

New tasks for stochastic processes?

Earlier collaborations with J. Berges, S. Borsanyi and D. Sexty

Present work in collaboration with E. Seiler

Stochastic processes in Quantum Field Theory

Procedure: Realize a sampling of field configurations by defining a supplementary (noisy) dynamics in a 5-th “time”.

Basic example: Parisi Wu stochastic quantization in Euclidean QFT

- proofs of equivalence with path integral formulation, proofs of convergence, etc
rely on the definition of a probability distribution over the space of field configurations via an associated Fokker-Planck equation
- can define a “perturbation theory” without gauge fixing
- for numerical studies has slight disadvantages compared to MC
(see also Damgaard and Hueffel eds., Stochastic Quantization; Namiki, ...)

Essential feature: uses a drift force to define the process (the 5-th time dynamics)

→ **versatility**

- can be directly related to expectation values
- can be directly defined from the set up of the problem without needing an action or a probability interpretation for the path integral

This may be of interest in cases where other approaches (e.g., MC) do not work.

In the following: point of view of numerical simulations.

Usual realizations: Langevin Equation and Random Walk.

Here in discretized form, Ito calculus, ϑ : 5-th “time”, $\delta\vartheta$: “time” step; for each d.o.f. $\varphi(x)$ (random variable), $K[\varphi]$: drift force,

Langevin equation:

$$\begin{aligned}\delta\varphi(x; \vartheta) &\equiv \varphi(x; \vartheta + \delta\vartheta) - \varphi(x; \vartheta) = K[\varphi(x; \vartheta)] \delta\vartheta + \eta(x; \vartheta) \\ \langle \eta(x; \vartheta) \rangle &= 0, \quad \langle \eta(x; \vartheta) \eta(x'; \vartheta') \rangle = 2 \delta\vartheta \delta_{x,x'} \delta_{\vartheta,\vartheta'}\end{aligned}$$

Random Walk:

$$\delta\varphi(x; \vartheta) = \pm\omega, \quad \text{with pbb : } \frac{1}{2}(1 \pm \frac{1}{2}\omega K[\varphi(x; \vartheta)]), \quad \omega = \sqrt{\delta\vartheta}$$

NB: since $\eta, \omega \propto \sqrt{\delta\vartheta}$ we need also second derivatives:

$$\delta f[\varphi(\vartheta)] = \partial_\varphi f[\varphi(\vartheta)] \delta\varphi(x; \vartheta) + \frac{1}{2} \partial_\varphi^2 f[\varphi(\vartheta)] [\delta\varphi(x; \vartheta)]^2$$

Relation to path integral and MC

If the drift is the gradient of a real action, bounded from below then there is a probability density $\rho(\varphi, \vartheta)$ satisfying an associated Fokker-Planck equation in the limit $\delta\vartheta \rightarrow 0$:

$$\partial_\vartheta \rho(\varphi, \vartheta) = \partial_\varphi (\partial_\varphi - K) \rho(\varphi, \vartheta), \quad K = -\partial_\varphi S$$

and we have $\rho(\varphi, \vartheta) \rightarrow \rho_{as}(\varphi) \propto e^{-S(\varphi)}$ for $\vartheta \rightarrow \infty$.

- expectation values can be calculated as averages over noise, equivalently as ϑ averages:

$$\langle f(\varphi) \rangle = \frac{1}{\Theta} \int_0^\Theta d\vartheta f(\varphi(\vartheta)) = \langle f(\varphi) \rangle + \mathcal{O}(1/\sqrt{\Theta})$$

- in practice $\delta\vartheta \neq 0$: $\rho_{as}(\varphi)$ has $\mathcal{O}(\delta\vartheta)$ corrections (controllable).

Beyond Euclidean action

1. There is no S (non-conservative drift)

Simulation of non-compact Y-M theory with stochastic gauge fixing
[E.Seiler, I.O.S., D.Zwanziger]

One adds a gauge fixing force tangent to orbits:

$$K_\mu^a[A_\nu^c] = K_{YM,\mu}^a[A_\nu^c] + K_{g.f.,\mu}^a[A_\nu^c]$$
$$K_{g.f.,\mu}^a(A) = \alpha D_\mu^{ab}(A) (\partial \cdot A)^b = -\frac{\delta S_{GF}}{\delta A_\mu^a} + g f^{abc} A_\mu^b (\partial \cdot A)^c$$

- $K_{g.f.}$ is not conservative since it contains a curl: no action,
- $K_{g.f.}$ is “restoring”,
- (generally for Y-M simulations): convergence is achieved in the orbit space, hence for gauge invariant observables, while the fields themselves - the potentials - diffuse along the orbits.

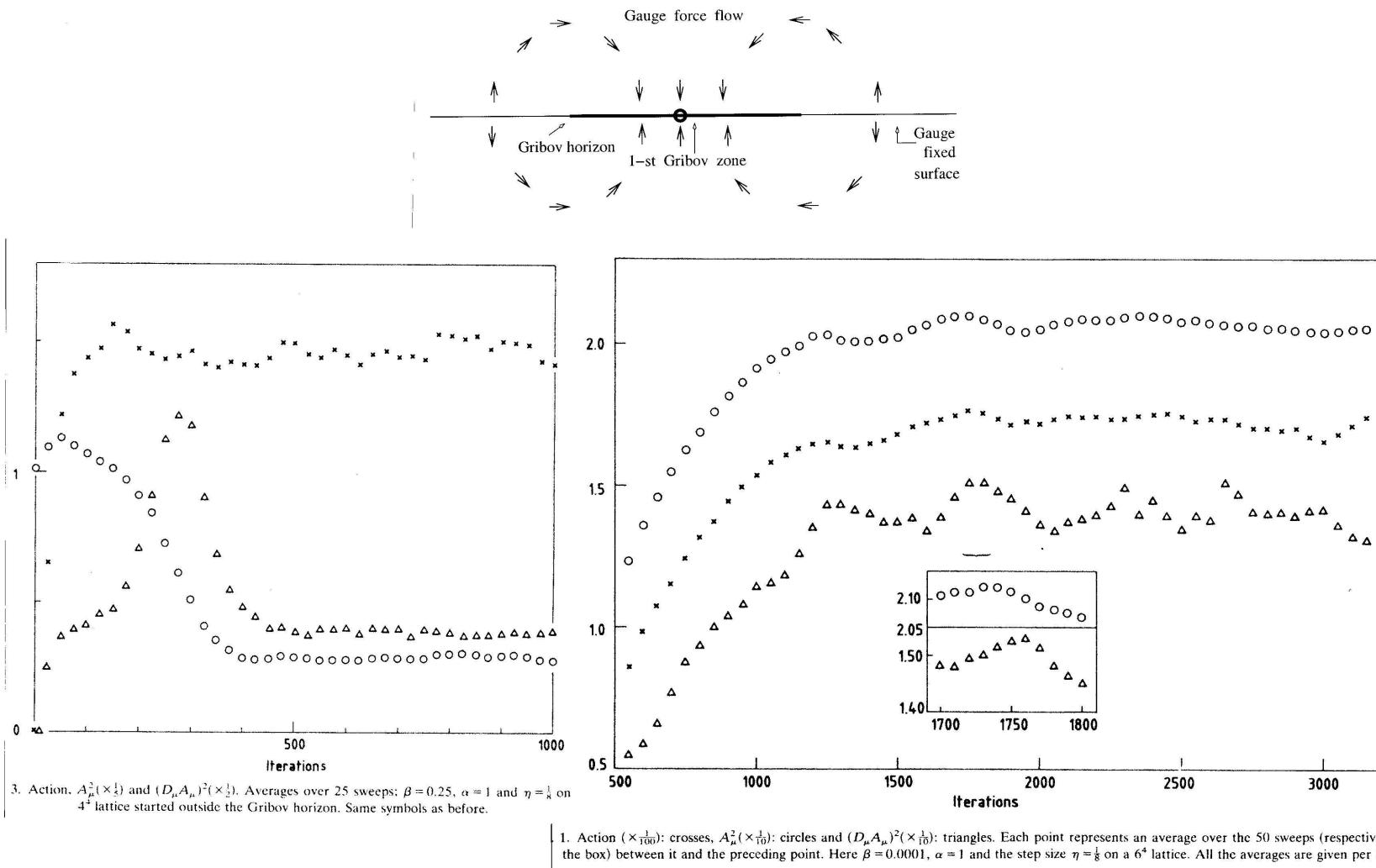


Figure 1: Stochastic gauge fixing (Random Walk simulation).

2. Complex action, in particular: Real time (Minkowski) problems

- reformulation of the stochastic quantization for the Minkowski path integral: Langevin equation with complex driving force
- implementation of the Nelson quantization procedure *directly in the real (physical) time*: no 5-th dimension, but needs the ground state wave function to define the driving force
- in both cases: (more or less formal) proofs of convergence and equivalence to the path integral formulation under certain conditions [Hueffel and Rumpf, Okamoto, etc] (the convergence is shown in the sense of tempered distributions, e.g. for correlation functions)
- interesting if Euclidean formulation not possible or ambiguous:

