

Hidden extended dynamical symmetries in non-equilibrium systems

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Contents :

- I. Ageing phenomena
physical ageing ; scaling behaviour and exponents
- II. Hidden dynamical symmetries
Local scaling with $z = 2$; stochastic field-theory ; computation of response and correlation functions
- III. Local scale-invariance for $z \neq 2$
Mass terms ; integrability ; test through responses and correlators in 2D disordered Ising model
- IV. Conclusions

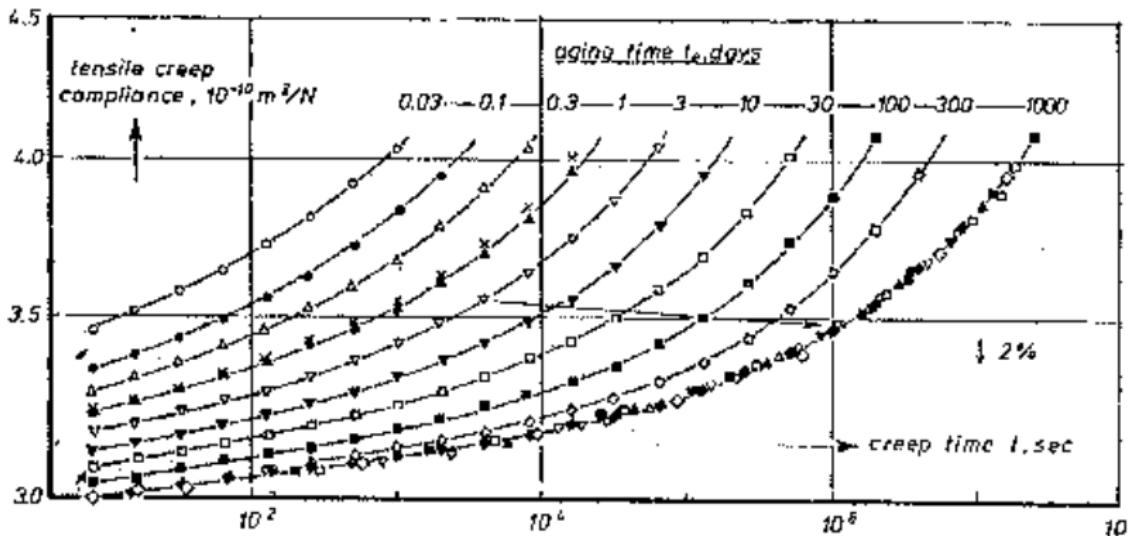
I. Ageing phenomena

- why do materials 'look old' after some time ?
- which (reversible) microscopic processes lead to such macroscopic effects ?
- physical ageing known since historical (or prehistorical) times
- systematic studies first in glassy systems

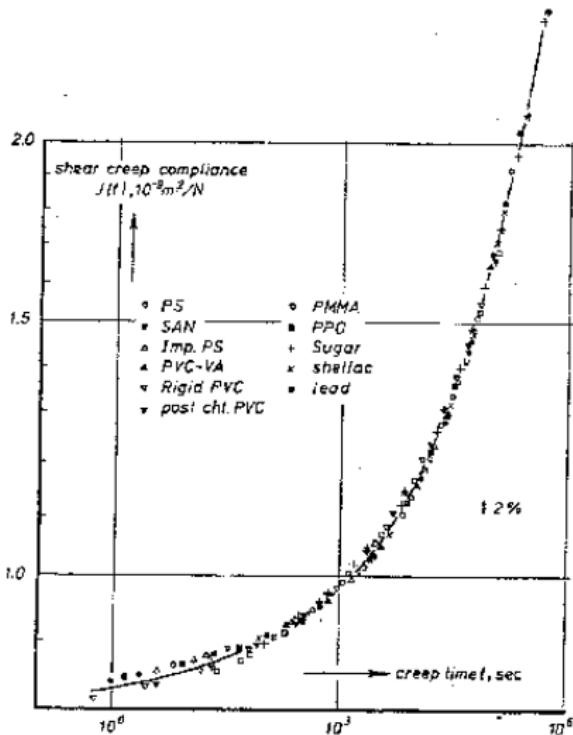
STRUIK 78

a priori behaviour should depend on entire prehistory
but evidence for reproducible and universal behaviour

- for better conceptual understanding : study ageing first in simpler systems (i.e. disordered ferromagnets)
- ageing : defining characteristics and symmetry properties :
 - ① slow dynamics (i.e. non-exponential relaxation)
 - ② breaking of time-translation invariance
 - ③ dynamical scaling
- new evidence for larger, local scaling symmetries



1. observe **slow relaxation** after quenching PVC from melt to low T
2. creep curves depend on **waiting time t_e** and **creep time t**
3. find master curve for all $(t, t_e) \rightarrow$ **dynamical scaling**
→ three defining properties of **physical ageing**



master curves of **distinct**
materials are **identical**

→ **Universality!**

good for theorists . . .

