

# Towards Functional Flows for Hierarchical Models

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# motivation

- **strongly coupled quantum fields**

lattice vs continuum implementations  
different numerical tools  
explicit links?

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- **strongly coupled quantum fields**

lattice vs continuum implementations  
different numerical tools  
explicit links?

- **critical phenomena in 3d scalar theories**

lattice implementation: hierarchical models  
continuum implementation: functional flows  
similarities, differences, links

# hierarchical model

- **lattice (scalar) fields**

lattice-regularised scalar field  $\varphi$ , lattice spacing  $a$

**questions:**

infrared physics

phase structure, phase transition, scaling exponents

- **Kadanoff blocking**

decimation step

$$a \rightarrow \ell a$$

interaction potential

$$v_k(\varphi) \rightarrow \ell^D v_{k/\ell}(\varphi)$$

field rescaling

$$\varphi \rightarrow \ell^{1-D/2} \varphi$$

hierarchical model: exact implementation!

# Dyson's hierarchical model

- **definition**

Dyson (1969), Baker (1972)

$$e^{-v_{k/\ell}(\varphi)} = \int d\xi \mu_\ell(\xi) \exp \left[ -\ell^3 v_k(\ell^{-1/2}\varphi + \xi) \right]$$

measure

$$\mu_\ell(\xi) = [\pi \sigma(\ell)]^{-1/2} \exp \left[ -\xi^2 / \sigma(\ell) \right]$$

$$\sigma(\ell) > 0$$

$$\ell \geq 1$$

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(norm)  $\int d\xi \mu_\ell(\xi) = 1$

(continuum)  $\lim_{\ell \rightarrow 1} \mu_\ell(\xi) = \delta(\xi)$

(interactions)  $\lim_{\ell \rightarrow 1} \ell \partial_\ell \mu_\ell(\xi) = -\delta''(\xi)$

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- **many rigorous results**

existence, uniqueness of fixed points

Felder (1987), Koch, Wittwer (1988)

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- **continuum transformation**

$$\ell \rightarrow 1 : \quad \ell \partial_\ell v = -3v + \tfrac{1}{2}\varphi v' - v'' + (v')^2$$

continuum Wilson-Polchinski RG

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- **numerical implementations**

3d O(N) symmetric theories

Koch, Wittwer (1988), Pinn, Pordt, Wieczerkowski (1994)

Godina et. al. (1997), Gottker-Schnettmann (1999), ..., Meurice (2007)

# Wilson's hierarchical model

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$$e^{-v_{k/\ell}(\varphi)} = \int d\xi \mu_\ell(\xi) \exp \left[ -\frac{1}{2} \ell^3 v_k(\ell^{-1/2} \varphi + \xi) + (\xi \leftrightarrow -\xi) \right]$$

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DL (2007), Meurice (2007)

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DL (2007), Meurice (2007)

- **numerical implementations**

3d O(N) symmetric theories

Wilson (1971), Meurice, Ordaz (1996), [Meurice (2007)]

# functional flows

- **continuum (scalar) fields**

continuum fields  $\varphi$ , momentum cutoff  $k$

**questions:**

infrared physics

phase structure, phase transition, scaling exponents

- **infinitesimal Kadanoff blocking**

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left( \Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \partial_t R_k$$

“exact” continuum RG at momentum scale  $k$

Wilsonian momentum cutoff, with

$$R_k(q^2) \rightarrow 0 \text{ for } k^2/q^2 \rightarrow 0, \quad R_k(q^2) > 0 \text{ as } q^2/k^2 \rightarrow 0, \quad R_k(q^2) \rightarrow \infty \text{ for } k \rightarrow \Lambda$$

# functional flows

- **continuum (scalar) fields**

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**questions:**

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- **local potential approximation**

$$\Gamma_k[\phi] = \int d^D p \left[ \frac{1}{2} \phi p^2 \phi + V_k(\phi) \right]$$

# functional flows

- **continuum (scalar) fields**

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**questions:**

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- **local potential approximation**

$$\partial_t u' = -2u' + \rho u'' - \int_0^\infty dy \frac{y^{5/2} r'(y)(3u'' + 2\rho u''')}{(y + y r(y) + u'' + 2\rho u'')^2}$$

$$u(\rho) = U_k/k^3, \quad \rho = \phi^2/(2k), \quad r(y) = R_k(q^2)/q^2, \quad y = q^2/k^2$$

# background field flows

- (generalised) proper-time flows

DL, Pawłowski (2002)

background fields  $\bar{\phi}$  with  $\Gamma_k[\phi] \rightarrow \Gamma_k[\phi, \bar{\phi}]$

introduce

$$x = \Gamma^{(2,0)}[\phi, \phi], \quad R(q^2) \rightarrow \bar{x} r[\bar{x}]$$

$$r_{\text{PT},m}[x] = \exp \left( \frac{1}{m} \left( \frac{mk^2}{x} \right)^m {}_2F_1[m, m; m+1; -\frac{mk^2}{x}] \right) - 1$$

then

$$\partial_t \Gamma_k = \text{Tr} \left( \frac{k^2}{k^2 + x/m} \right)^m + \mathcal{O}(\partial_t x)$$

# background field flows

- (generalised) proper-time flows    DL, Pawłowski (2002)