Non-equilibrium dynamics

Simple problems

Scalar field simulations on the lattice

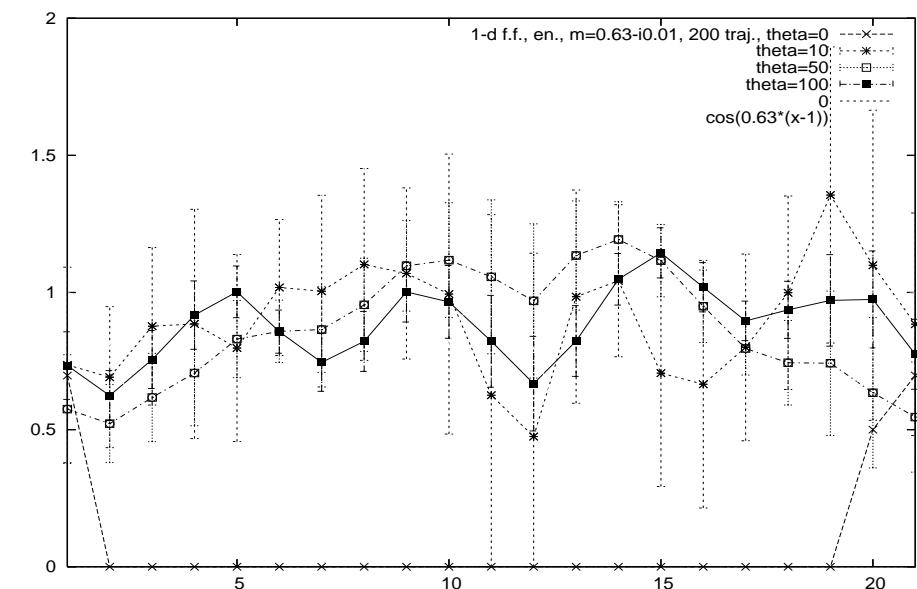
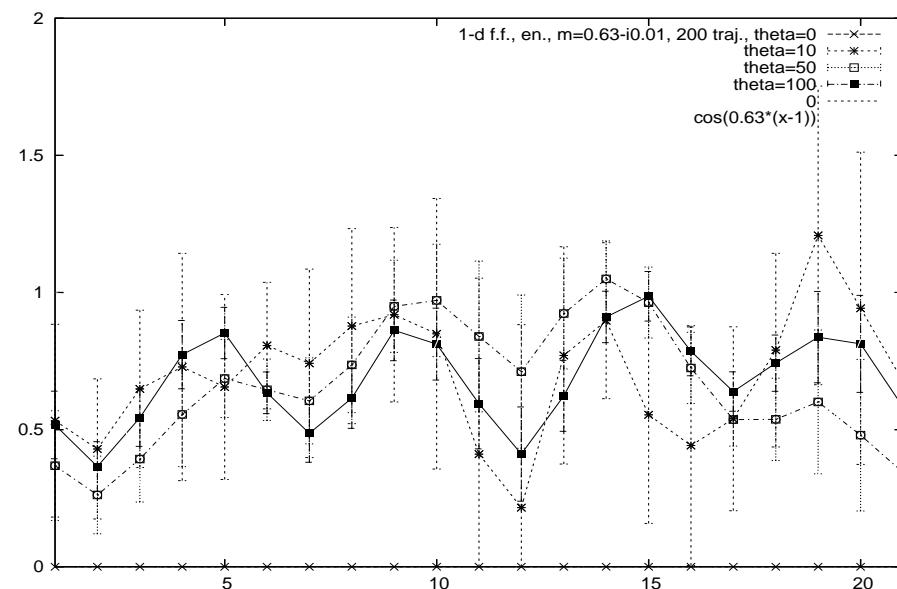
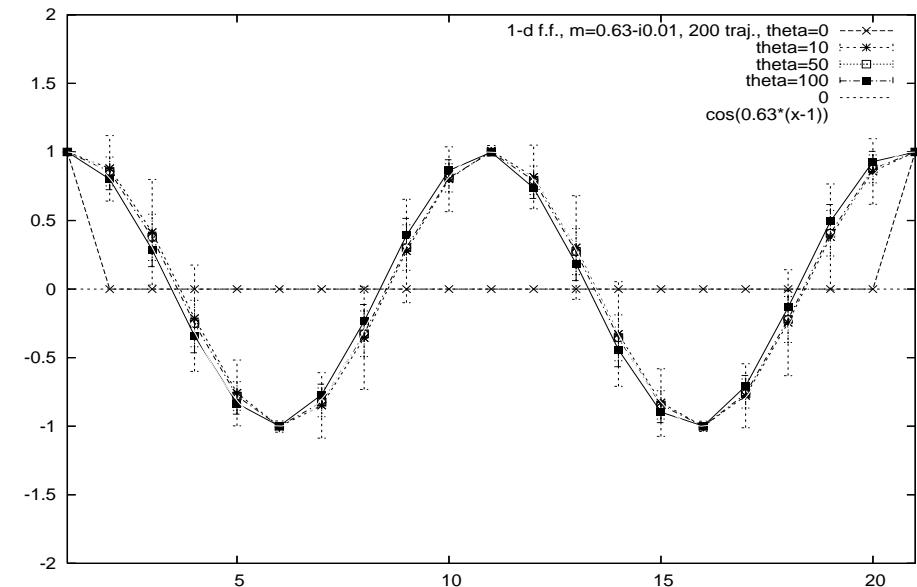
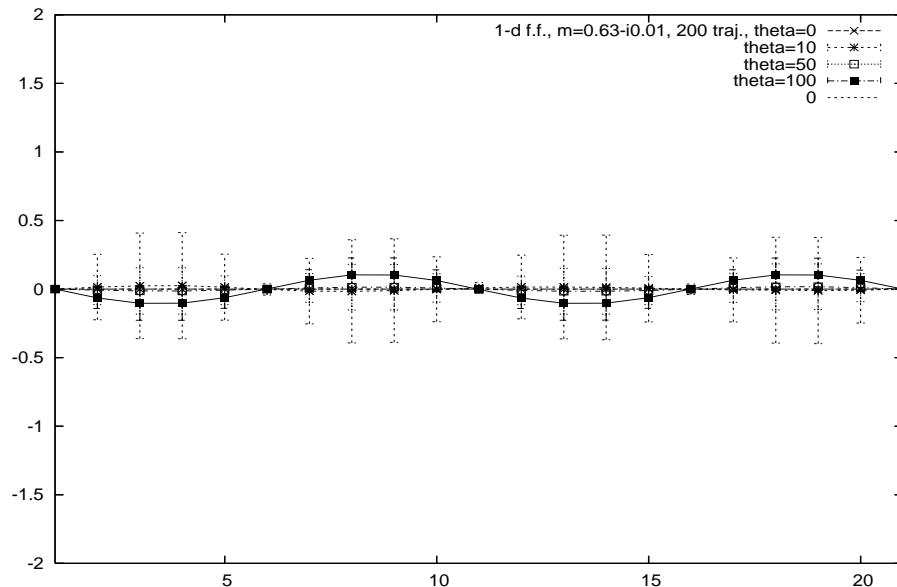
$$\hat{\varphi} = a\varphi, \hat{m} = am, \hat{\mathbf{x}} = \mathbf{x}/a, \hat{t} = t/a_t, \gamma = a/a_t$$

$$\hat{\vartheta} = \vartheta/a^2, \epsilon = \delta\vartheta/a^2,$$

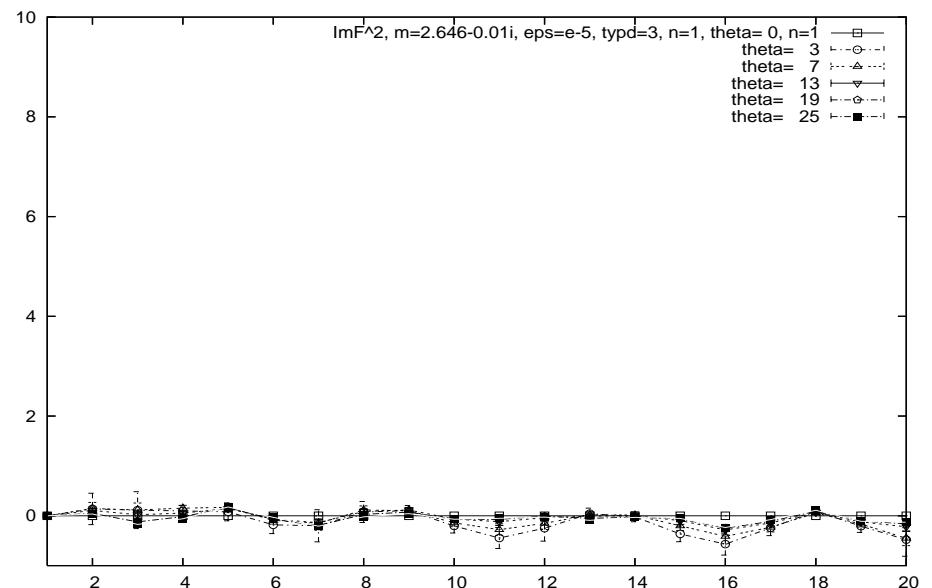
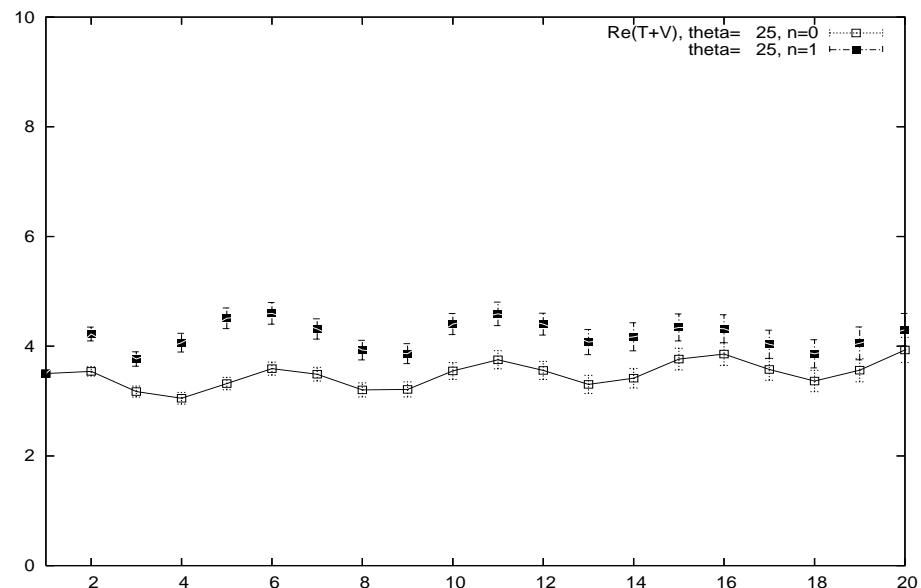
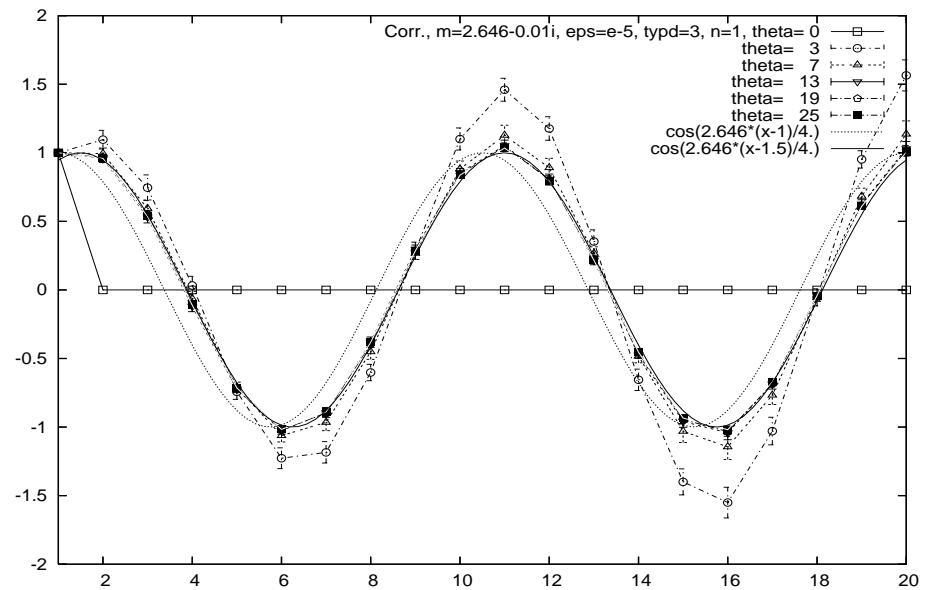
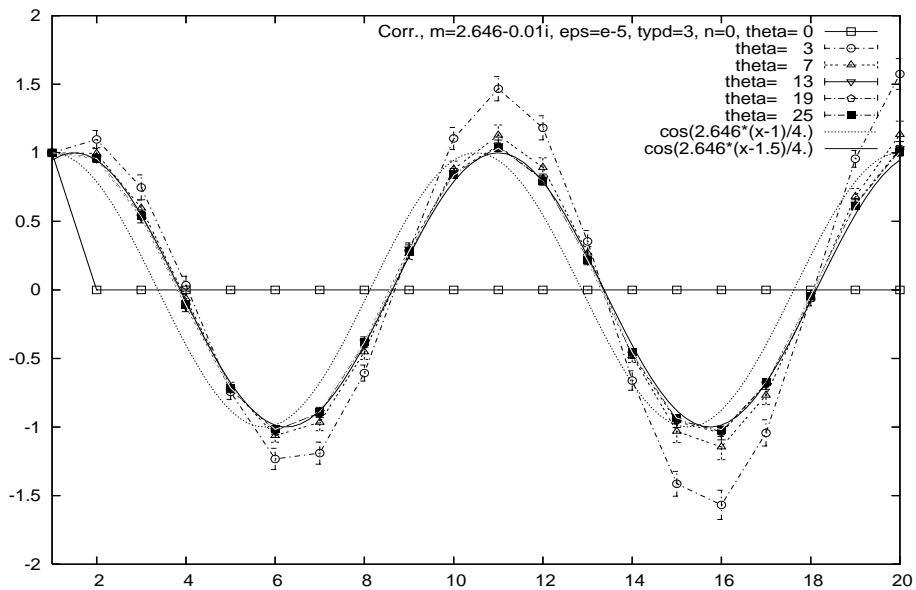
$$\hat{\eta} = \sqrt{a^3 a_t \delta \vartheta} \eta = \sqrt{\epsilon/\gamma} a^3 \eta, \quad \langle \hat{\eta}(\hat{x}, \hat{\vartheta}) \hat{\eta}(\hat{x}', \hat{\vartheta}') \rangle_\eta = 2 \delta_{\hat{x}, \hat{x}'} \delta_{\hat{\vartheta}, \hat{\vartheta}'}$$