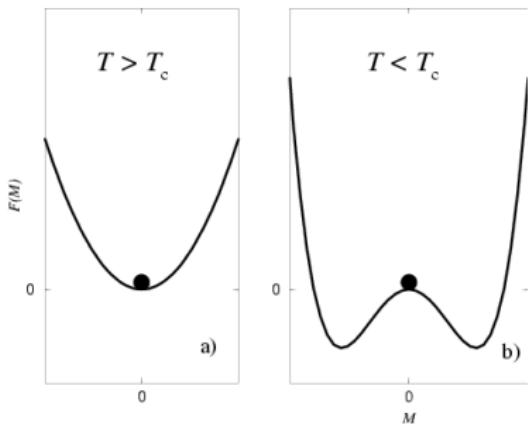
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conceptual confirmation in phase-ordering : Allen-Cahn equation

easier to study : ageing in simple systems without disorder
consider a simple magnet (ferromagnet, i.e. Ising model)

- ① prepare system initially at high temperature $T \gg T_c > 0$
- ② quench to temperature $T < T_c$ (or $T = T_c$)
→ non-equilibrium state
- ③ fix T and observe dynamics

BRAY 94



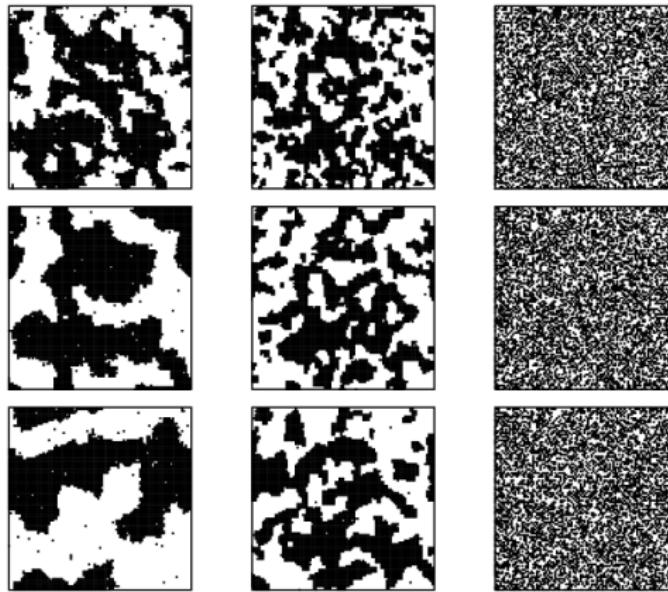
competition :

at least 2 equivalent ground states
local fields lead to rapid local ordering
no global order, relaxation time ∞

formation of ordered domains, of linear size $L = L(t) \sim t^{1/z}$

dynamical exponent z

universal Allen-Cahn equation $v = -(d - 1)K$ for domain walls



Snapshots of spin configurations in several 2D/3D Ising models quenched to $T < T_c$, for three different times $t = 25, 100, 225$.

Left : pure **Middle : disordered** **Right : 3D spin glass**

Scaling behaviour & exponents

single relevant time-dependent length scale $L(t) \sim t^{1/z}$

BRAY 94, JANSSEN ET AL. 92, CUGLIANDOLO & KURCHAN 90s, GODRÈCHE & LUCK 00, ...

$$\text{correlator } C(t, s; \mathbf{r}) := \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{0}) \rangle = s^{-b} f_C \left(\frac{t}{s}, \frac{\mathbf{r}}{(t-s)^{1/z}} \right)$$

$$\text{response } R(t, s; \mathbf{r}) := \left. \frac{\delta \langle \phi(t, \mathbf{r}) \rangle}{\delta h(s, \mathbf{0})} \right|_{h=0} = s^{-1-a} f_R \left(\frac{t}{s}, \frac{\mathbf{r}}{(t-s)^{1/z}} \right)$$

No fluctuation-dissipation theorem : $R(t, s; \mathbf{r}) \neq T \partial C(t, s; \mathbf{r}) / \partial s$

values of exponents : equilibrium correlator \rightarrow classes **S** and **L**

$$C_{\text{eq}}(\mathbf{r}) \sim \begin{cases} \exp(-|\mathbf{r}|/\xi) \\ |\mathbf{r}|^{-(d-2+\eta)} \end{cases} \implies \begin{cases} \text{class S} \\ \text{class L} \end{cases} \implies \begin{cases} a = 1/z \\ a = (d-2+\eta)/z \end{cases}$$

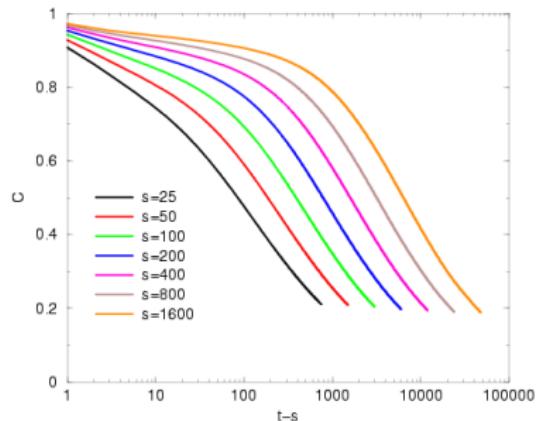
if $T < T_c$: $z = 2$ and $b = 0$ if $T = T_c$: $z = z_c$ and $b = a$

for $y \rightarrow \infty$: $f_{C,R}(y, \mathbf{0}) \sim y^{-\lambda_{C,R}/z}$, $\lambda_{C,R}$ independent exponents