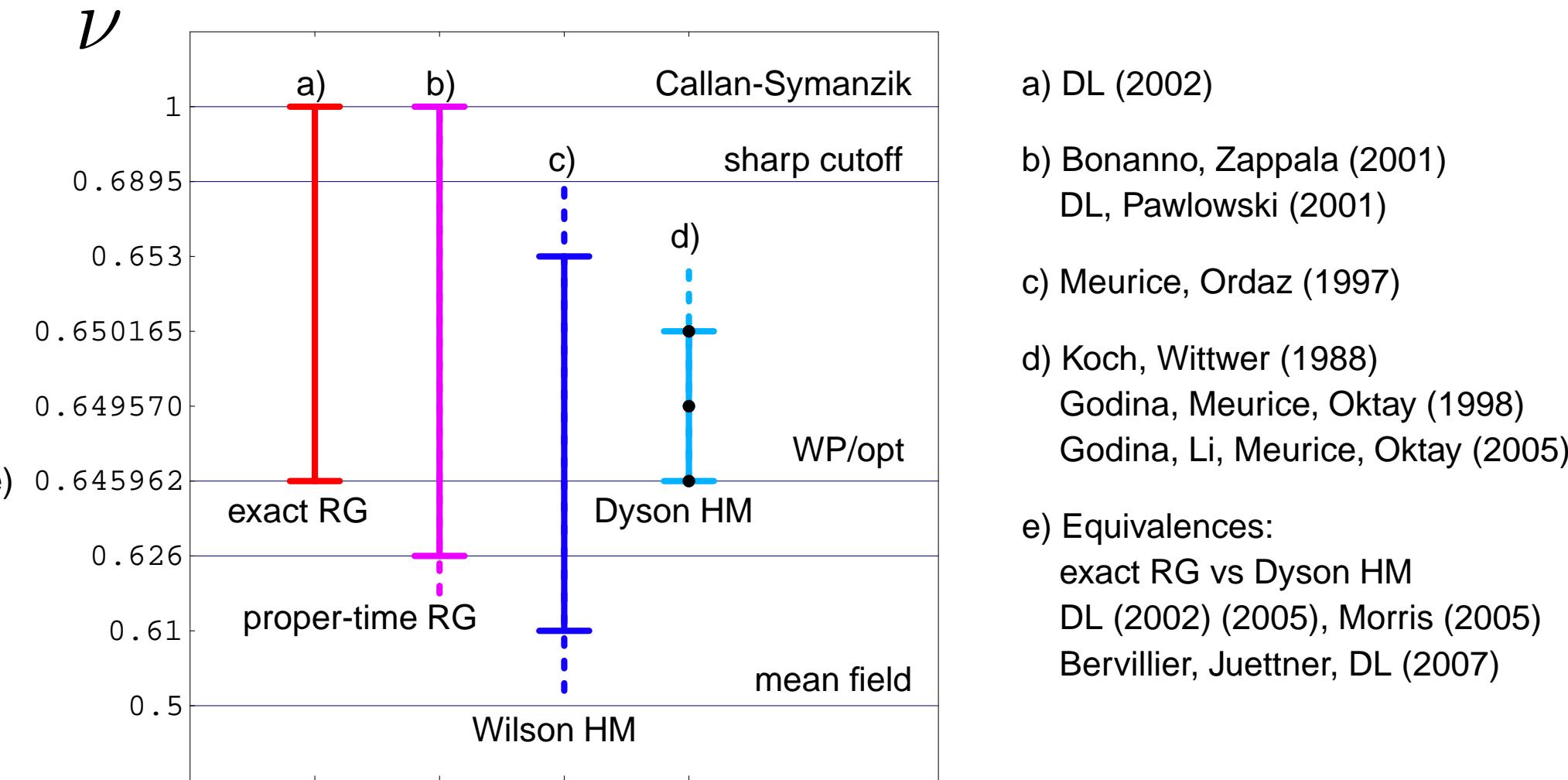
background fields  $\bar{\phi}$  with  $\Gamma_k[\phi] \rightarrow \Gamma_k[\phi, \bar{\phi}]$

- local potential approximation

neglect  $\partial_t x$  : standard proper-time flow    Liao (1997)

$$\partial_t u' = -2u' + \rho u'' - \frac{3u'' + 2\rho u'''}{(m + u' + 2\rho u'')^{m-1/2}}$$

# comparison



# correlations of scaling exponents

- **functional RG study**

Ising universality class

local potential approximation  $u(\rho)$

$$\partial_t u' = -2u' + \rho u'' - \int_0^\infty dy \frac{y^{5/2} r'(y) (3u'' + 2\rho u''')}{(y + y r(y) + u'' + 2\rho u'')^2}$$

