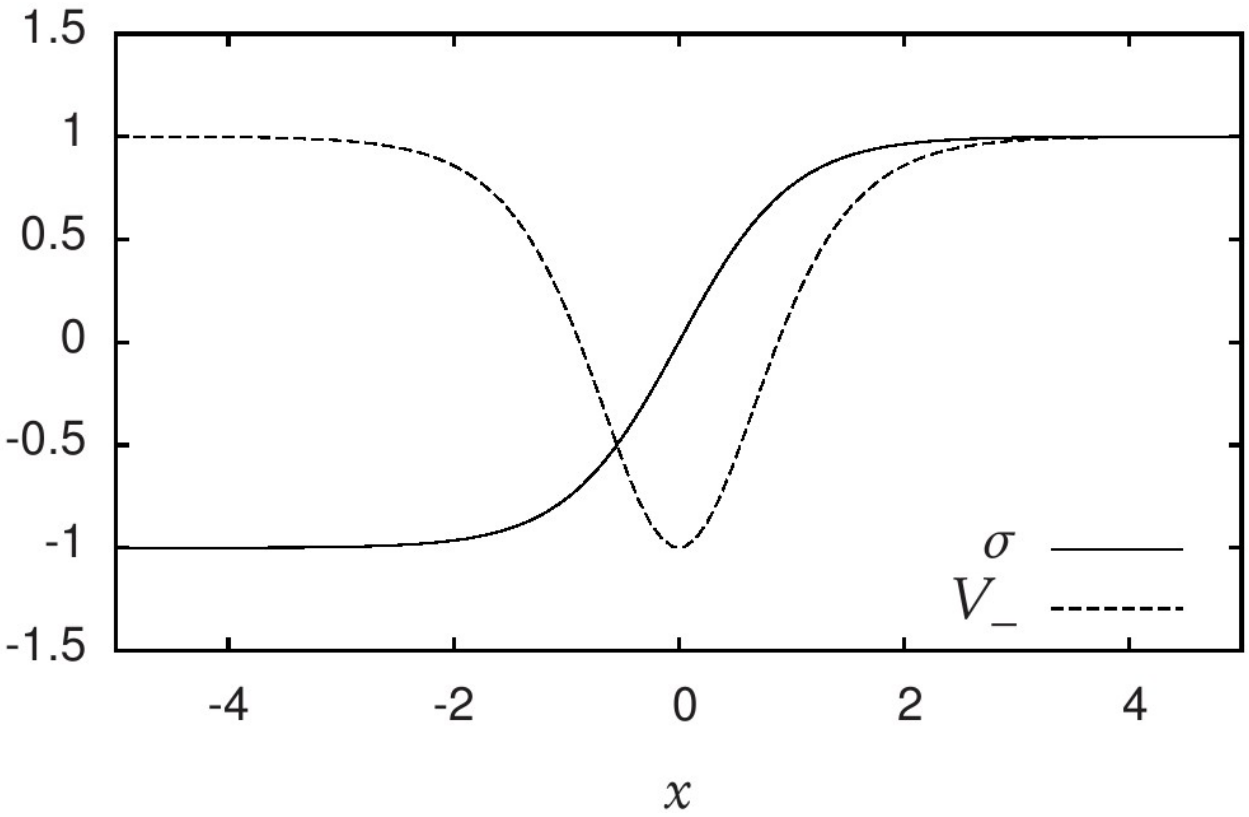


$$E/\text{period} = -\frac{1}{4\pi} \begin{cases} 6.8174\ldots & \text{exact} \\ 6.818 \pm 0.001 & \text{worldline numerics} \end{cases}$$