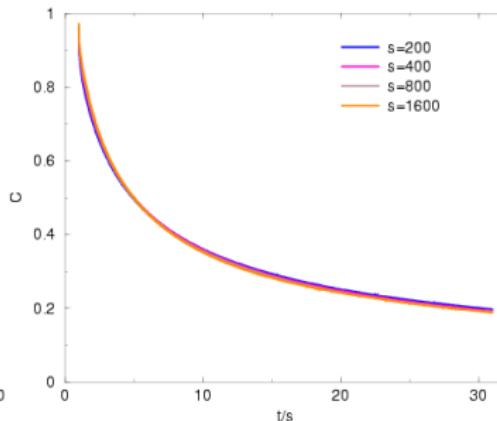
Question : general arguments to find form of scaling functions ?

Tests of dynamical scaling : 2D Ising model, $T < T_c$

2D Ising, $T=1.5$



2D Ising, $T=1.5$



Values of λ_C for $T < T_c$: $1.25(2)$ in $2D$,

$1.59(2)$ in $3D$

Values of λ_C for $T = T_c$: $1.59(2)$ in $2D$,

$2.78(4)$ in $3D$

Yeung-Desai-Rao inequality for $T < T_c$: $\boxed{\lambda_C \geq d/2}$

Question : how to find the scaling functions $f_R(y)$ and $f_C(y)$?

II. Hidden dynamical symmetries

A) Langevin equation

(model A of HOHENBERG & HALPERIN 77)

$$2\mathcal{M} \frac{\partial \phi}{\partial t} = \Delta \phi - \frac{\delta \mathcal{V}[\phi]}{\delta \phi} + \eta$$

order-parameter $\phi(t, \mathbf{r})$ non-conserved

\mathcal{M} : kinetic coefficient

\mathcal{V} : Landau-Ginsbourg potential

η : gaussian noise, centered and with variance

$$\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2T \delta(t - t') \delta(\mathbf{r} - \mathbf{r}')$$

fully disordered initial conditions (centred gaussian noise)

B) master equation

E.G. GLAUBER 63

i.e. kinetic Ising model with heat-bath dynamics

random initial state

→ relaxation towards equilibrium stationary states

Local scaling with $z = 2 \rightarrow$ LSI

Question : extended dynamical scaling for given $z \neq 1$? MH 92, 94, 02
motivation :

1. conformal invariance in equilibrium critical phenomena, $z = 1$
2. Schrödinger-invariance of simple diffusion, $z = 2$

LIE 1881, NIEDERER 72, HAGEN 71, KASTRUP 68

$$t \mapsto \frac{\alpha t}{\gamma t + \delta}, \quad \mathbf{r} \mapsto \frac{\mathcal{R}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta}, \quad \alpha\delta = 1$$

Lie algebra $\mathfrak{age}_1 := \langle X_{1,0}, Y_{\pm 1/2}, M_0 \rangle$ generators : (no TTI!)

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n r\partial_r - \frac{n(n+1)}{4}\mathcal{M}t^{n-1}r^2 - \left[\frac{x}{2}(n+1) + n\xi\right]t^n$$

$$Y_m = -t^{m+1/2}\partial_r - \left(m + \frac{1}{2}\right)\mathcal{M}t^{m-1/2}r$$

$$M_n = -t^n \mathcal{M}$$

also contains 'phase changes' in the wave function ! (projective)

Stochastic field-theory

Langevin equations do **not** have non-trivial dynamical symmetries !
compare results of **deterministic** symmetries to **stochastic** models ?
go to stochastic field-theory, action

JANSSEN, DE DOMINICIS, . . . 70s-80s

$$\mathcal{J}[\phi, \tilde{\phi}] = \underbrace{\int \tilde{\phi}(2\mathcal{M}\partial_t - \Delta)\phi + \tilde{\phi}\mathcal{V}'[\phi]}_{\mathcal{J}_0[\phi, \tilde{\phi}] : \text{deterministic}} - T \underbrace{\int \tilde{\phi}^2}_{+ \mathcal{J}_b[\tilde{\phi}] : \text{noise}} - \int \tilde{\phi}_{t=0} C_{init} \tilde{\phi}_{t=0}$$

$\tilde{\phi}$: response field ;

$$C(t, s) = \langle \phi(t)\phi(s) \rangle, R(t, s) = \langle \phi(t)\tilde{\phi}(s) \rangle$$

averages : $\langle A \rangle_0 := \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} A[\phi, \tilde{\phi}] \exp(-\mathcal{J}_0[\phi, \tilde{\phi}])$

masses :

$$\mathcal{M}_\phi = -\mathcal{M}_{\tilde{\phi}}$$

Theorem : IF \mathcal{J}_0 is Galilei- and spatially translation-invariant,
then Bargman superselection rules