momentum cutoff  $R = y \cdot r(y), y = q^2/k^2$

scaling solution  $\partial_t u'_* = 0$

scaling exponents  $\partial_t (u'_* + \delta u'_n) = \omega_n \delta u'_n$

analyse  $\{\omega_n(R)\}$

# correlations of scaling exponents


$$r_{\text{mod}} = 1/(\exp[c(y + (b - 1)y^b)/b] - 1), \quad c = \ln 2$$

$$r_{\text{opt},n} = b(1/y - 1)^n \theta(1 - y), \quad n = 1$$

$$r_{\text{mexp}} = b/((b + 1)^y - 1)$$

$(r_{\text{PT}})$  proper time flow

$$r_{\text{exp}} = 1/(\exp cy^b - 1)$$

$$r_{\text{mix}} = \exp[-b(\sqrt{y} - 1/\sqrt{y})]$$

$$r_{\text{mix,opt}} = \exp[-\frac{1}{b}(y^b - y^{-b})]$$

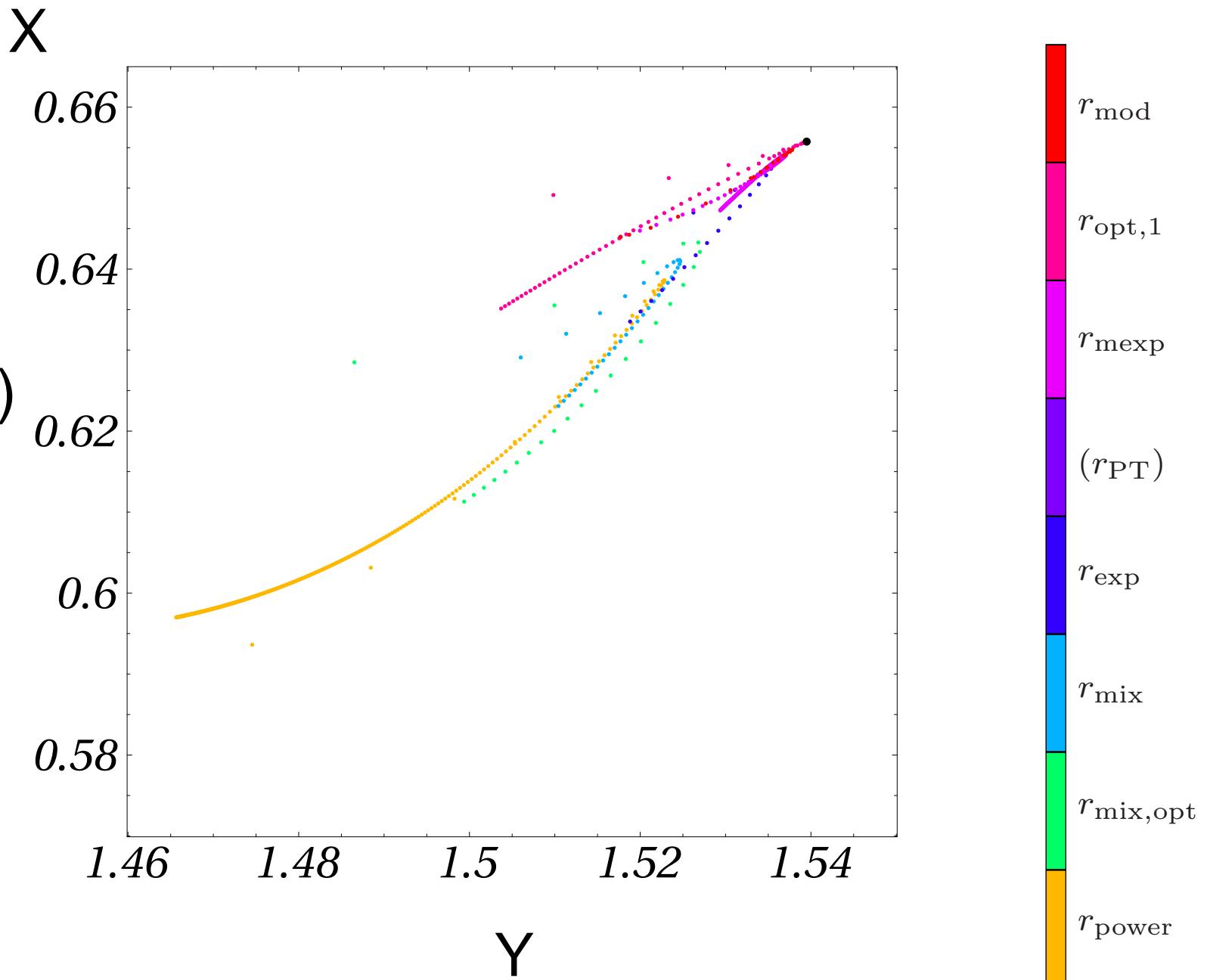
$$r_{\text{power}} = y^{-b}$$

# correlations of scaling exponents

$X = \omega$