$$\begin{aligned} \hat{\varphi}(\hat{x}; \hat{\vartheta} + \epsilon) &= \hat{\varphi}(\hat{x}; \hat{\vartheta}) + \sqrt{\epsilon \gamma} \hat{\eta}(\hat{x}; \hat{\vartheta}) \\ &\quad - i \epsilon \left(\square_\gamma \hat{\varphi}(\hat{x}; \hat{\vartheta}) + \hat{m}^2 \hat{\varphi}(\hat{x}; \hat{\vartheta}) + \lambda \hat{\varphi}(\hat{x}; \hat{\vartheta})^3 \right). \\ \square_\gamma \hat{\varphi}(\hat{x}; \hat{\vartheta}) &= \gamma^2 \left(\hat{\varphi}(\hat{x} + \hat{e}_0; \hat{\vartheta}) + \hat{\varphi}(\hat{x} - \hat{e}_0; \hat{\vartheta}) - c_t \hat{\varphi}(\hat{x}; \hat{\vartheta}) \right) \\ &\quad - \sum_i \left(\hat{\varphi}(\hat{x} + \hat{e}_i; \hat{\vartheta}) + \hat{\varphi}(\hat{x} - \hat{e}_i; \hat{\vartheta}) - 2 \hat{\varphi}(\hat{x}; \hat{\vartheta}) \right) \end{aligned}$$

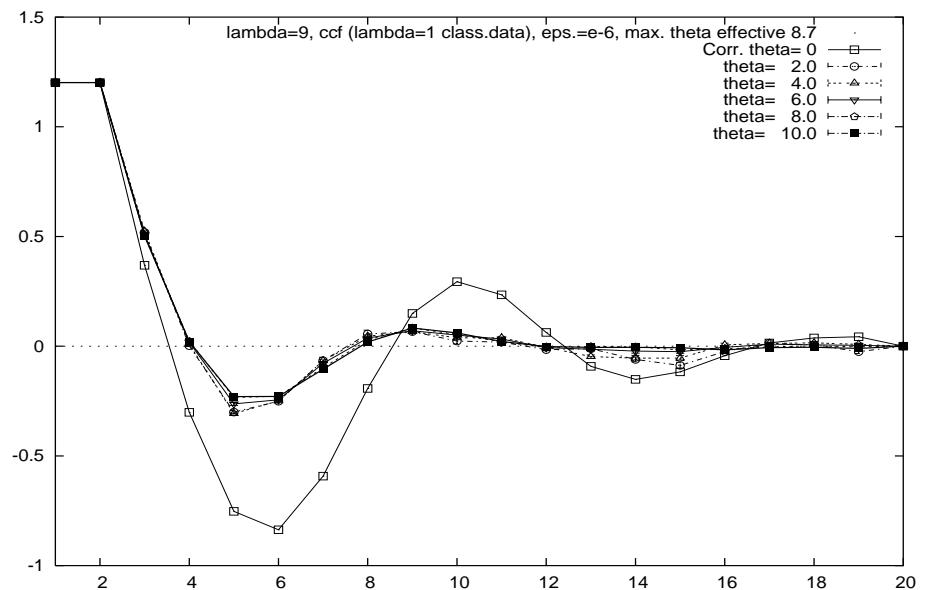
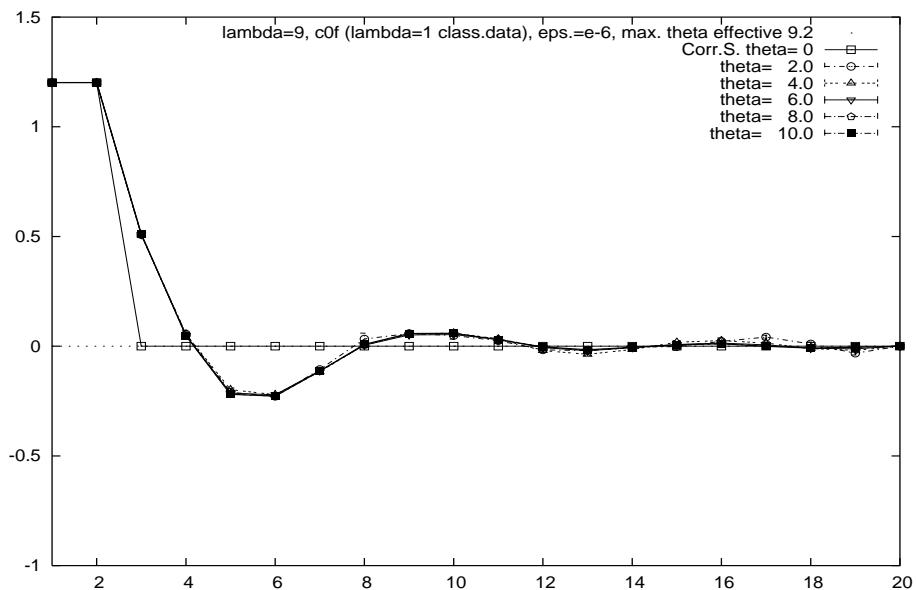
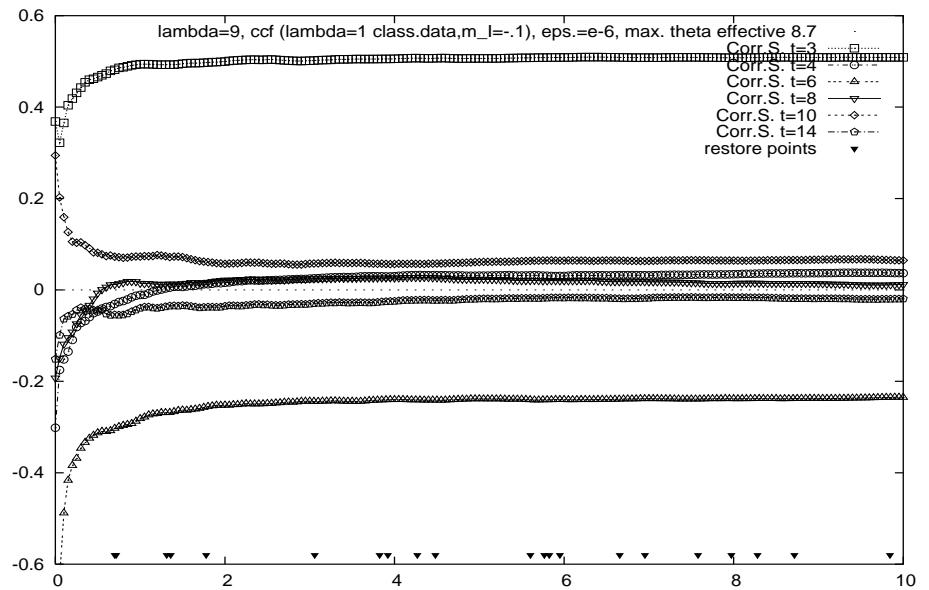
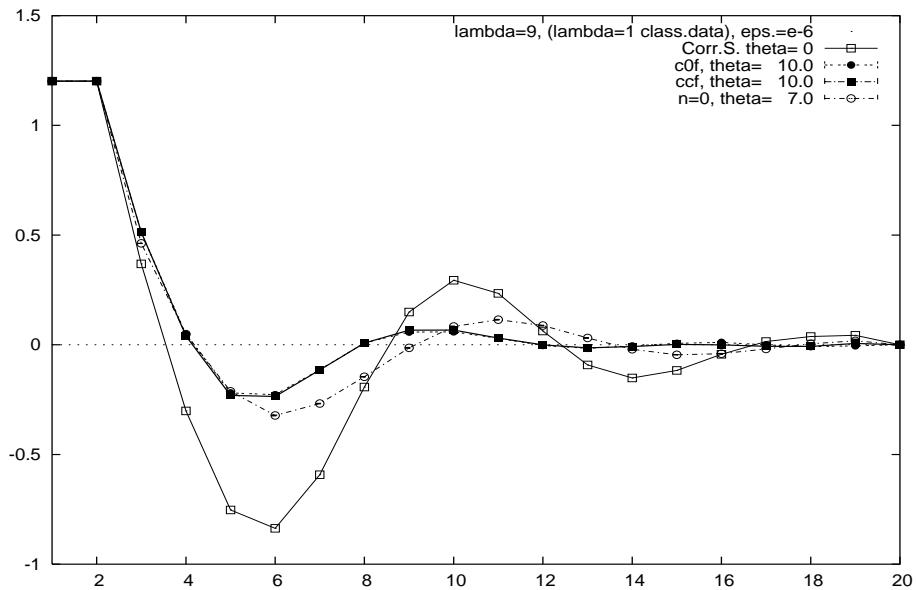
1-dim Harmonic oscillator, $N_t = 21, M = 0.63 - i0.01, \lambda = 0$, p.b.c..