GN Model Studies in $D = 1 + 1$

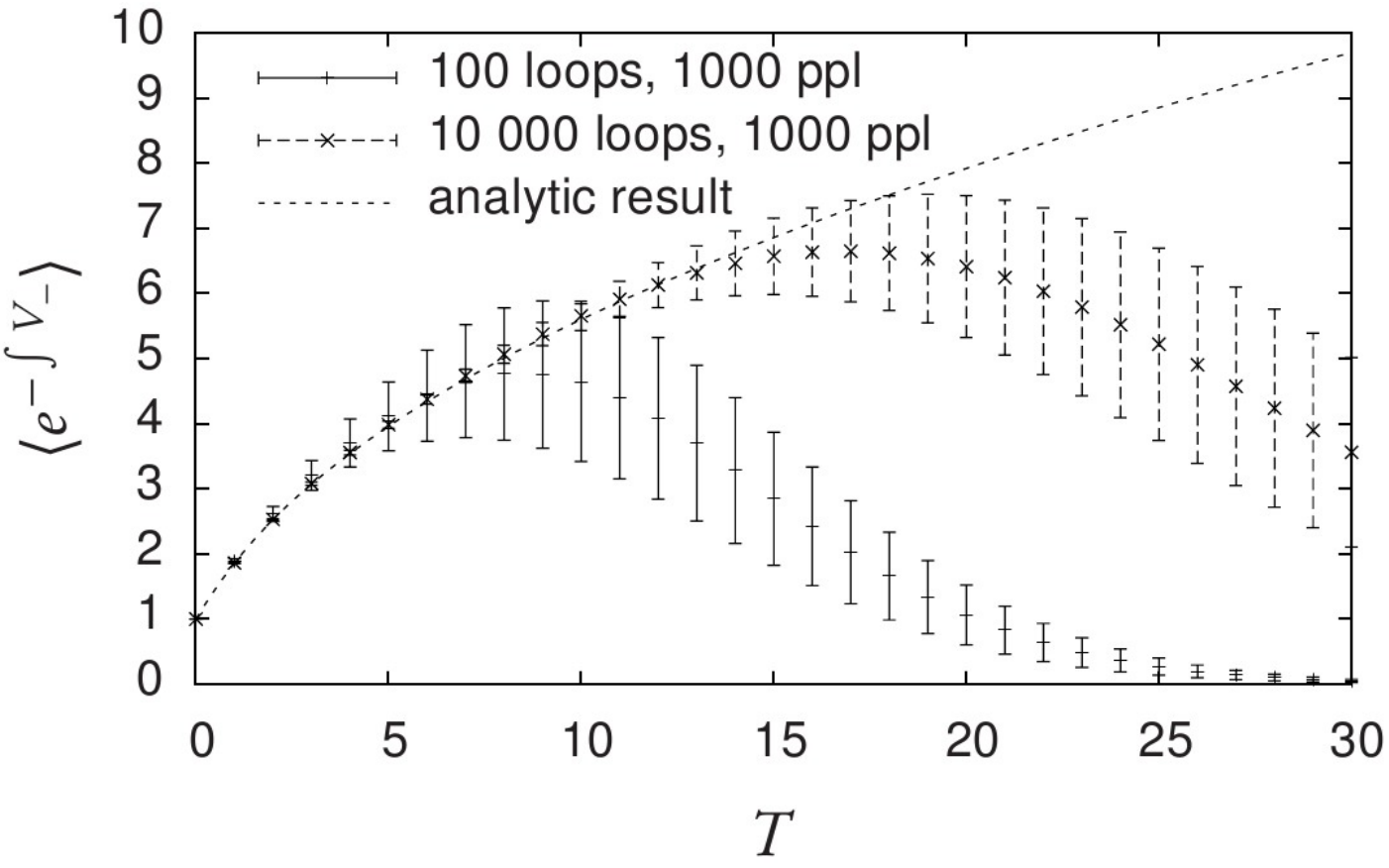
▷ extremal case: single kink

$$\sigma = \tanh x$$



GN Model Studies in $D = 1 + 1$

▷ worldline numerics vs. exact result



GN Model Studies in $D = 1 + 1$

- zero mode:

$$\mathrm{tr}_x \exp\{-TH_\pm\} = \frac{1}{\sqrt{4\pi T}} \Big\langle \exp\{-TH_\pm\} \Big\rangle_x \,, \quad H_\pm \;=\; -\partial^2 + \sigma^2 \pm \sigma'(x)$$

- spectral decomposition:

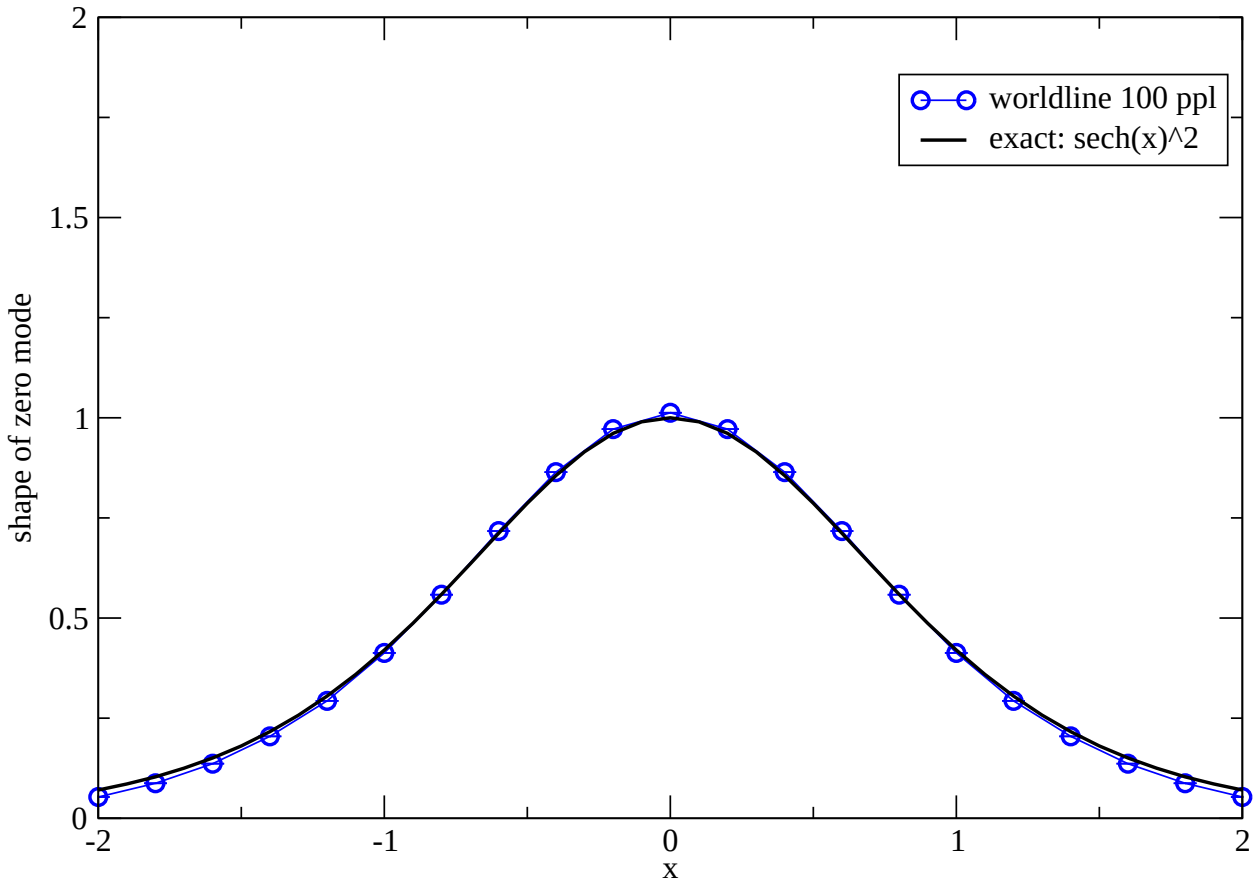
$$\Big\langle \exp\{-TH_-\} \Big\rangle_x \;=\; \sqrt{4\pi T} \left(|\psi_0(x)|^2 + \sum_i |\psi_i(x)|^2 \, \mathrm{e}^{-TE_i} \right)$$