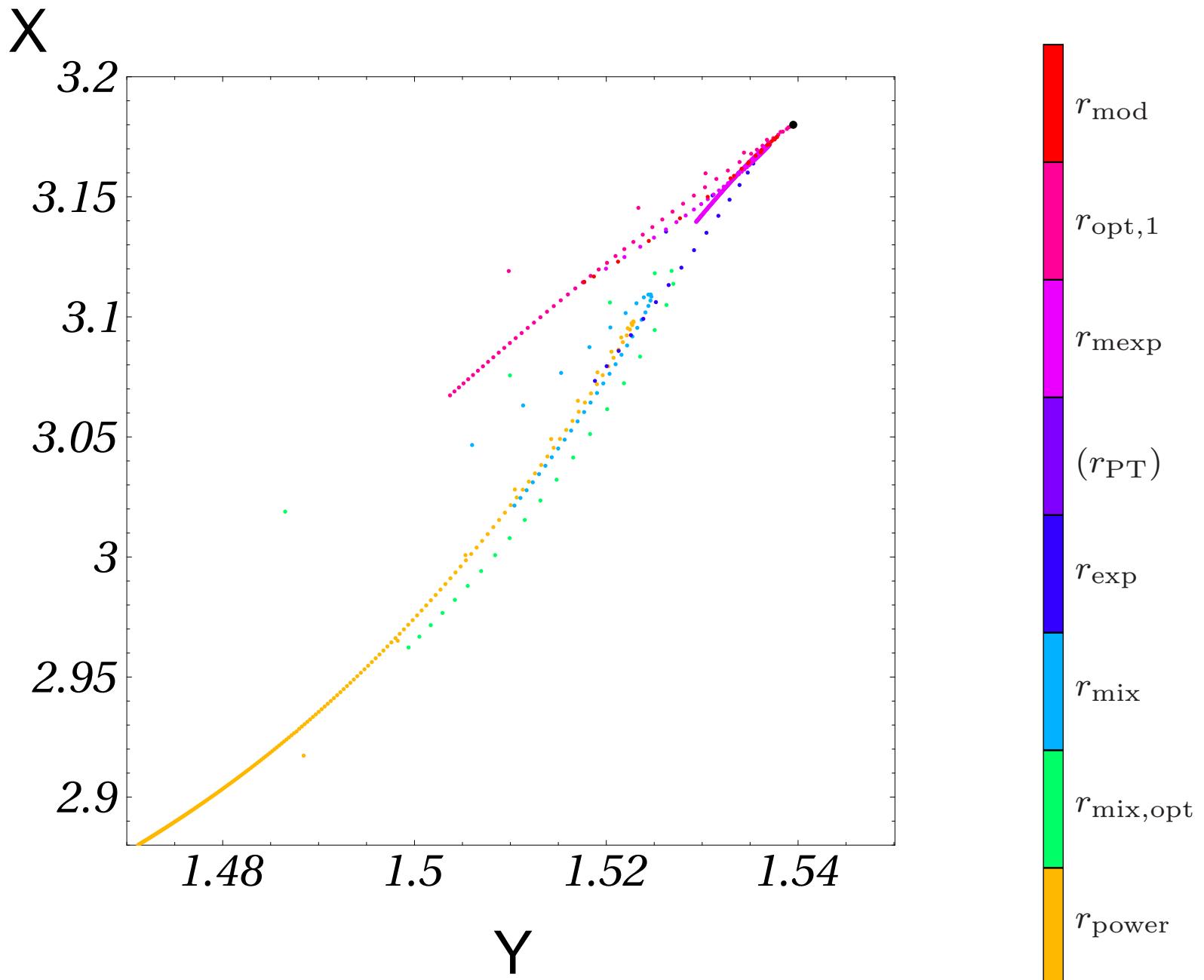
$Y = |\omega_0|$

$(\nu = |\omega_0|^{-1})$



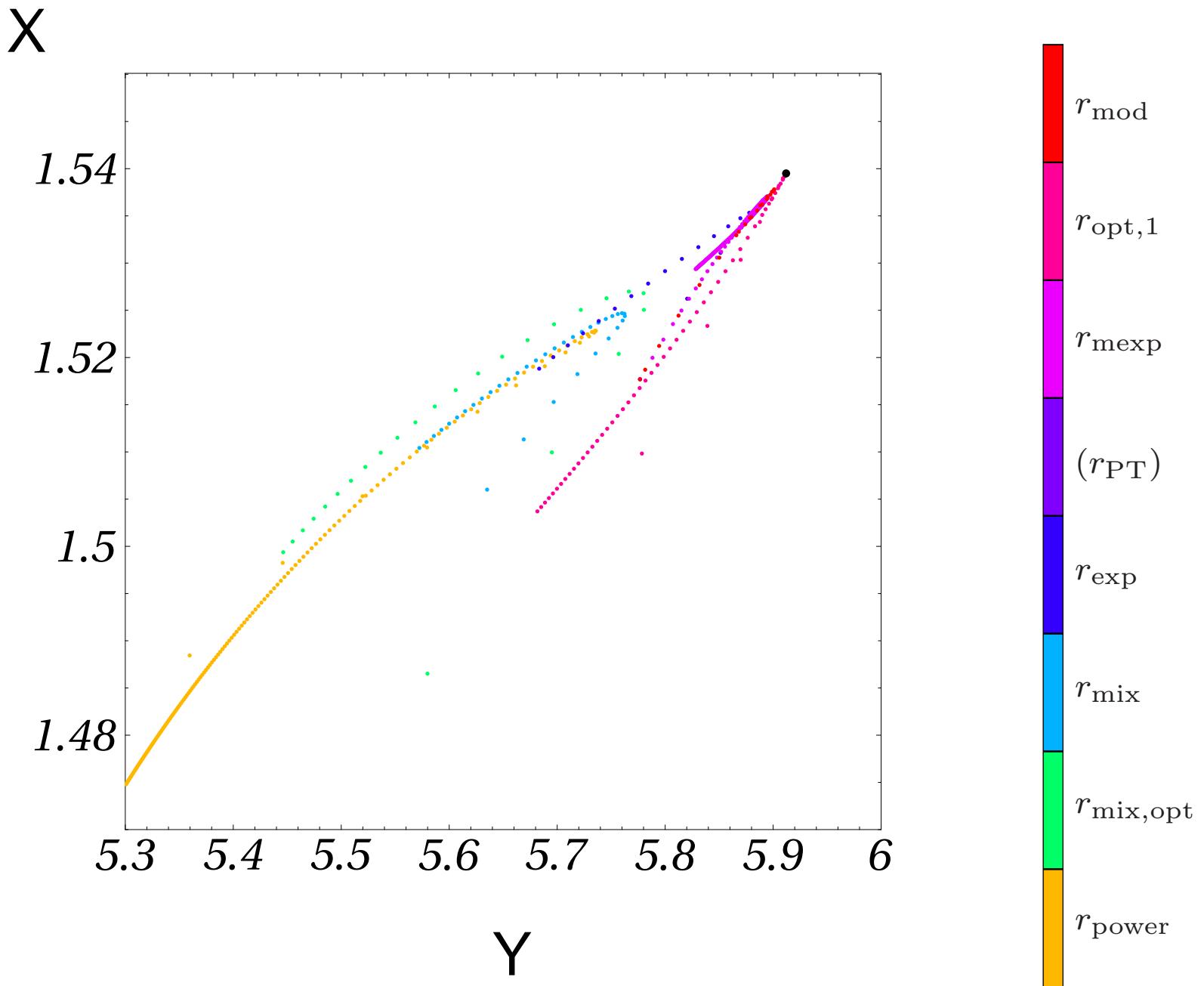
# correlations of scaling exponents

$X = \omega_2$   
 $Y = |\omega_0|$



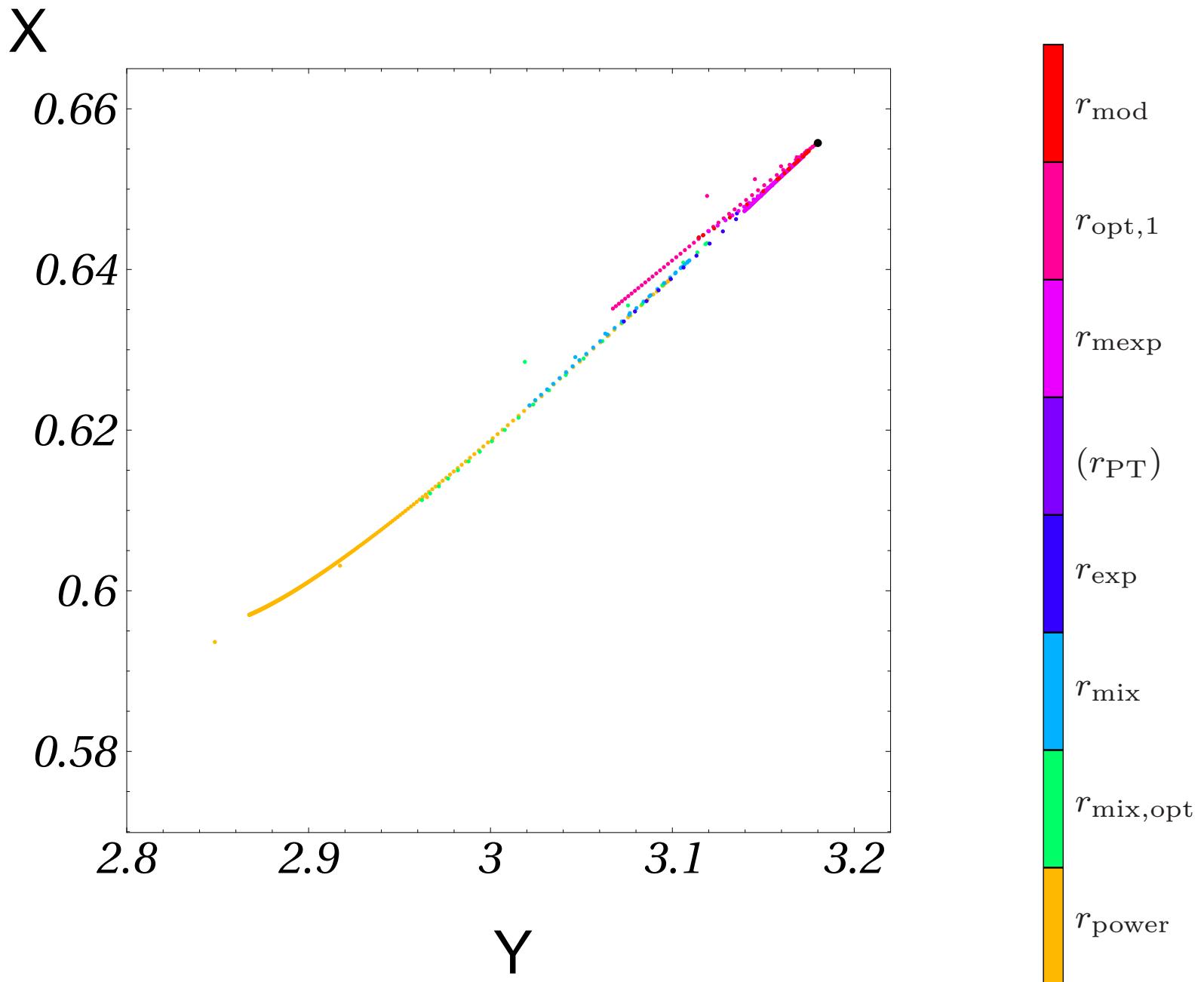
# correlations of scaling exponents

X =  $|\omega_0|$   
Y =  $\omega_3$



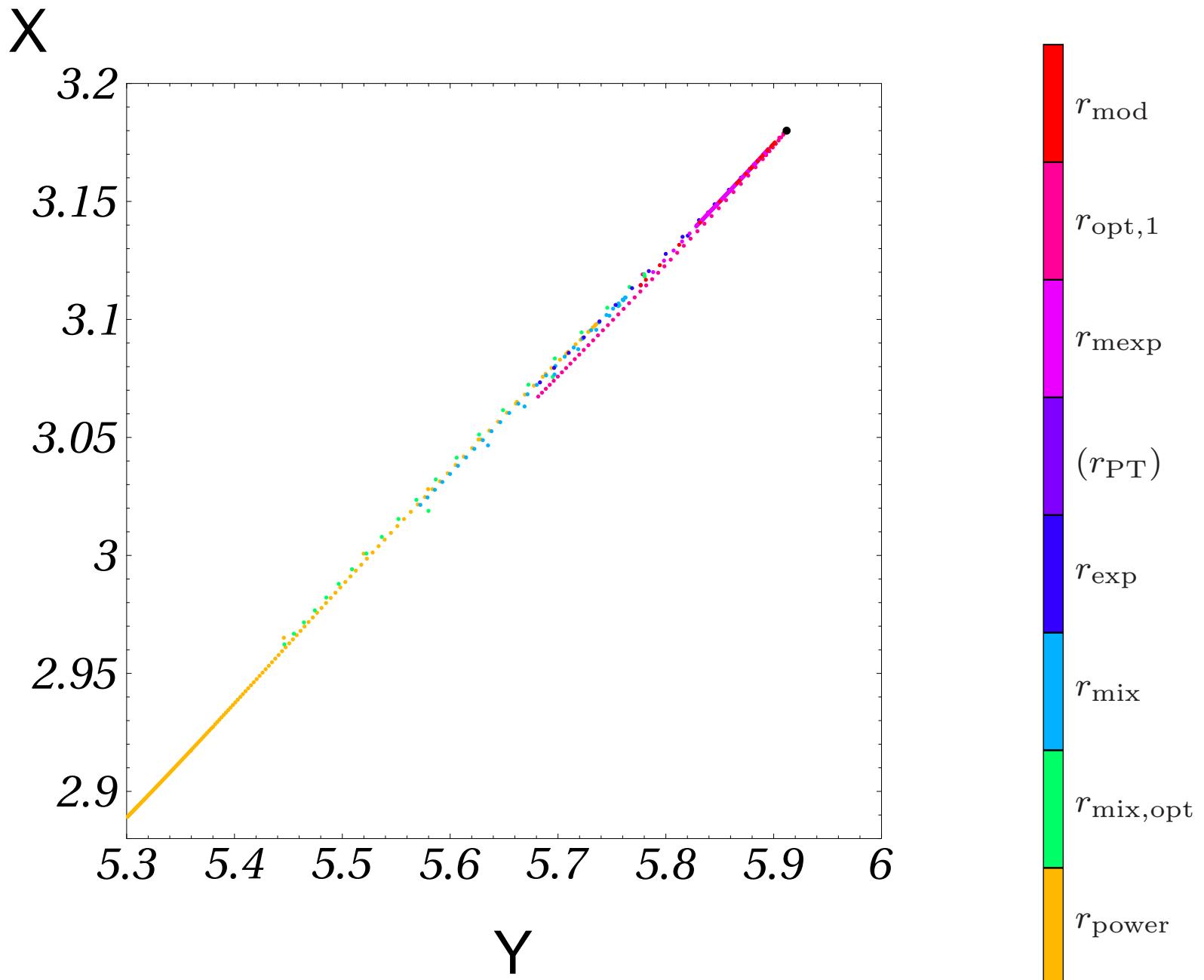