BARGMAN 54

$$\left\langle \phi_1 \cdots \phi_n \tilde{\phi}_1 \cdots \tilde{\phi}_m \right\rangle_0 \sim \delta_{n,m} \quad (1)$$

computation of a response function

PICONE & MH 04

$$\begin{aligned} R(t, s) &= \left\langle \phi(t) \tilde{\phi}(s) \right\rangle = \left\langle \phi(t) \tilde{\phi}(s) e^{-\mathcal{J}_b[\tilde{\phi}]} \right\rangle_0 \\ &= \left\langle \phi(t) \tilde{\phi}(s) \right\rangle_0 = R_0(t, s) \end{aligned}$$

Bargman eq. (1) \implies response function does not depend on noise!

left side : computed in stochastic models

right side : local scale-symmetry of deterministic equation

application to ageing : age_d-covariant two-point response function

$$R(t, s; \mathbf{r}) = r_0 s^{-1-a} \left(\frac{t}{s} \right)^{1+\textcolor{blue}{a}' - \lambda_R/z} \left(\frac{t}{s} - 1 \right)^{-1-\textcolor{blue}{a}'} \exp \left(-\frac{\mathcal{M}}{2} \frac{\mathbf{r}^2}{t-s} \right) \quad (2)$$

confirmed in many phase-ordering systems

reviews : MH, J. Phys. Cond Matt. **19**, 065101 ('07)

MH & BAUMANN, J. Stat. Mech. P07015 ('07)

Correlation functions for $z = 2$

find $C(t, s) = \langle \phi(t)\phi(s) \rangle = \langle \phi(t)\phi(s)e^{-\mathcal{J}_b[\tilde{\phi}]} \rangle_0$ from Bargman rule

$$C(t, s) = \frac{a_0}{2} \int_{\mathbb{R}^d} d\mathbf{R} R_0^{(4)}(t, s, 0; \mathbf{R}) \quad \text{initial}$$

$$+ \frac{T}{2\mathcal{M}} \int_0^\infty du \int_{\mathbb{R}^d} d\mathbf{R} R_0^{(4)}(t, s, u; \mathbf{R}) \quad \text{thermal}$$

$$R_0^{(4)}(t, s, u; \mathbf{r}) = \left\langle \phi(t, \mathbf{y})\phi(s, \mathbf{y})\tilde{\phi}^2(u, \mathbf{r} + \mathbf{y}) \right\rangle_0$$

Four-point $R_0^{(4)}$ function must be found from local scaling

Theorem : LSI with $z = 2 \implies \lambda_C = \lambda_R$

PICONE & MH 04

agrees with a different argument of BRAY 94 – and explicit models test $C(t, s)$ explicitly in Ising/Potts models

MH ET AL. 04, LORENZ & JANKE 07

what appear to be essential features of local scaling?

- **no** local scaling in **full** Langevin equation
 - noise terms only compatible with translation-invariance
- local scaling in **deterministic** part → reduction formulæ
- **hidden** local scaling symmetry, at least when $z = 2$
- Galilei-invariance of deterministic part
 - together with scaling, implies full ageing-invariance

MH & UNTERBERGER 03

- physical origin of Galilei-invariance?
- testable predictions for responses and correlators

III. Local scale-invariance for $z \neq 2$

Extend known cases $z = 1, 2 \implies$ **axioms of LSI** :

MH 02, BAUMANN & MH 07

- ① Möbius transformations in time (generator X_n)

$$t \mapsto t' = \frac{\alpha t}{\gamma t + \delta} ; \quad \alpha\delta = 1$$

require commutator : $[X_n, X_{n'}] = (n - n')X_{n+n'}$

- ② Dilatation generator : $X_0 = -t\partial_t - \frac{1}{z}\mathbf{r} \cdot \partial_{\mathbf{r}} - \frac{x}{z}$
Implies simple power-law scaling $L(t) \sim t^{1/z}$ (no glasses!).
- ③ Spatial translation-invariance \rightarrow 2^e family Y_m of generators.
- ④ X_n contain phase terms from the scaling dimension $x = x_\phi$
- ⑤ X_n, Y_m contain further 'mass terms' (Galilei !)
- ⑥ finite number of independent conditions for n -point functions.

Extend to $z \neq 1, 2$ by generators with mass terms, for $d = 1$:

$$Y_{1-1/z} := -t\partial_r - \mu zr\nabla_{\mathbf{r}}^{2-z} - \gamma z(2-z)\partial_r\nabla_{\mathbf{r}}^{-z} \quad \text{Galilei}$$

$$\begin{aligned} X_1 := & -t^2\partial_t - \frac{2}{z}tr\partial_r - \frac{2(x+\xi)}{z}t - \mu r^2\nabla_{\mathbf{r}}^{2-z} \quad \text{special} \\ & -2\gamma(2-z)r\partial_r\nabla_{\mathbf{r}}^{-z} - \gamma(2-z)(1-z)\nabla_{\mathbf{r}}^{-z} \end{aligned}$$

- depend on two parameters γ, μ and on two dimensions x, ξ
- contains fractional derivative $(\widehat{f} : \text{Fourier transform})$

$$\nabla_{\mathbf{r}}^\alpha f(\mathbf{r}) := i^\alpha \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^\alpha e^{i\mathbf{r}\cdot\mathbf{k}} \widehat{f}(\mathbf{k})$$

- some properties : $\nabla_{\mathbf{r}}^\alpha \nabla_{\mathbf{r}}^\beta = \nabla_{\mathbf{r}}^{\alpha+\beta}, [\nabla_{\mathbf{r}}^\alpha, r_i] = \alpha \partial_{r_i} \nabla_{\mathbf{r}}^{\alpha-2}$
 $\nabla_{\mathbf{r}}^\alpha \exp(i\mathbf{q}\cdot\mathbf{r}) = i^\alpha |\mathbf{q}|^\alpha \exp(i\mathbf{q}\cdot\mathbf{r})$

Fact 1 : simple algebraic structure :

$$[X_n, X_{n'}] = (n - n')X_{n+n'} \quad , \quad [X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}$$

→ Generate Y_m from $Y_{-1/z} = -\partial_r$.