- shape of the zero mode:

$$|\psi_0(x)|^2 = \frac{1}{\cosh^2 x}$$

GN Model Studies in $D = 1 + 1$

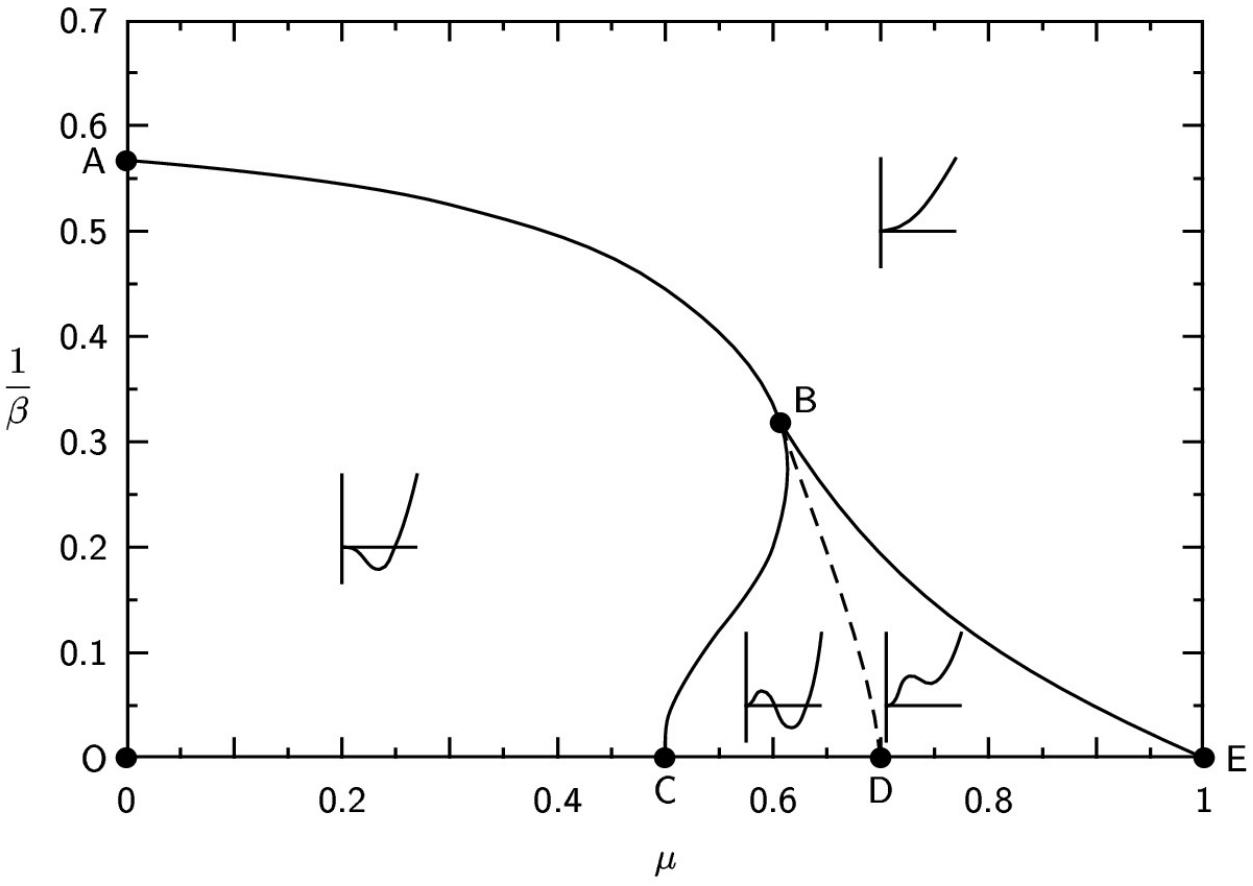
▷ mapping out the zero mode from $\sqrt{T}$ tail:



$\Longrightarrow$ zero modes detectable by worldline numerics

Phase Diagram of the GN_{1+1} Model at large N

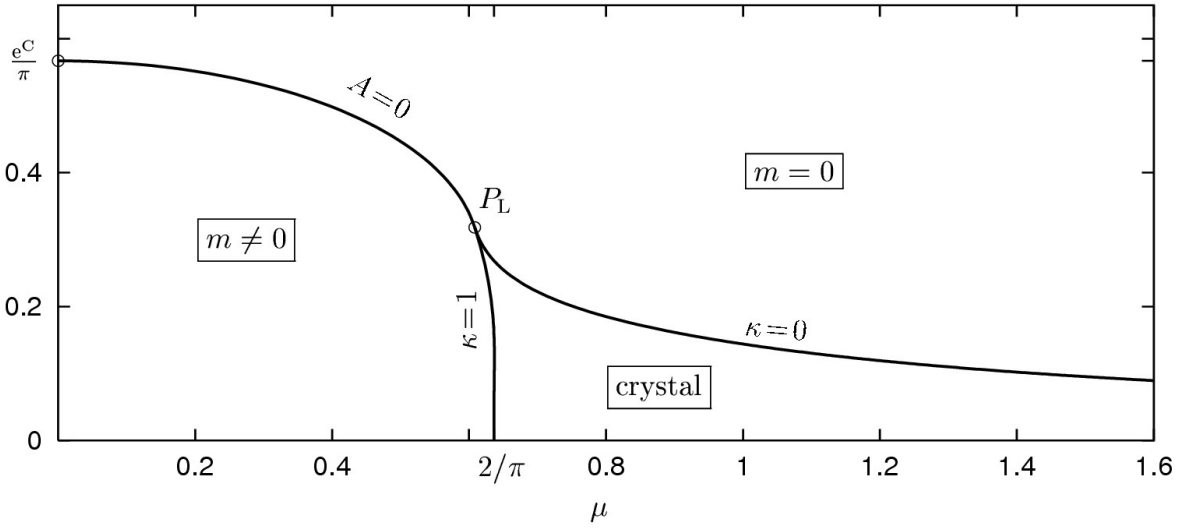
▷ "old" phase diagram, σ =const (WOLFF'85)



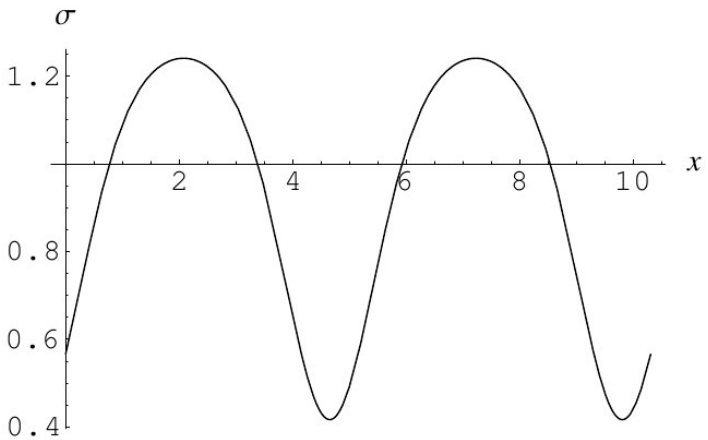
[THIES:HEP-TH/0601049]

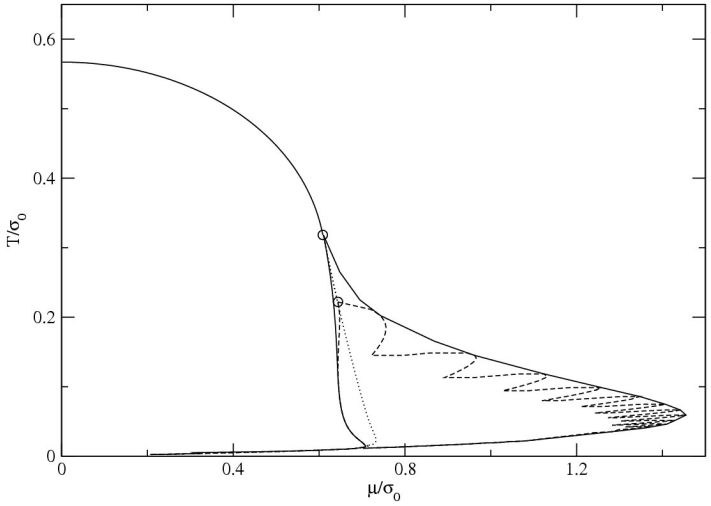
Phase Diagram of the GN_{1+1} Model at large N