# correlations of scaling exponents

$X = \omega$   
 $Y = \omega_2$



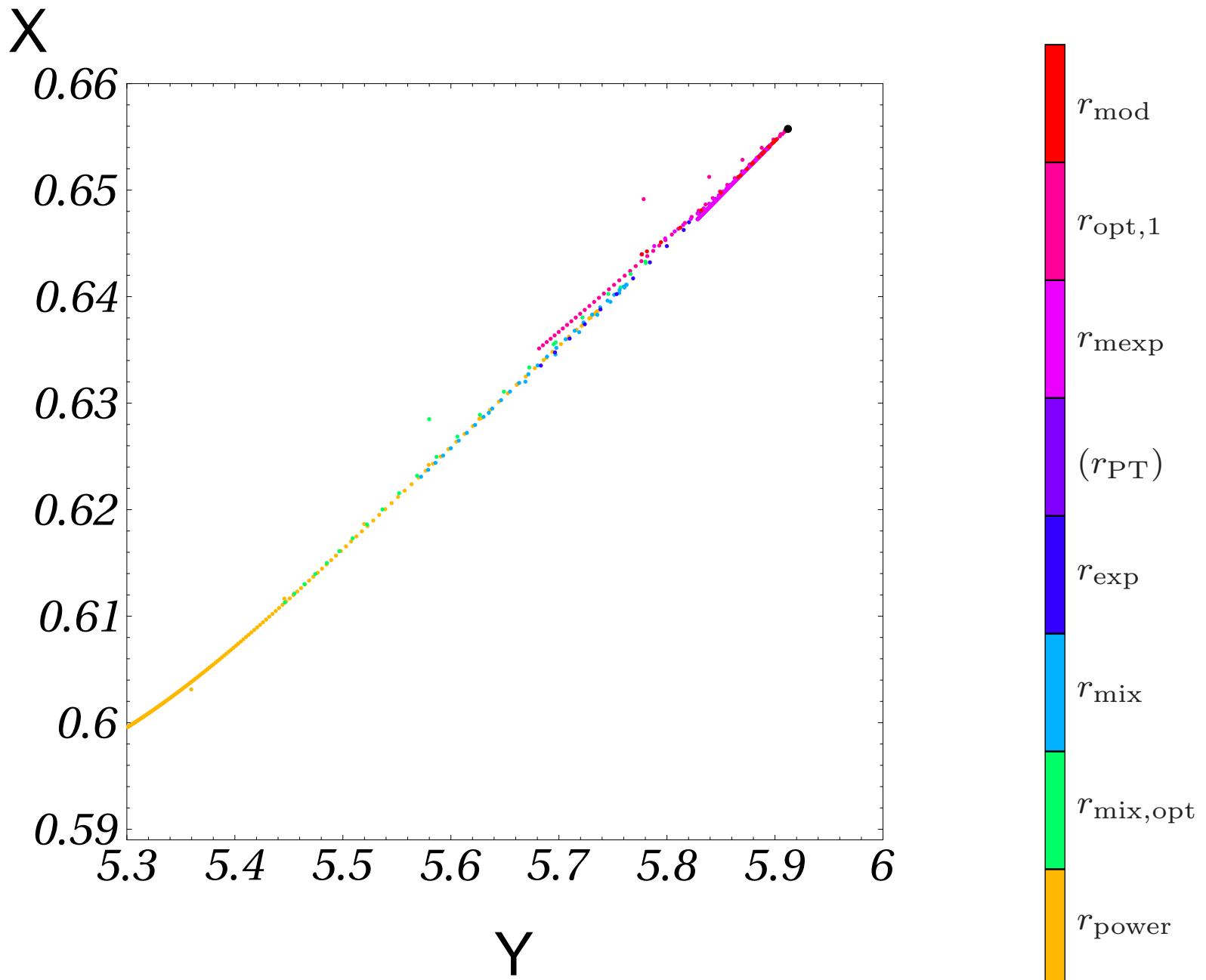
# correlations of scaling exponents

$X = \omega_2$   
 $Y = \omega_3$



# correlations of scaling exponents

$X = \omega$   
 $Y = \omega_3$



# distance of scaling exponents

$$\rho(x, y) = \sqrt{(x_{\text{opt}} - x)^2 + (y_{\text{opt}} - y)^2} \equiv 10^{-N_\rho(x, y)}$$