Fact 2 : LSI-invariant Schrödinger operator :

$$\mathcal{S} := -\mu\partial_t + z^{-2}\nabla_r^2$$

Let $x_0 + \xi = 1 - 2/z + (2 - z)\gamma/\mu$. Then $[\mathcal{S}, Y_m] = 0$ and

$$[\mathcal{S}, X_0] = -\mathcal{S} \quad , \quad [\mathcal{S}, X_1] = -2t\mathcal{S} + \frac{2\mu}{z}(x - x_0)$$

⇒ $\boxed{\mathcal{S}\phi = 0}$ is **LSI-invariant** equation, if $x_\phi = x_0$.

Physical assumption (hidden) : equations of motion remain of first order in ∂_t , even after renormalisation.

Fact 3 : non-trivial conservation laws :

iterated commutator with $G := Y_{1-1/z}$, $\text{ad}_G = [., G]$

$$M_\ell := (\text{ad}_G)^{2\ell+1} Y_{-1/z} = a_\ell \mu^{2\ell+1} \nabla_{\mathbf{r}}^{(2\ell+1)(1-z)+1}$$

For $z = 2$, $a_\ell = 0$ if $\ell \geq 1$. For a n -point function

$F^{(n)} = \langle \phi_1 \dots \phi_n \rangle$, $M_\ell F^{(n)} = 0$ gives in momentum space

$$\left(\sum_{i=1}^n \mu_i^{2\ell-1} |\mathbf{k}_i|^{2\ell-(2\ell-1)z} \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) = 0$$

$$\left(\sum_{i=1}^n \mathbf{k}_i \right) \widehat{F}^{(n)}(\{t_i, \mathbf{k}_i\}) = 0$$

\implies momentum conservation & conservation of $|\mathbf{k}|^\alpha$!

analogous to relativistic factorisable scattering

ZAMOLODCHIKOV² 79, 89

equil. analogy : 2D Ising model at $T = T_c$ in magnetic field

Consequence : a $\mathbb{I}\mathbb{S}\mathbb{I}$ -covariant $2n$ -point function $F^{(2n)}$ is only non-zero, if the ‘masses’ μ_i can be arranged in pairs $(\mu_i, \mu_{\sigma(i)})$ with $i = 1, \dots, n$ such that $\boxed{\mu_i = -\mu_{\sigma(i)}}.$

generalised Galilei-invariance with $z \neq 2 \implies$ integrability

Corollary 1 : Bargman rule : $\langle \phi_1 \dots \phi_n \tilde{\phi}_1 \dots \tilde{\phi}_m \rangle_0 \sim \delta_{n,m}$

Corollary 2 : treat (linear) stochastic equations with $\mathbb{I}\mathbb{S}\mathbb{I}$ -invariant deterministic part, reduction formulæ

Corollary 3 : response function noise-independent

$$R(t, s; \mathbf{r}) = R(t, s) \mathcal{F}^{(\mu_1, \gamma_1)}(|\mathbf{r}|(t-s)^{-1/z})$$

$$R(t, s) = r_0 s^{-a} \left(\frac{t}{s}\right)^{1+a'-\lambda_R/z} \left(\frac{t}{s}-1\right)^{-1-a'}$$

$$\mathcal{F}^{(\mu, \gamma)}(\mathbf{u}) = \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^\gamma \exp(i\mathbf{u} \cdot \mathbf{k} - \mu|\mathbf{k}|^z)$$

Corollary 4 :

Correlators obtained from factorised 4-point responses.

How to test the foundations of LSI

theory is built on :

- a) simple scaling – domain sizes $L(t) \sim t^{1/z}$
- b) invariance under Möbius transformation $t \mapsto t/(\gamma t + \delta)$
- c) Galilei-invariance generalised to $z \neq 2$

together with spatial translation-invariance

⇒ extended Bargman rules

⇒ factorisation of $2n$ -point functions

Möbius transformation	autoresponse $R(t, s)$
generalised Galilei-invariance	space-time response $R(t, s; \mathbf{r})$
factorisation	two-time correlation function

Tests of LSI for $z \neq 2$:

- spherical model with conserved order-parameter, $T = T_c$,
 $z = 4$ BAUMANN & MH 06
- Mullins-Herring model for surface growth, $z = 4$ RÖTHLEIN, BAUMANN, PLEIMLING 06
- spherical model with long-ranged interactions, $T \leq T_c$,
 $0 < z = \sigma < 2$ CANNAS ET AL. 01 ; BAUMANN, DUTTA, MH 07
- **2D Ising model with disorder**, $T < T_c$ (non-frustrated)
Hamiltonian $\mathcal{H} = - \sum_{(i,j)} J_{ij} \sigma_i \sigma_j$
uniform disorder $J_{ij} \in [1 - \varepsilon/2, 1 + \varepsilon/2]$, $0 \leq \varepsilon \leq 2$
 $\Rightarrow T_c(\varepsilon) \approx T_c(0)$
kinetics : non-conserved order-parameter, heat bath