▷ "revised" phase diagram, $\sigma = \sigma(x)$ (THIES&URLICHS'03; SCHNETZ,THIES,URLICHS'04)



[THIES:HEP-TH/0601049]

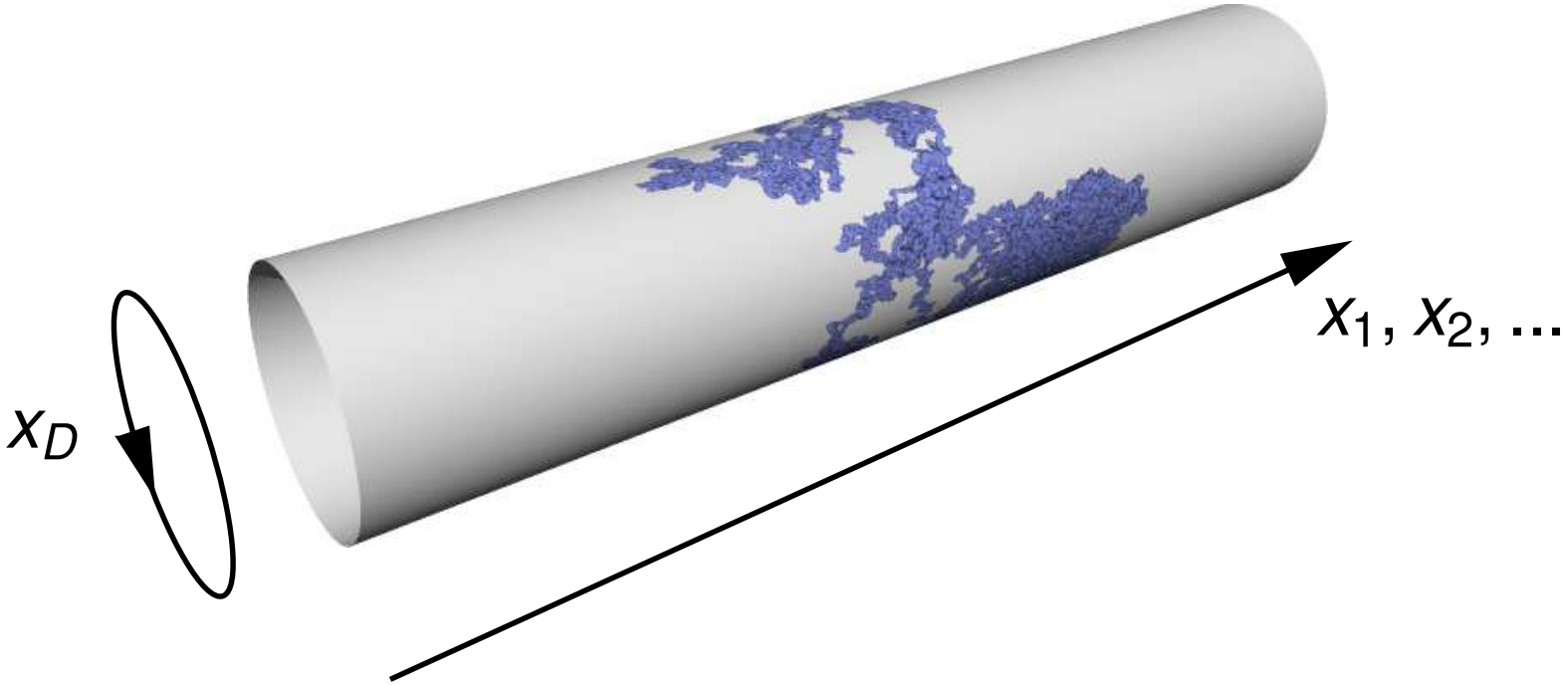




(DEFORCAND&WENGER'06)