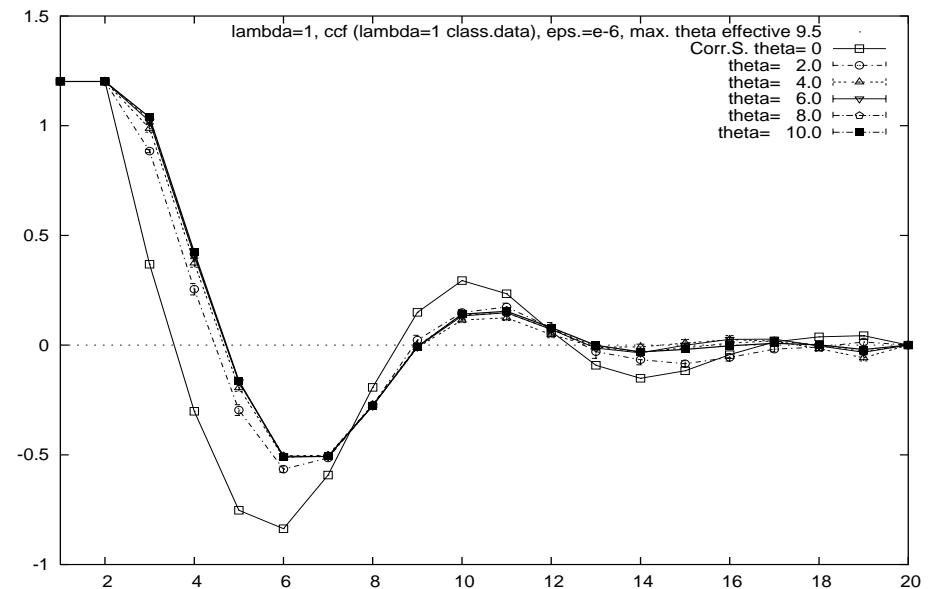
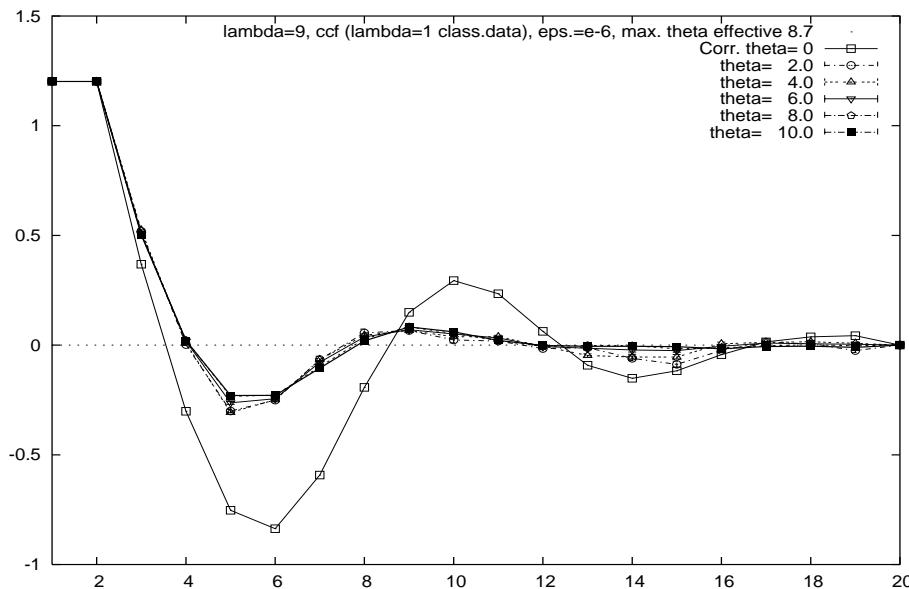
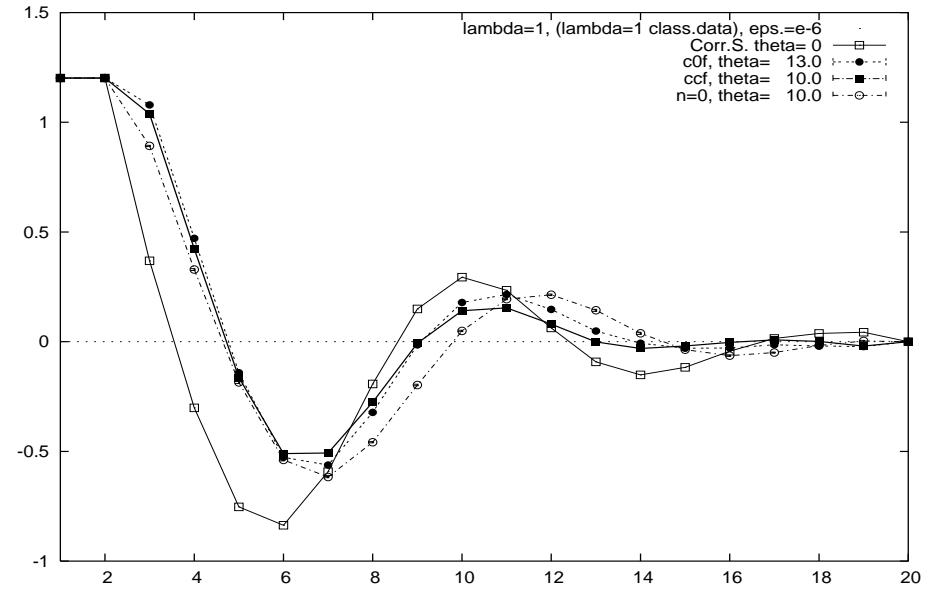
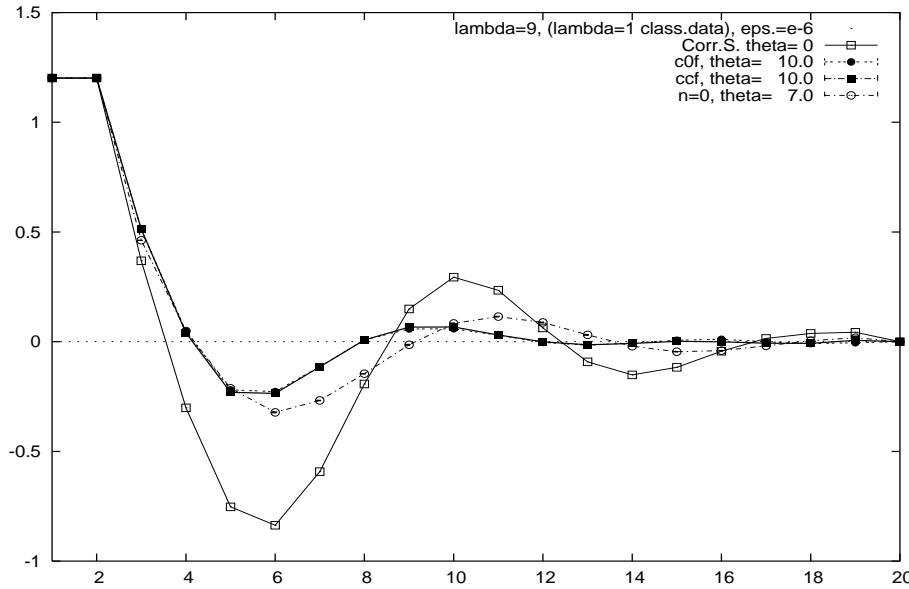
$3+1$ dim free field, $12^3 \cdot 20$, $\gamma = 4$, $M = 2.63 - i0.01$, $\lambda = 0$



$3+1$ dim , φ^4 theory, $8^3.20, M = 0, \lambda = 9, \gamma = 4$



$3+1$ dim , φ^4 theory, $8^3 \cdot 20, M = 0, \lambda = 9/\lambda = 1, \gamma = 4$



Questions

- convergence
- systematic discretization errors
- run-away trajectories

Gauge Theory in real time

The process is defined in the group algebra, the links are recalculated and used to derive the drift:

$$U'_{x,\mu} = \exp \left\{ i\lambda_a (\epsilon K_{x\mu a}[U] + \sqrt{\epsilon} \eta_{x\mu a}) \right\} U_{x,\mu},$$