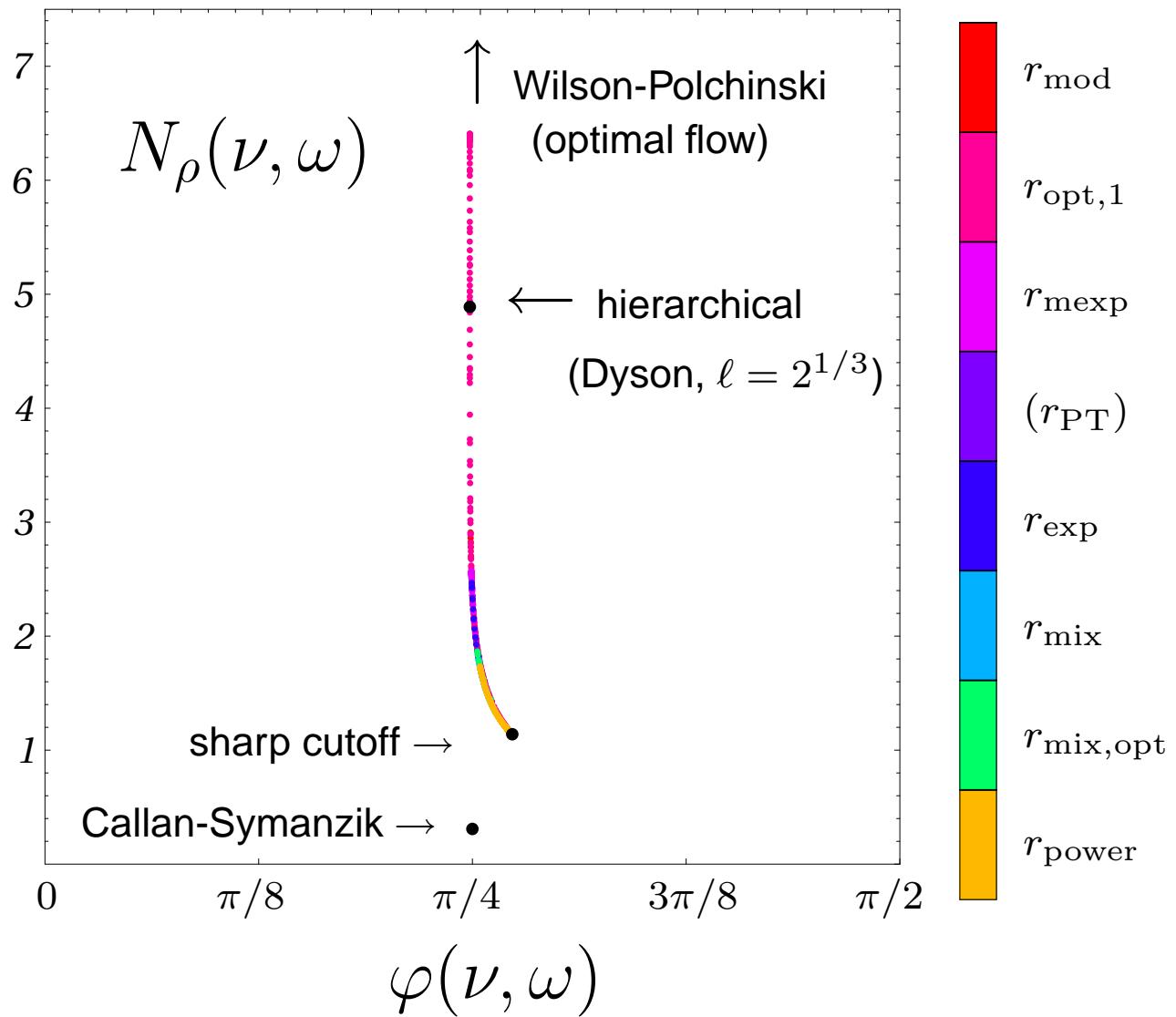
$$\varphi(x, y) = \arctan(x/y)$$

$$\rho_{\text{opt}} = 0$$

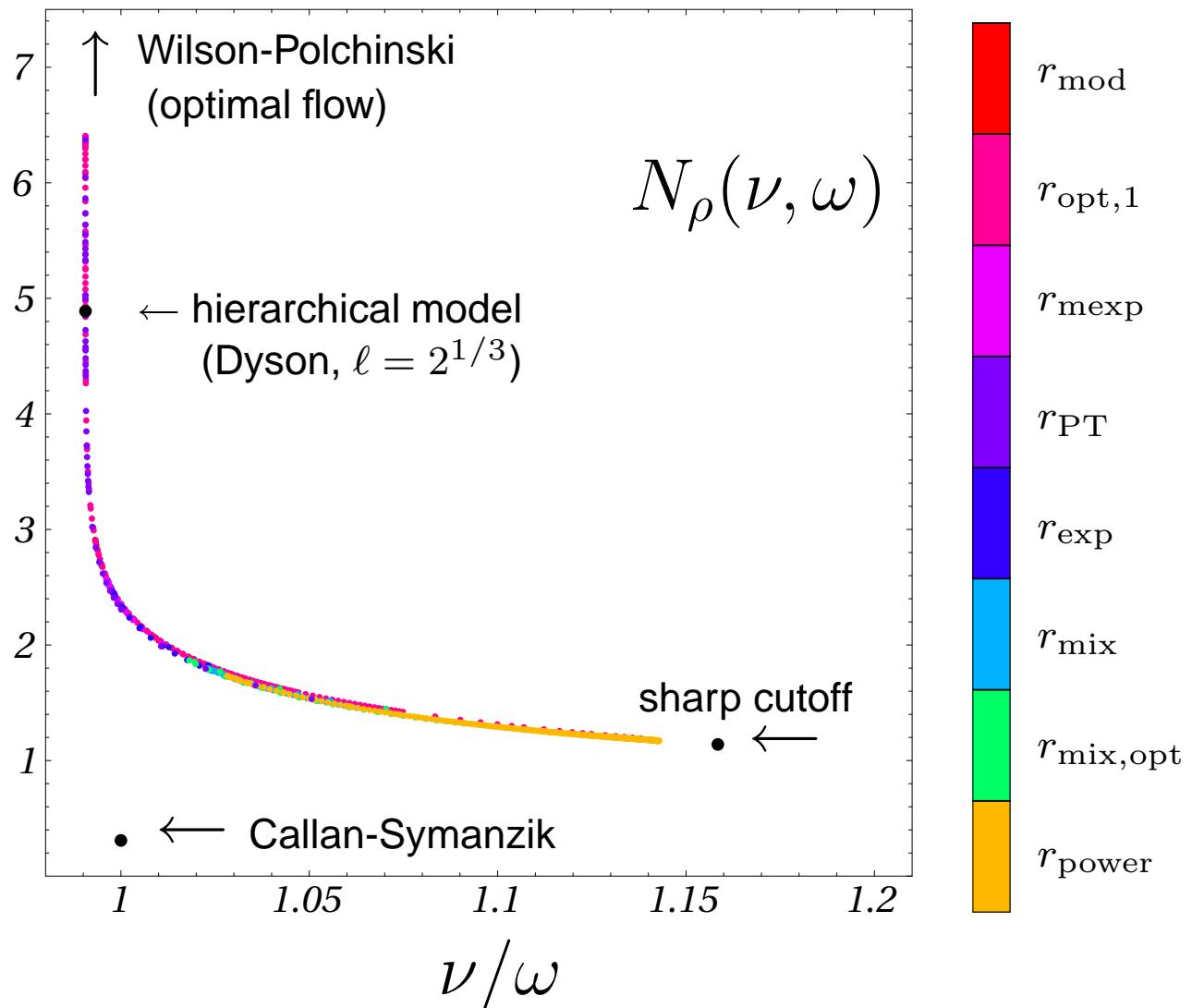
$$N_{\rho, \text{opt}} = \infty$$

$$\varphi(\nu, \omega)_{\text{opt}} = 0.78066$$

# distance of scaling exponents



# distance of scaling exponents



# matching (continuous HM)

method	cutoff	parameter	$\nu$	$\omega$
hierarchical model	Dyson	$(\ell = 1)$	0.649 561 773 880 <sup>a</sup>	0.655 745 939 193 <sup>a</sup>
functional RG	$r_{\text{opt},n}$	$(n = 1, b = 1)$	0.649 561 773 880 <sup>a</sup>	0.655 745 939 193 <sup>a</sup>
	$r_{\text{compact}}^b)$	$(b \rightarrow 0)$	0.649 561 773 880	0.655 745 939 193
	$r_{\text{int}}^c)$	$(b \rightarrow 1)$	0.649 561 773 880	0.655 745 939 193
	$r_{\text{PT},m}$	$(m = 5/2)$	0.649 561 773 880 <sup>a</sup>	0.655 745 939 193 <sup>a</sup>

a) C. Bervillier, A. Jüttner and D. F. Litim (2007)

b)  $r_{\text{compact}} = y^{-1} \exp[-\exp(-1/y)/(b-y)] \theta(b-y)$  with  $b \geq 0$ .

c)  $r_{\text{int}} = \exp(-y) \theta(1-y) \theta(y-b)$  with  $b \in [0, 1]$ .

# matching (discrete HM)

method	cutoff	parameter	$\nu$	$\omega$