T and μ on the Worldline

▷ sum over worldline windings & $e^{\pm\beta\mu}$ factors

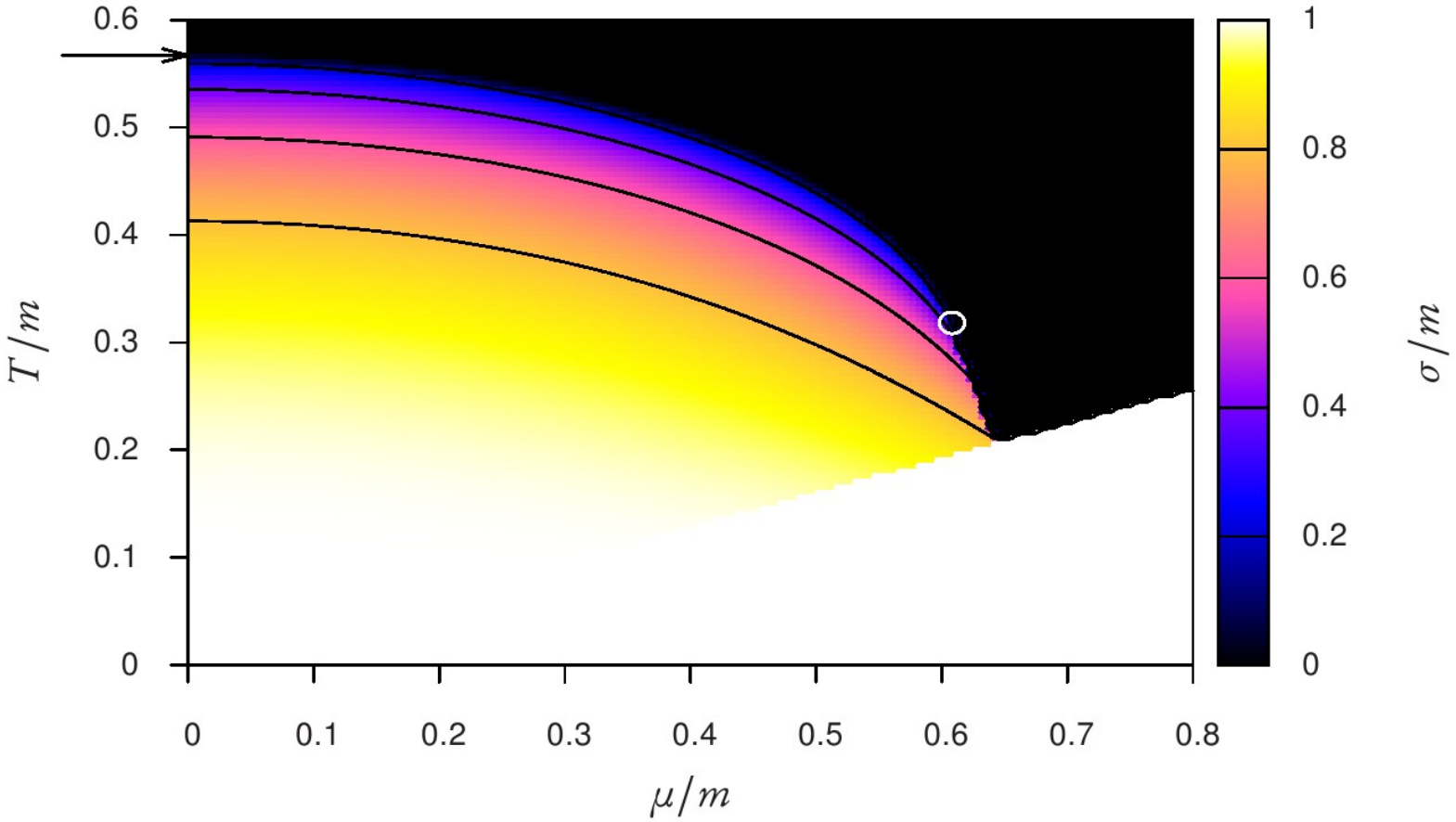


$$\int dT \int \mathcal{D}x \to \int dT \left(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-\frac{\beta^2 n^2}{4T}} \cosh\beta\mu n\right) \int \mathcal{D}x$$