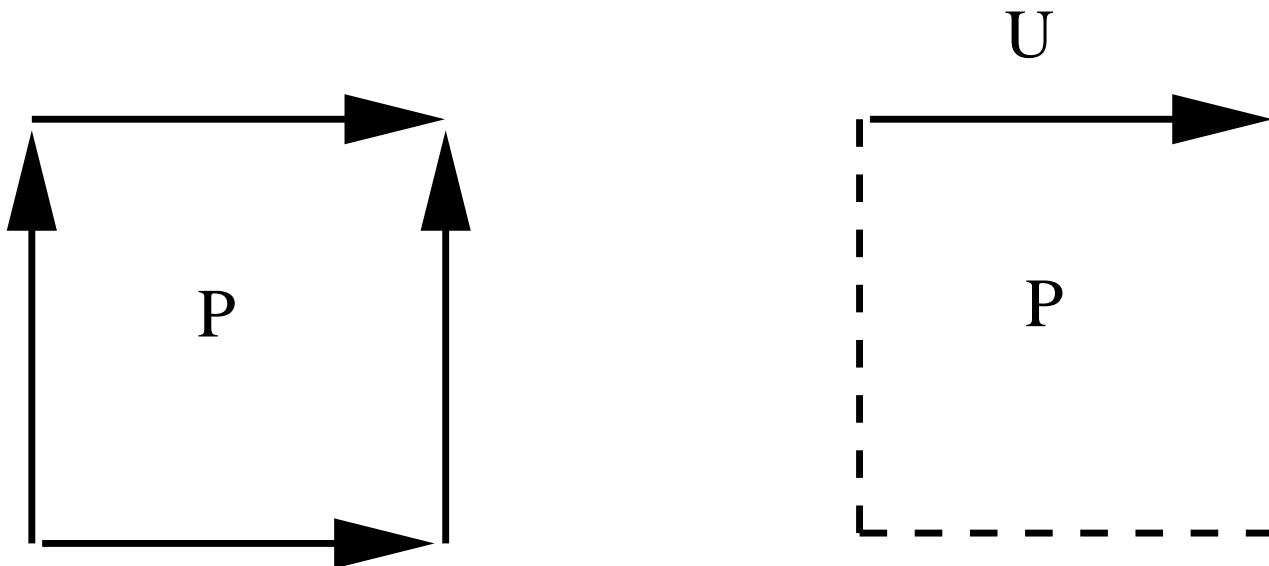
$$K_{x\mu a} = -\frac{1}{2N} \sum_{\nu \neq \mu} \beta \text{Tr} \left(\lambda_a U_{x,\mu} R_{x,\mu\nu} - \bar{R}_{x,\mu\nu} U_{x,\mu}^{-1} \lambda_a \right).$$

with λ_a the $SU(N)$ generators and $R_{x,\mu\nu}, \bar{R}_{x,\mu\nu}$ the “rest-Plaquette” staples. The process runs in $SL(N, C)$. One can use various procedure to ensure this: reduced Haar measure, projection onto SL , etc.

For $SU(2)$ we can use $U = a_0 + i \vec{\sigma} \vec{a}$, with a_ν complex, $a_0^2 + \vec{a}^2 = 1$ and we also have $U^{-1} = a_0 - i \vec{\sigma} \vec{a}$.

Tests on soluble models

One Plaquette U(1) with reweighting



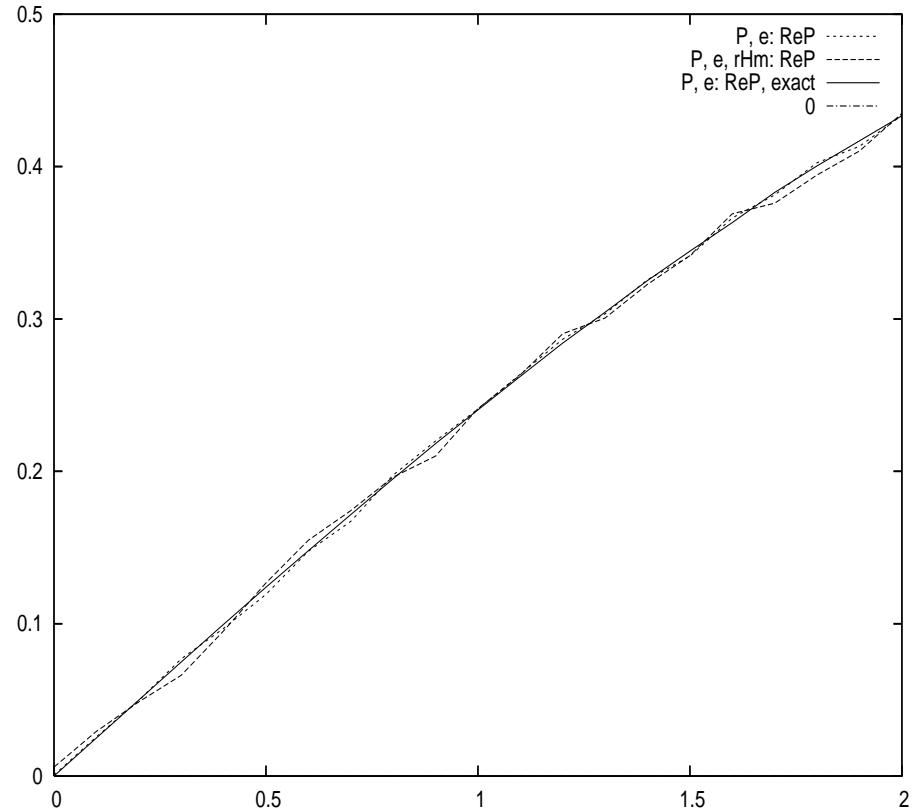
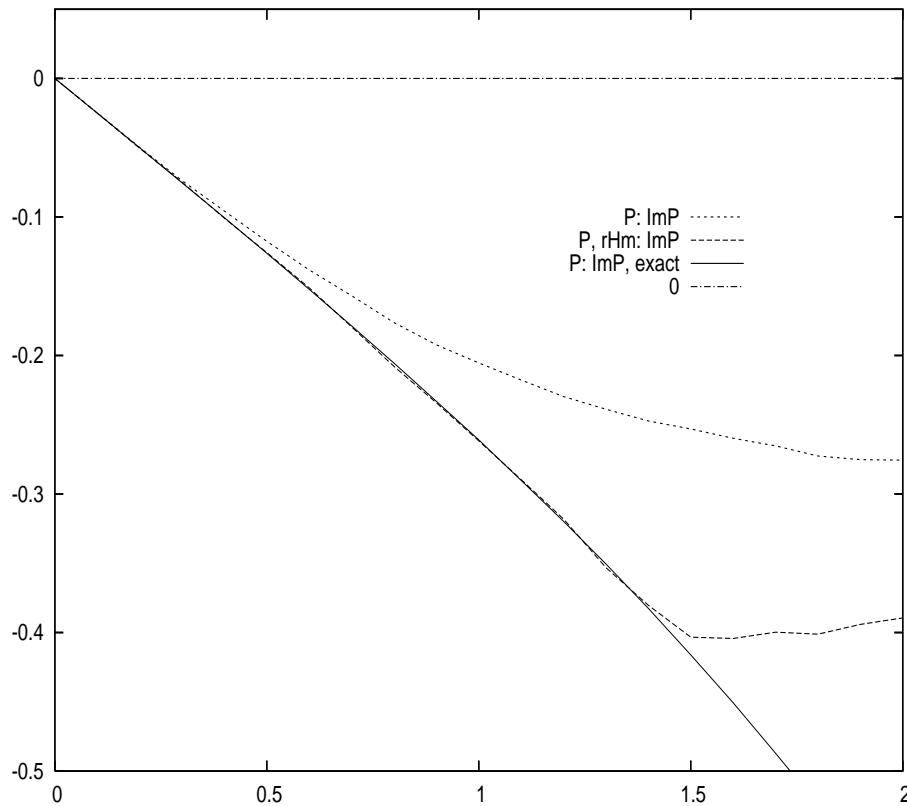
$$\begin{aligned}
U &= e^{i\varphi}, \quad P = \cos \varphi, \\
Z_p &= \int_0^{2\pi} d\varphi w_p^{-1} e^{-S_p}, \quad S_p = i\beta \cos \varphi, \quad w_p = e^{-ip\varphi}, \\
\langle O \rangle_p &= \frac{1}{Z_p} \int_0^{2\pi} d\varphi w_p^{-1} e^{-S_p} O(\varphi), \quad \langle O \rangle_0 = \frac{\langle O \rangle_p}{\langle w_p \rangle_p}
\end{aligned}$$

One Plaquette SU(2) with reweighting

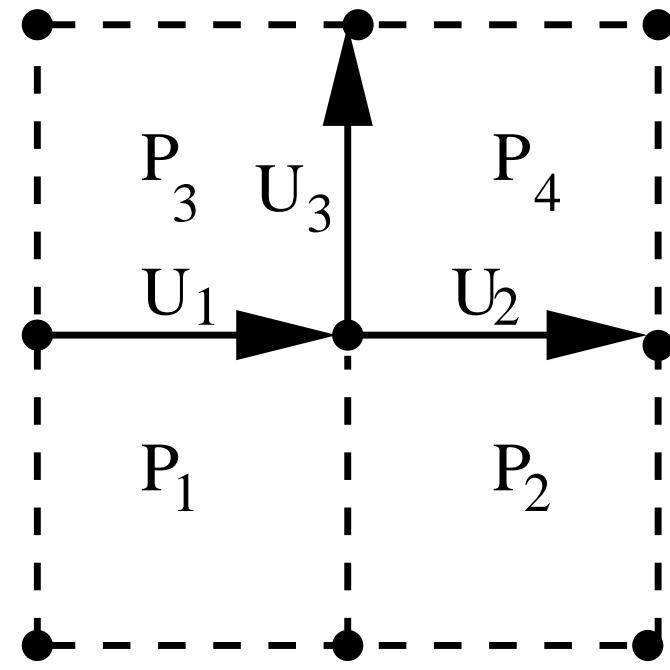
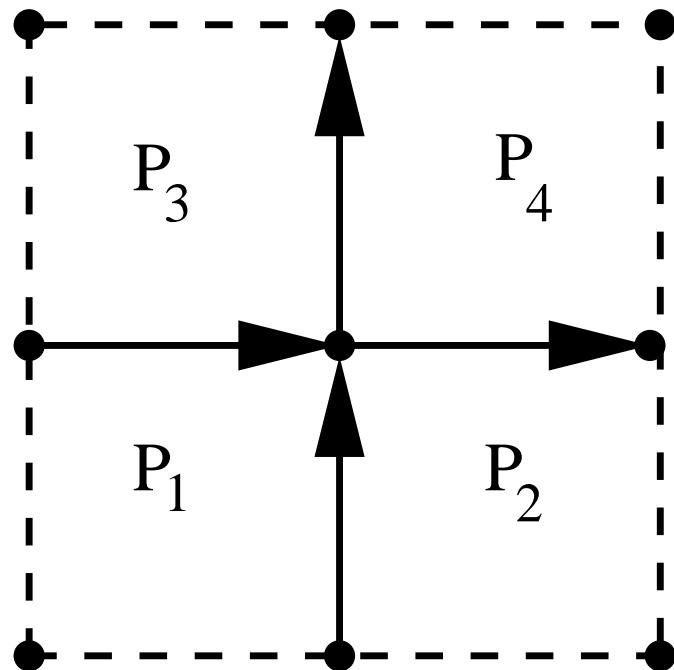
We use the reduced Haar measure, hence:

$$\begin{aligned} Z_p &= \int_0^{2\pi} d\varphi \sin^2\left(\frac{\varphi}{2}\right) w_p^{-1} e^{-S_p}, \\ S_p &= i\beta \cos\left(\frac{\varphi}{2}\right), \quad w_p = e^{-i p \frac{\varphi}{2}}, \end{aligned}$$

Minkowski vs Euclidean problem. Reduced Haar measure or projection.