Phase-ordering in disordered Ising models

pure systems : dynamics through moving domain walls

universal description through Allen-Cahn eq. :

$$\dot{L} = v = (1 - d)K \sim L^{-1} \implies L(t) \sim t^{1/2}$$

disorder : pins domain walls, need thermal activation

HENLEY & HUSE 85

$$\frac{dL(t)}{dt} = D(T, L)L(t)^{-1}, \quad D(T, L) = D_0 \exp(-E_B(L)/T)$$

a) power-law $E_B(L) = E_0 L^\psi$: $\implies L(t) \sim (\ln t)^{1/\psi}$

HENLEY & HUSE 85

b) log law $E_B(L) = \epsilon \ln(1 + L)$: $\implies L(t) \sim t^{1/z}$

with temperature- and disorder-dependent dynamical exponent

$$z = 2 + \epsilon/T$$

R. PAUL, S. PURI & H. RIEGER 04

Confirmations of algebraic scaling of the domain sizes :

- ① direct simulations of $L(t)$ in bond- and site-disordered 2D Ising models
gives **empirical** identification $\epsilon = \varepsilon$ for bond-disorder

PAUL, PURI, RIEGER 04/05

- ② analytical studies of SOS-model on disordered substrate

$$\mathcal{H} = \sum_{(i,j)} (h_i - h_j)^2 , \quad h_i = \underbrace{n_i}_{\in \mathbb{Z}} + \underbrace{d_i}_{disorder, \in [0,1]}$$

continuum limit described by Cardy-Ostlund model

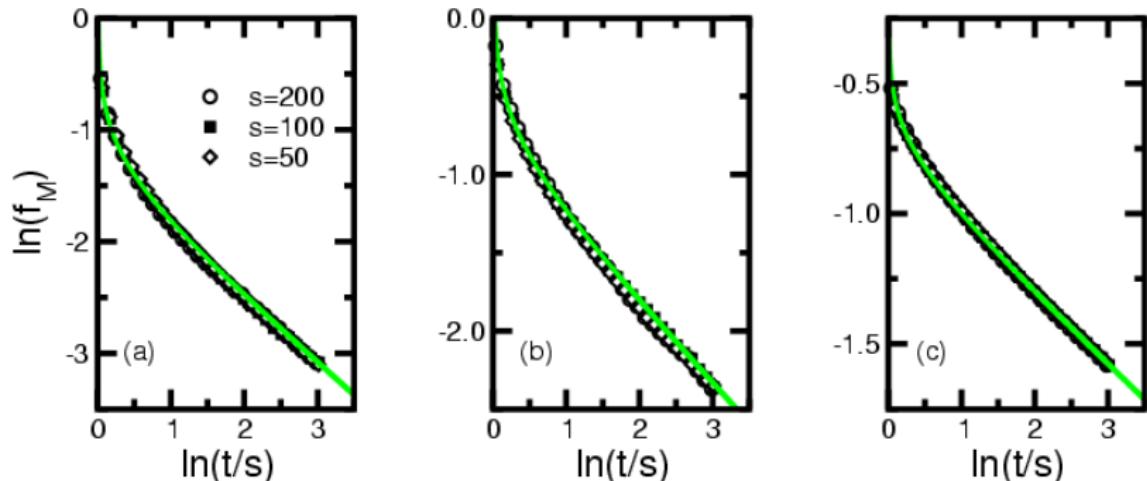
$$\mathcal{H} = \int d\mathbf{r} [(\nabla \phi(\mathbf{r}))^2 - g \cos(2\pi[\phi(\mathbf{r}) - \xi(\mathbf{r})])]$$

such that $z = 2 + (2\pi^2/9)(T_g/\textcolor{red}{T})$ if $T \ll T_g$

SCHEHR & LE DOUSSAL 04/05

change contrôles parameters to vary z

Practical tests of LSI, I : autoresponse



(a) $\varepsilon = 0.5, T = 0.6$ (b) $\varepsilon = 1, T = 1$ (c) $\varepsilon = 2, T = 0.6$