BUT: $\beta\mu < \pi$

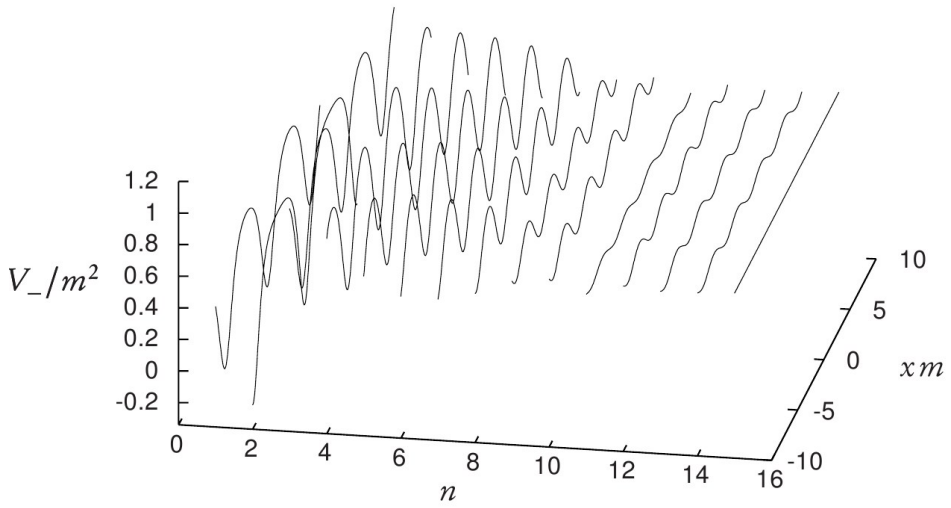
Phase Diagram on the Worldline – work in progress

▷ σ =const.



Phase Diagram on the Worldline – work in progress

▷ minimization of the free energy, e.g., for $T = 0.25m$, $\mu = 0.56m$:



▷ $\beta\mu > \pi$?: heat kernel = $\mathcal{L}$[density of states]

$$K(T) = \mathrm{Tr}e^{-TH} = \int_0^\infty dE^2\, e^{-E^2 T}\, \rho(E)$$

$$\Longrightarrow \quad \mathcal{F} = \frac{1}{2}\int dE^2\, \rho(E) \ln[(1 + e^{-\beta(E+\mu)})(1 + e^{-\beta(E-\mu)})]$$

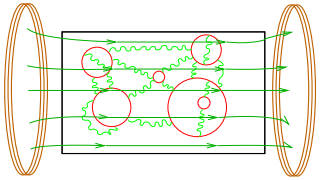
Summary

Worldline Numerics works (. . . also for fermions!)

Outlook

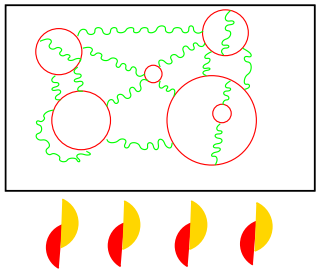
- phase diagram studies also in higher dimensional ψ^4 models
- . . . beyond large N
- abelian gauge theories with fermions (e.g., QED_3, . . .)
- QCD ?

Further Worldline Applications



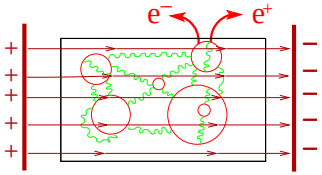
Heisenberg-Euler effective actions, spinor QED, flux tubes, quantum-induced vortex interactions

(HG, Langfeld'01; Langfeld, Moyaerts, HG'02)



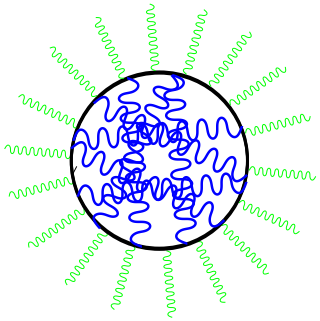
thermal fluctuations, free energies

(HG, Langfeld'02)



"spontaneous vacuum decay", Schwinger pair production in inhomogeneous electric fields

(HG, Klingmüller'05)



nonperturbative effective actions

(HG, Sánchez–Guillén, Vázquez'05)