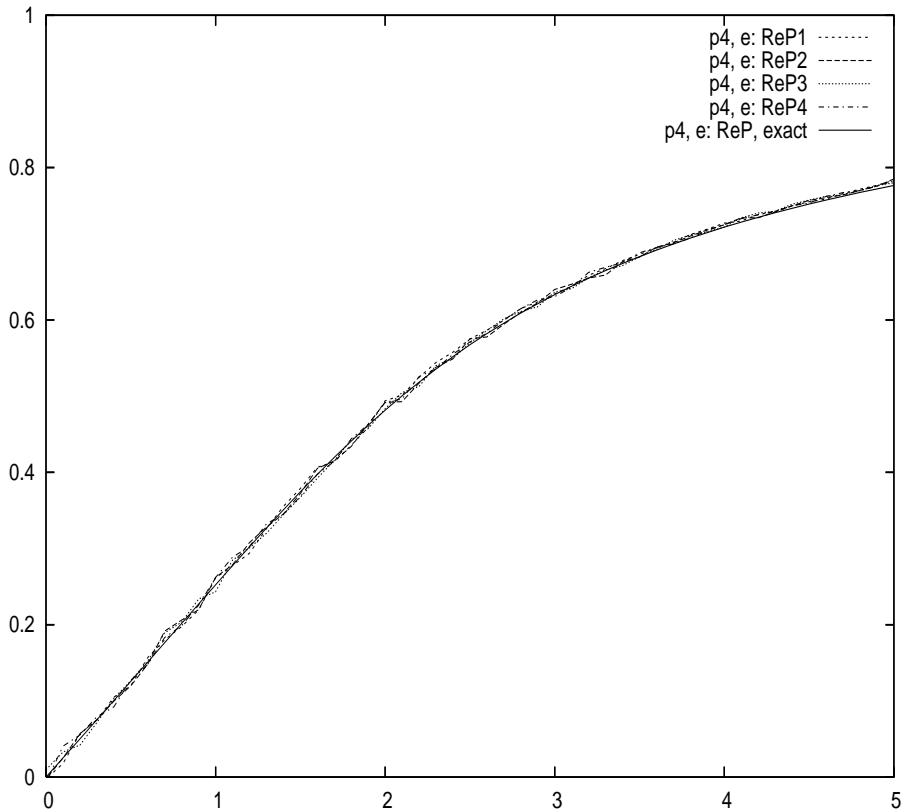
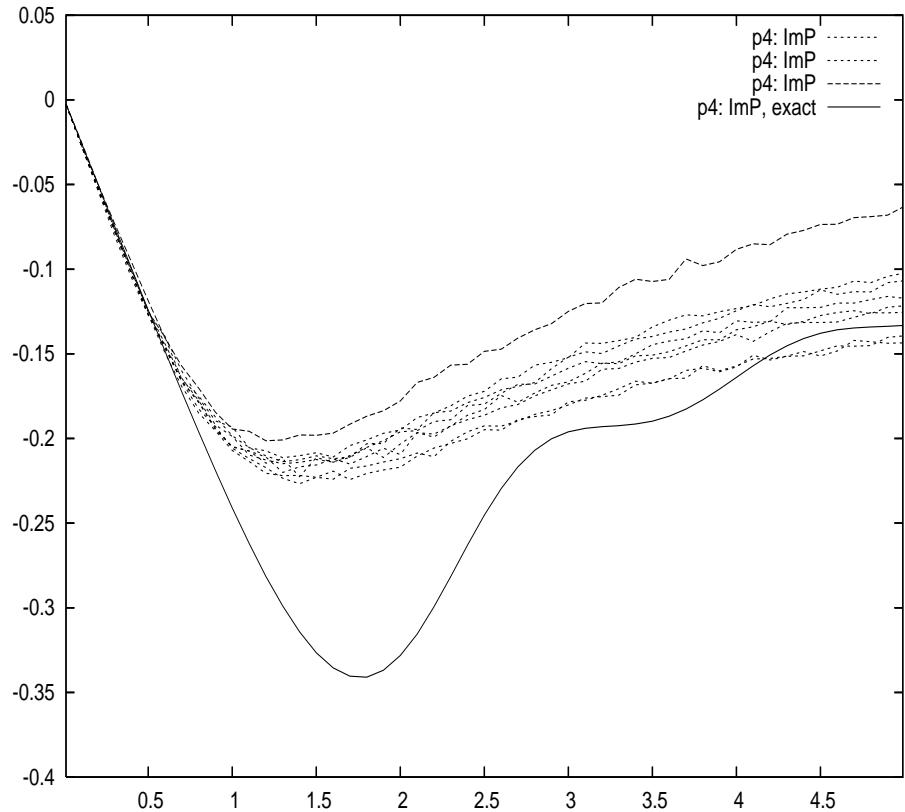


Four Plaquettes SU(2) with reweighting



Updating in the group parameters with projection on determinant=1.

Minkowski vs Euclidean problem. Projection or “gauge” force.



Trying to control the process - the One Plaquette U(1) model

Notations

$\langle O \rangle$: exact EW (from integration)

\overline{O} : average over the LE process (noise, time, ensemble)

$\mathcal{E} \{O\}$: EW with real distribution $P(x, y, t)$

$\langle O \rangle$: EW with complex distribution $\rho(x)$

Integral

$$Z_p(\beta) = \int_{-\pi}^{\pi} dx e^{i p x - i \beta \cos x} \quad (1)$$

We consider the analytic functions

$$f(z) = e^{i q z}, \quad z = x + i y, \quad f(x) \equiv f(z)|_{y=0} \quad (2)$$

Exact expectation values, direct and using p as reweighting

$$\langle e^{i q x} \rangle_p = \frac{\int_{-\pi}^{\pi} dx e^{i q x} e^{i p x - i \beta \cos x}}{\int_{-\pi}^{\pi} dx e^{i p x - i \beta \cos x}} \quad (3)$$

$$\langle e^{i q x} \rangle_0 = \frac{\int_{-\pi}^{\pi} dx e^{i q x} e^{-i \beta \cos x}}{\int_{-\pi}^{\pi} dx e^{-i \beta \cos x}} = \frac{\langle e^{i (q-p) x} \rangle_p}{\langle e^{i (-p) x} \rangle_p} \quad (4)$$

Complex Langevin eq. for complex measure

$$e^{i p z - i \beta \cos z} dz \quad (5)$$

$$S = -i p z + i \beta \cos z, \quad K = -\partial_z S = i p + i \beta \sin z \quad (6)$$

With real variables we have

$$z = x + i y \quad (7)$$

$$K = i(p + \beta \sin x \cosh z) - \beta \sinh y \cos x \quad (8)$$

Lang. eq.:

$$\dot{z} = i p + i \beta \sin z + \eta \quad (9)$$

$$\dot{x} = -\beta \cos x \sinh y + \eta \quad (10)$$

$$\dot{y} = p + \beta \sin x \cosh y \quad (11)$$

$$Im \eta = 0, \quad \overline{\eta(t) \eta(t')} = 2 \lambda \delta(t - t') \quad (12)$$

One builds averages of $f(z)$ over the LE process

$$\overline{f(z)} = \frac{1}{T} \sum_{t=1}^T f(x_t + i y_t) \quad (13)$$

$\overline{f(z)}_p$ should reproduce the ensemble averages over the distribution realized by the LE process, and thus the exact EW (3,4). Using reweighting (for $p = 0$) it holds, numerically:

$$\langle e^{i q x} \rangle_p = \overline{\{e^{i q z}\}_p}, \quad p \neq 0 \quad (14)$$

$$\langle e^{i q x} \rangle_0 = \frac{\overline{\{e^{i(q-p)z}\}_p}}{\overline{\{e^{-ipz}\}_p}} \quad (15)$$

In the tables $t = 1000$, $\delta t = 10^{-5}$. Agreement/disagreement appears systematical; simple rule: use $p = 2$ for positive modes and $p = -2$ for negative. See also Figs. 2 and 3.

| $\beta = 1.0$ | q= -3.0 | | q= -2.0 | | q= -1.0 | | q= 1.0 | | q= 2.0 | |
|---------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|--------|
| p | Re | Im | Re | Im | Re | Im | Re | Im | Re | Im |
| -3.0 | .227E-04 | 0.114E-02 | -.129E-01 | 0.298E-03 | -.176E-02 | -.127 | -.108 | 5.87 | -22.5 | -.997 |
| exact | -.395E-15 | 0.107E-02 | -.128E-01 | 0.602E-14 | 0.966E-15 | -.127 | -.651E-14 | 5.87 | -22.5 | 0.631E |
| -2.0 | 0.924E-04 | 0.232E-02 | -.216E-01 | 0.858E-03 | -.343E-02 | -.170 | -.109 | 3.83 | -6.67 | -.629 |
| exact | 0.310E-15 | 0.217E-02 | -.216E-01 | -.628E-15 | -.135E-15 | -.170 | -.215E-14 | 3.83 | -6.66 | -.341E |
| -1.0 | 0.289E-03 | 0.721E-02 | -.450E-01 | 0.273E-02 | -.725E-02 | -.259 | -.373E-01 | 1.74 | 1.21 | -.183 |
| exact | 0.107E-16 | 0.563E-02 | -.445E-01 | -.231E-15 | 0.339E-15 | -.261 | 0.410E-15 | 1.74 | 1.00 | -.193E |
| 0.0 | 20.7 | 13.9 | 1.06 | -.247 | 0.146E-01 | 0.168E-03 | 0.639E-01 | -.350E-03 | 0.972 | -.445 |
| exact | -.372E-16 | 0.256E-01 | -.150 | -.729E-16 | 0.149E-15 | -.575 | -.138E-15 | -.575 | -.150 | 0.363E |
| 1.0 | -5.78 | -11.1 | 1.21 | 0.193 | 0.435E-01 | 1.74 | 0.720E-02 | -.257 | -.439E-01 | -.285E |
| exact | 0.109E-15 | -.261 | 1.00 | 0.188E-15 | -.309E-15 | 1.74 | -.593E-15 | -.261 | -.445E-01 | 0.994E |
| 2.0 | -.149E+04 | -66.6 | -6.95 | 0.382 | 0.116 | 3.83 | 0.350E-02 | -.169 | -.213E-01 | -.857E |
| exact | -.280E-14 | 3.83 | -6.66 | -.135E-14 | 0.952E-15 | 3.83 | 0.185E-15 | -.170 | -.216E-01 | 0.355E |
| 3.0 | -4.42 | -37.2 | -22.5 | 0.974 | 0.101 | 5.87 | 0.183E-02 | -.126 | -.128E-01 | -.296E |
| exact | -.148E-12 | -39.1 | -22.5 | 0.981E-13 | 0.336E-13 | 5.87 | -.182E-15 | -.127 | -.128E-01 | 0.237E |
| 4.0 | -8.26 | -178. | -46.5 | 1.33 | 0.101 | 7.90 | 0.112E-02 | -.101 | -.852E-02 | -.112E |
| exact | -.606E-11 | -178. | -46.4 | 0.163E-11 | 0.292E-12 | 7.90 | -.402E-13 | -.101 | -.845E-02 | 0.196E |
| 5.0 | -14.4 | -463. | -78.5 | 1.72 | 0.100 | 9.91 | 0.764E-03 | -.837E-01 | -.608E-02 | -.410E |
| exact | 0.265E-09 | -460. | -78.3 | -.451E-10 | -.569E-11 | 9.92 | -.649E-14 | -.838E-01 | -.602E-02 | -.208E |