Thermoremanent susceptibility

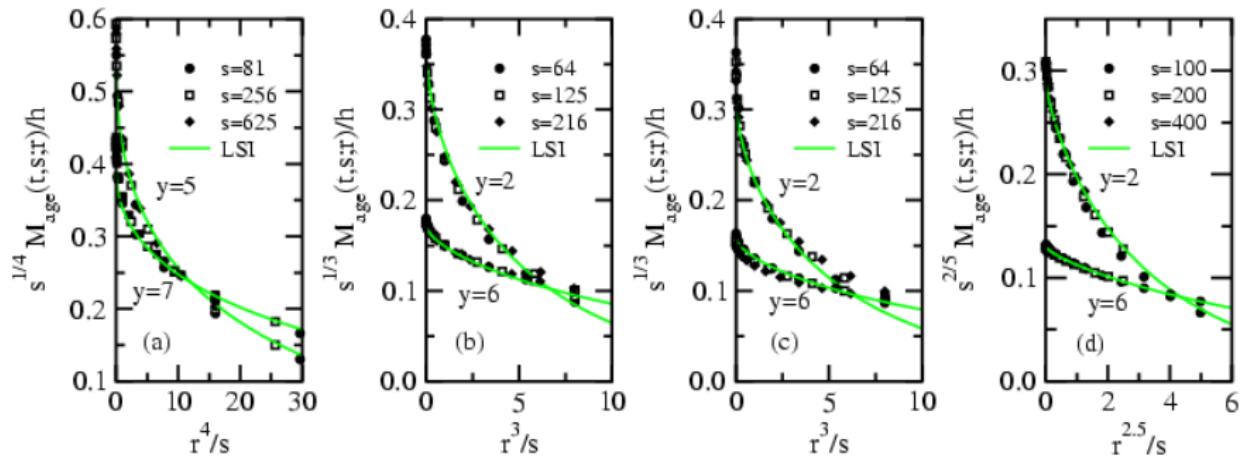
$$\chi_{\text{TRM}}(t, s) = \int_0^s du R(t, u) = s^{-a} f_M(t/s) + O(s^{-\lambda_R/z})$$

Full **curve** : LSI-prediction, with $a = a' = 1/z$.

Confirm $z = 2 + \varepsilon/T$, in agreement with PAUL, PURI & RIEGER 04/05.

MH & Pleimling, Europhys. Lett. **76**, 561 (2006).

Practical tests of LSI, II : space-time response



(a) $\varepsilon = 2, T = 1$ (b) $\varepsilon = T = 1$ (c) $\varepsilon = T = 0.5$ (d) $\varepsilon = 0.5, T = 1$

$$\chi_{\text{TRM}}(t, s; \mathbf{r}) = \int_0^s du R(t, u; \mathbf{r}) = s^{-a} r_0 f_M(t/s, \mathbf{r} s^{-1/z}) + O(t^{-\lambda_R/z})$$

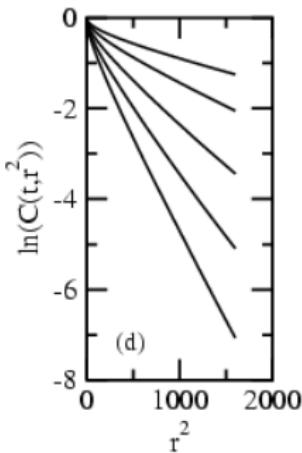
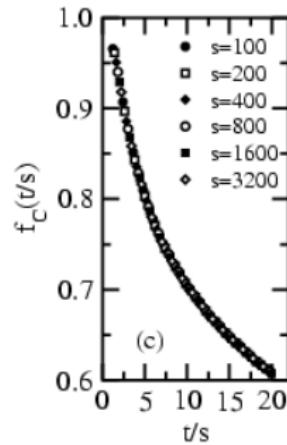
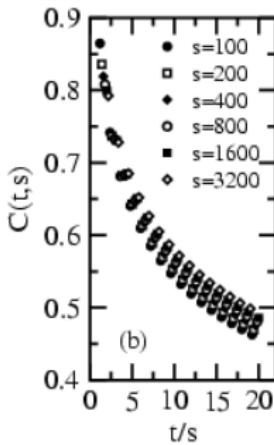
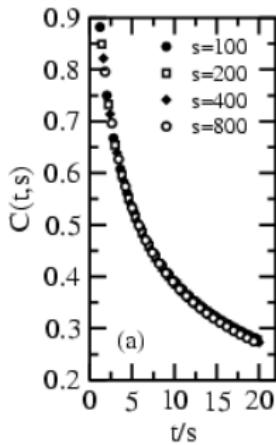
Scaling function f_M only depends on ratio $\varepsilon/T \Rightarrow$ universality

Full **curve** : LSI-prediction, with $y = t/s$ fixed and $a = a' = 1/z$.

first test of ‘Galilei-invariance’ for $z \neq 2$ in a non-linear model

Baumann, MH & Pleimling, arXiv :0709.3228

Practical tests of LSI, III : autocorrelation



(a) $\varepsilon = 0.5, T = 1$, (b,c) $\varepsilon = 2, T = 1$ (d) $t = [200, 300, 500, 1000, 2000]$

No simple scaling with $y = t/s$ for $z \gtrsim 4$!