hierarchical model	Dyson	$(\ell = 2^{1/3})$	0.649 570 <sup>a</sup>	0.655 736 <sup>a</sup>
	$r_{\text{opt},n}$	$(n = 1, b = 1.048)$	0.649 570(9)	0.655 736(6)
	$r_{\text{opt},n}$	$(n = 1, b = 0.9545)$	0.649 570(9)	0.655 736(9)
	$r_{\text{opt},n}$	$(n = 1.135, b = 1)$	0.649 570(6)	0.655 736(8)
	$r_{\text{opt},n}$	$(n = 1.1, b = 1.028)$	0.649 570(6)	0.655 736(8)
functional RG	$r_{\text{compact}}$	$(b = 0.04775)$	0.649 570(9)	0.655 736(9)
	$r_{\text{int}}$	$(b = 0.944)$	0.649 570(9)	0.655 736(8)
	$r_{\text{int}}$	$(b = 0.9444)$	0.649 570(7)	0.655 736(9)
	$r_{\text{PT},m}$	$(m = 2.499785)$	0.649 564(9)	0.655 736(1)
	$r_{\text{PT},m}$	$(m = 2.49944)$	0.649 570(1)	0.655 720(6)

a) J. J. Godina, Y. Meurice and M. B. Oktay (1998)

# conclusions

- **functional flows**

local potential approximation

scaling exponents very strongly correlated

huge redundancy

physical observables: extremal points distinguished

linked with optimisation/stability

- **functional flows vs hierarchical models**

discrete HM data match continuous RG data

results point towards fundamental link  $\ell = \ell(R)$

cutoff dependence  $\Leftrightarrow$  discretisation artefact