| $\beta = 1.0$ | q= -3.0 | | q= -2.0 | | q= -1.0 | | q= 1.0 | | q= 2.0 | |
|---------------|-----------|-----------|-----------|-----------|-----------|-------|-----------|-------|-----------|-----------|
| p | Re | Im | Re | Im | Re | Im | Re | Im | Re | Im |
| -3.0 | 0.251E-02 | 0.263E-01 | -.154 | 0.119E-01 | -.305E-01 | -.595 | -.281 | -1.11 | -8.67 | -2.70 |
| -2.0 | 0.290E-02 | 0.253E-01 | -.149 | 0.140E-01 | -.374E-01 | -.571 | -.370E-01 | -.750 | -6.86 | -14.0 |
| -1.0 | 0.212E-02 | 0.258E-01 | -.149 | 0.735E-02 | -.123E-01 | -.575 | -.120 | -.694 | -5.43 | -2.58 |
| 1.0 | 11.4 | -136. | -6.43 | 3.16 | 0.128 | -.694 | 0.144E-01 | -.574 | -.147 | -.782E-02 |
| 2.0 | 0.209E+10 | -.912E+09 | 0.755E+06 | -.183E+06 | 213. | 21.3 | 0.136E-01 | -.550 | -.143 | -.788E-02 |
| 3.0 | -903. | -.312E+04 | -24.3 | 78.3 | 1.24 | -1.76 | 0.451E-01 | -.600 | -.156 | -.158E-01 |
| 4.0 | -891. | -.560E+04 | -75.8 | 375. | 5.65 | -10.5 | -.409E-01 | -.724 | -.189 | 0.732E-02 |
| 5.0 | 0.782E+05 | 0.248E+06 | 979. | -.443E+04 | -54.7 | 57.9 | 0.198 | 0.165 | 0.452E-01 | -.558E-01 |
| exact | -.372E-16 | 0.256E-01 | -.150 | -.729E-16 | 0.149E-15 | -.575 | -.138E-15 | -.575 | -.150 | 0.363E-16 |

One can define a FP Equation for a complex “distribution” $\rho(z)$, z : complex ($\lambda = 1$ is the quantum problem, $\lambda = 0$ the classical limit):

$$\dot{\rho}_p(z, t) = (\lambda \partial_z^2 - i \partial_z (p + \beta \sin z)) \rho_p(z, t). \quad (16)$$

Formally this has as asymptotic solution e^{-S} .

Alternatively one can define a joint, true probability distribution $P(x, y, t)$ for the real variables x, y :

$$\begin{aligned} \dot{P}_p(x, y, t) &= \\ &(\lambda \partial_x^2 + \beta \partial_x \cos x \sinh y - \partial_y(p + \beta \sin x \cosh y)) P_p(x, y, t) \end{aligned} \quad (17)$$

With arbitrary analytic $f(z)$ we have

$$\int dx dy P_p(x, y, t) f(x + iy) = \int dx \rho_p(x, t) f(x) \quad (18)$$

Complex FP Equation

$$\dot{\rho}_p(z, t) = (\lambda \partial_z^2 - \partial_z (i p + i \beta \sin z)) \rho_p(z, t) \quad (19)$$

$$\dot{\rho}_p(x, t) = (\lambda \partial_x^2 - i \partial_x (p + \beta \sin x)) \rho_p(x, t) \quad (20)$$

FT:

$$\tilde{\rho}_p(k, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx e^{ikx} \rho_p(x, t) \quad \rho_p(x, t) = \sum_k e^{-ikx} \tilde{\rho}_p(k, t) \quad (21)$$

For calculation x is discretized in N points: $x = \frac{2\pi}{N} l$ and the FPE in momentum space reads:

$$\begin{aligned} \dot{\tilde{\rho}}_p(k, t) &= \\ -(\lambda k^2 + k p) \tilde{\rho}_p(k, t) &+ \frac{i}{2} \beta k \tilde{\rho}_p(k+1, t) - \frac{i}{2} \beta k \tilde{\rho}_p(k-1, t) \end{aligned} \quad (22)$$

EW from ρ averages:

$$\langle e^{i q x} \rangle_p = \frac{\int_{-\pi}^{\pi} dx e^{i q x} \rho_p(x, t)}{\int_{-\pi}^{\pi} dx \rho_p(x, t)} = \frac{\tilde{\rho}_p(q, t)}{\tilde{\rho}_p(0, t)} \quad (23)$$

$$\langle e^{i q x} \rangle_0 = \frac{\int_{-\pi}^{\pi} dx e^{i q x} \rho_0(x, t)}{int_{-\pi}^{\pi} dx \rho_0(x, t)} = \frac{\tilde{\rho}_0(q, t)}{\tilde{\rho}_0(0, t)} \quad (24)$$

$$= \frac{\int_{-\pi}^{\pi} dx e^{i (q-p) x} \rho_p(x, t)}{\int_{-\pi}^{\pi} dx e^{i (-p) x} \rho_p(x, t)} = \frac{\tilde{\rho}_p(q - p, t)}{\tilde{\rho}_p(-p, t)} \quad (25)$$

The modes of ρ_0 can be calculated using (22) for $0 \leq \beta < 2.3$, see Fig. 2. Then we can calculate the modes of *all* ρ_p , p : integer, using:

$$\tilde{\rho}_p(k, t) = \frac{\tilde{\rho}_0(k + p, t)}{\tilde{\rho}_0(p, t)}, \quad \rho_p(x, t) = \frac{e^{-ipx}}{\tilde{\rho}_0(p, t)} \rho_0(x, t) \quad (26)$$

See Fig. 3. Thus the complex ‘distribution’ $\rho_p(x, t)$ can be calculated directly by solving the FPE for $p = 0$, and then for all p using (26), p : integer and $0 \leq \beta < 2.3$. $\rho_p(x)$ calculated as asymptotic solution from the FPE (22) reproduces very well the original measure (5,6):

$$\rho_p(x) = \lim_{t \gg 1} \rho_p(x, t) = e^{-S_p(x)} \quad (27)$$

(Figs. 4, 5). This thus ensures – (3,4),(14,15), (23,24,25), (27):

$$\langle e^{iqx} \rangle_p = \langle e^{iqx} \rangle_p = \overline{\{e^{iqz}\}_p} \quad (28)$$

where for $p = 0$ we must use reweighting in the LE and for $p \neq 0$ we must use ‘reweighting’ of the FPE.

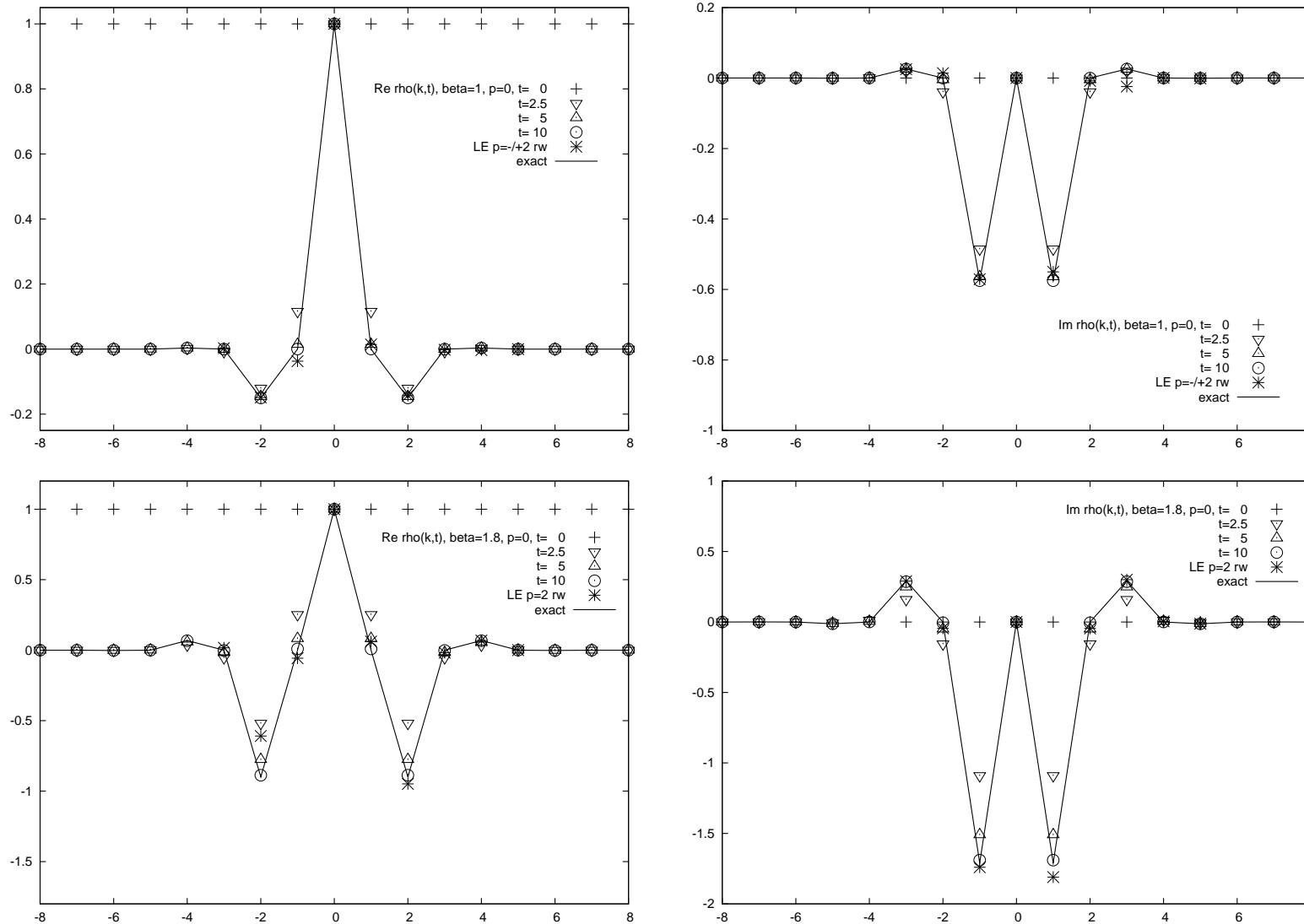


Figure 2: $\tilde{\rho}_0(k, t)$ (Re, Im) for $p = 0$ and $\beta = 1$ and 1.8 from FPE (22) with $N = 100$ x-discretization, $\delta t = 10^{-6}$, for $t = 0, 2.5, 5, 10$. Solid line: exact measure (5). Stars: stochastic (LE) data.