P, S & R 06, H & P 06

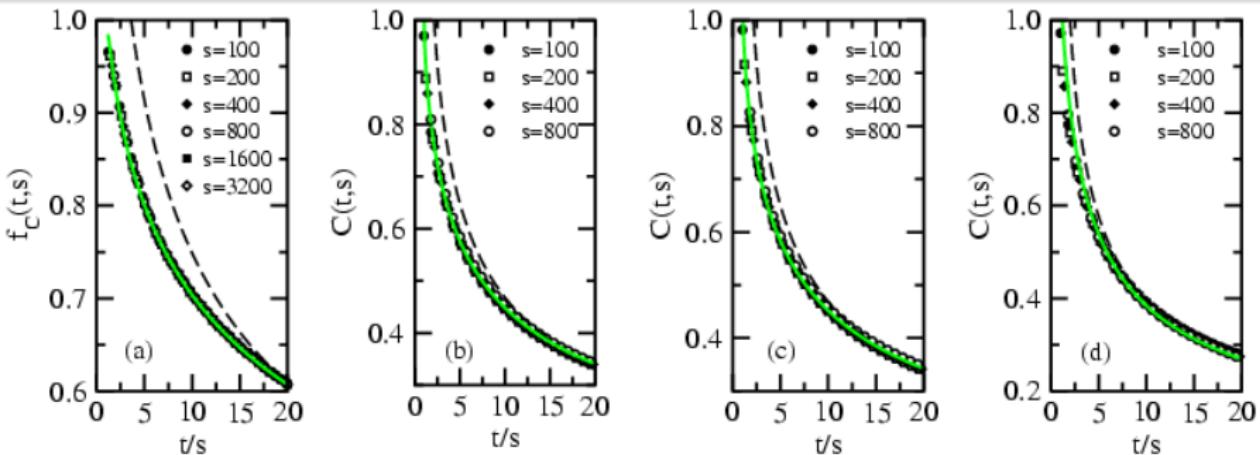
↘ indication for ‘superageing’?

PAUL, SCHEHR, RIEGER 07

1. subtract finite-time correction, $C(t,s) = f_C(t/s) - s^{-b'} g_C(t/s)$
 2. then scaling of $C(t,s)$ according to **simple ageing** with $y = t/s$
- * Scaling function f_C only depends on ratio $\varepsilon/T \implies$ **universality**
 - * ageing sets in at time scale $\tau = t - s \sim s^\zeta$
 - use $C(s + \tau, s; r) \sim \exp(-\nu r^2 s^{-2/z})$

ZIPPOLD, KÜHN, HORNER 00

generalised from OHTA, JASNOW, KAWASAKI 82



(a) $\varepsilon = 2, T = 1$ (b) $\varepsilon = T = 1$ (c) $\varepsilon = T = 0.5$ (d) $\varepsilon = 0.5, T = 1$

Dashed line : LSI with fully disordered initial correlator

Full **curve** : LSI prediction

$$f_C(y) = c_2 y^\rho \int_{\mathbb{R}^d} \frac{d\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{2\beta} \exp \left(-\alpha |\mathbf{k}|^z (y - 1) - \frac{\mathbf{k}^2}{4\nu} \right)$$

with $\beta = \lambda_C - \lambda_R$, $\rho = (2\beta + d - \lambda_C)/z$. Use ‘initial’ correlator $C(s + \tau, s; \mathbf{r}) \sim \exp(-\nu \mathbf{r}^2 s^{-2/z})$: asymptotics enough for $z > 2$

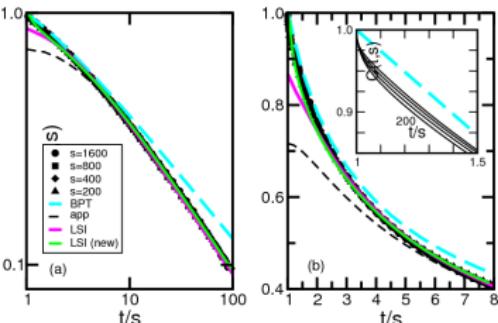
Retour to the **non-disordered Ising model**, at $T < T_c$: $z = 2$
 requires a more precise form of the 'initial' correlator :

Ohta, Jasnow, Kawasaki '82

$$C(t, t; \mathbf{r}) = \frac{2}{\pi} \arcsin \left(\exp \left[-\frac{\mathbf{r}^2}{L(t)^2} \right] \right)$$

hence autocorrelator in the scaling limit

$$\begin{aligned} C(ys, s) &= C_0 y^\rho (y-1)^{-\rho - \lambda_C/z} \int_0^\infty dx e^{-x} f_\nu \left(\sqrt{\frac{x}{y-1}} \right) \\ f_\nu(\sqrt{u}) &= \int_0^\infty dv \arcsin(e^{-\nu v}) J_0(\sqrt{uv}) \end{aligned}$$



of practical importance :
 'good' choice of 'initial correlations'
 $C_{\text{ini}}(\mathbf{r}) = c_0 \delta(\mathbf{r})$ not sufficient

IV. Conclusions

- ① look for extensions of dynamical scaling in ageing systems

recently, scaling derived for phase-ordering ARENZON ET AL. 07

- ② here : **hypothesis** of **generalised Galilei-invariance**
- ③ leads to Bargman rule if $z = 2$
and further to 'integrability' if $z \neq 1, 2$.
- ④ **hidden** dynamical symmetry of deterministic part of (linear)
Langevin equations
- ⑤ Tests : derive two-time response and correlation functions
- ⑥ LSI exactly proven for linear Langevin equations
very good numerical evidence for non-linear systems

Some questions (the list could/should be extended) :

- how to physically justify Galilei-invariance ?
- how to extend to non-linear equations ?
- choice of the type of fractional derivative ?
- what is the algebraic (non-Lie !) structure of LSI ?
- treatment of master equations with LSI ?