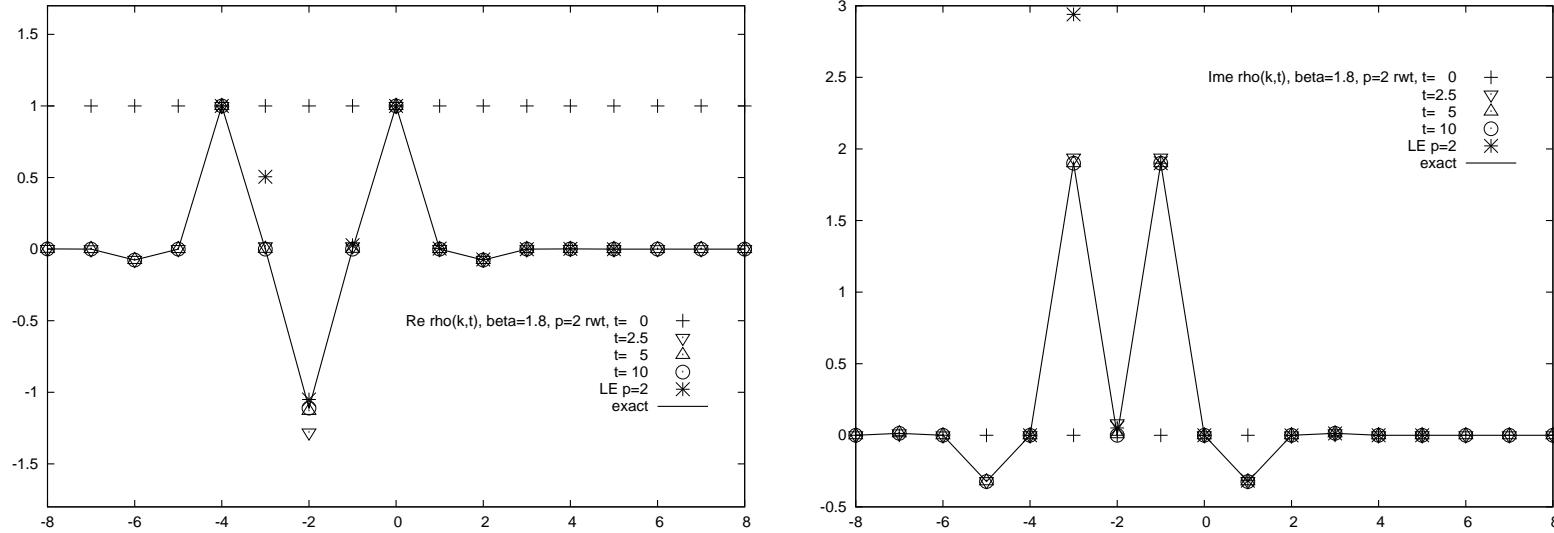


Figure 3: $\tilde{\rho}_p(k, t)$ (Re, Im) for $p = 2$ and $\beta = 1.8$ from FPE (22) at $p = 0$ and ‘reweighting’ according to (??), compared with the exact measure (5) (solid line: discretized, FT) and the stochastic data from the LE simulation (stars) – see tables; $N = 100$ x-discretization, $\delta t = 10^{-6}$, $t = 0, 2.5, 5, 10$.

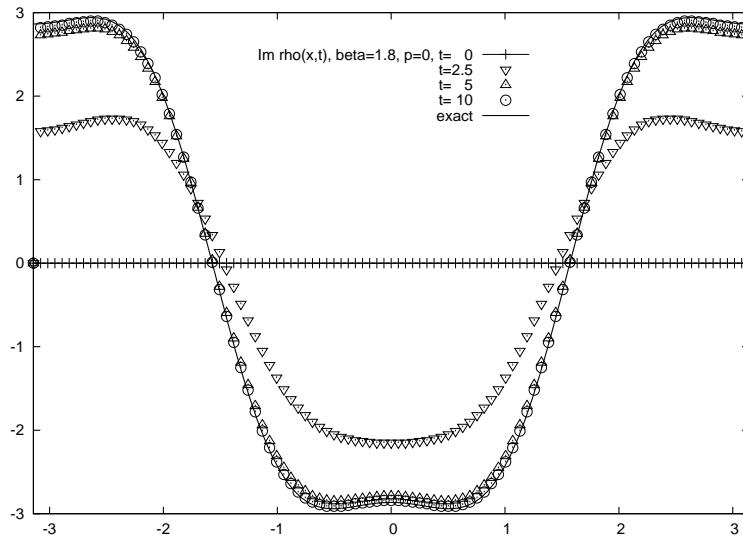
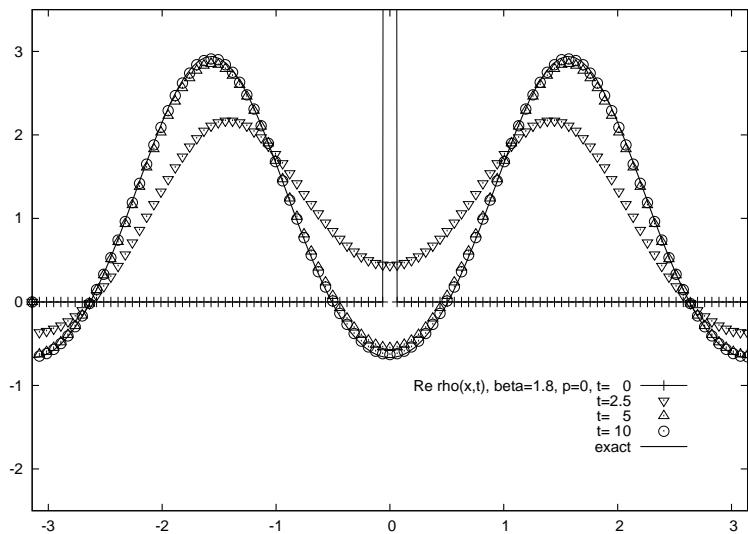


Figure 4: $\rho_0(x, t)$ (Re, Im), $p = 0$, $\beta = 1.8$ with $N = 100$ x-discretization, $\delta t = 10^{-6}$, for $t = 0, 2.5, 5, 10$ compared with exact measure (5) (discretized).

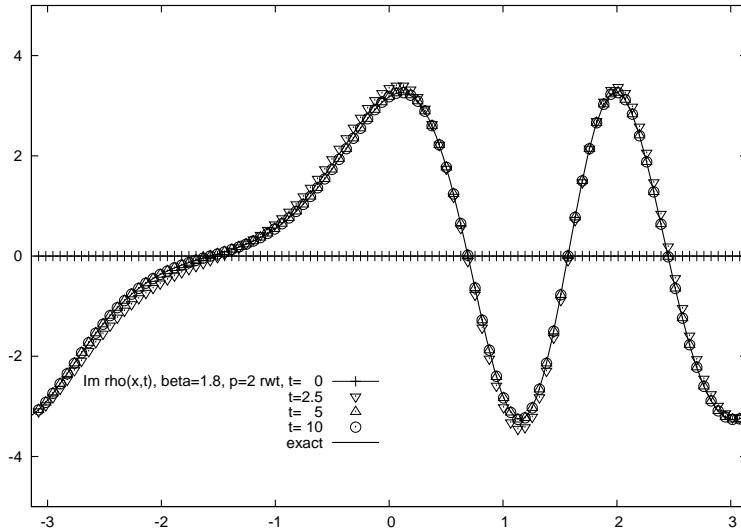
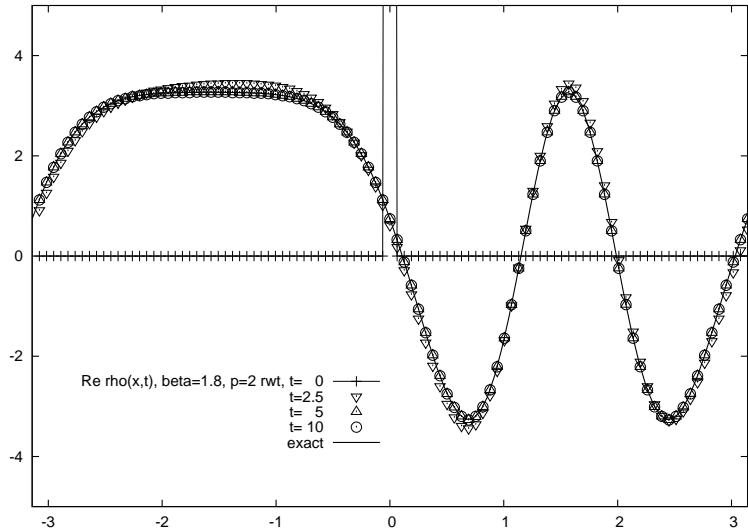


Figure 5: $\rho_p(x,t)$ (Re, Im), $p = 2$, $\beta = 1.8$ obtained from the FPE at $p = 0$ followed by 'reweighting' according to (26), compared with exact measure (5) (discretized); $N = 100$ x-discretization, $\delta t = 10^{-6}$, $t = 0, 2.5, 5, 10$.

Notice:

- The FPE (22) diverges for $p > 0 (< 0)$ for the negative (positive) modes, and generally for $\beta > 2.3$. One finds that $Re Z_0(\beta)$ becomes negative at $\beta \sim 2.3$, see Fig. 6 – which might explain the divergence of the FPE?
- The LE (9) seems to converge for any β , at least for some q 's, if p is large enough [*Berges and Sexty*].

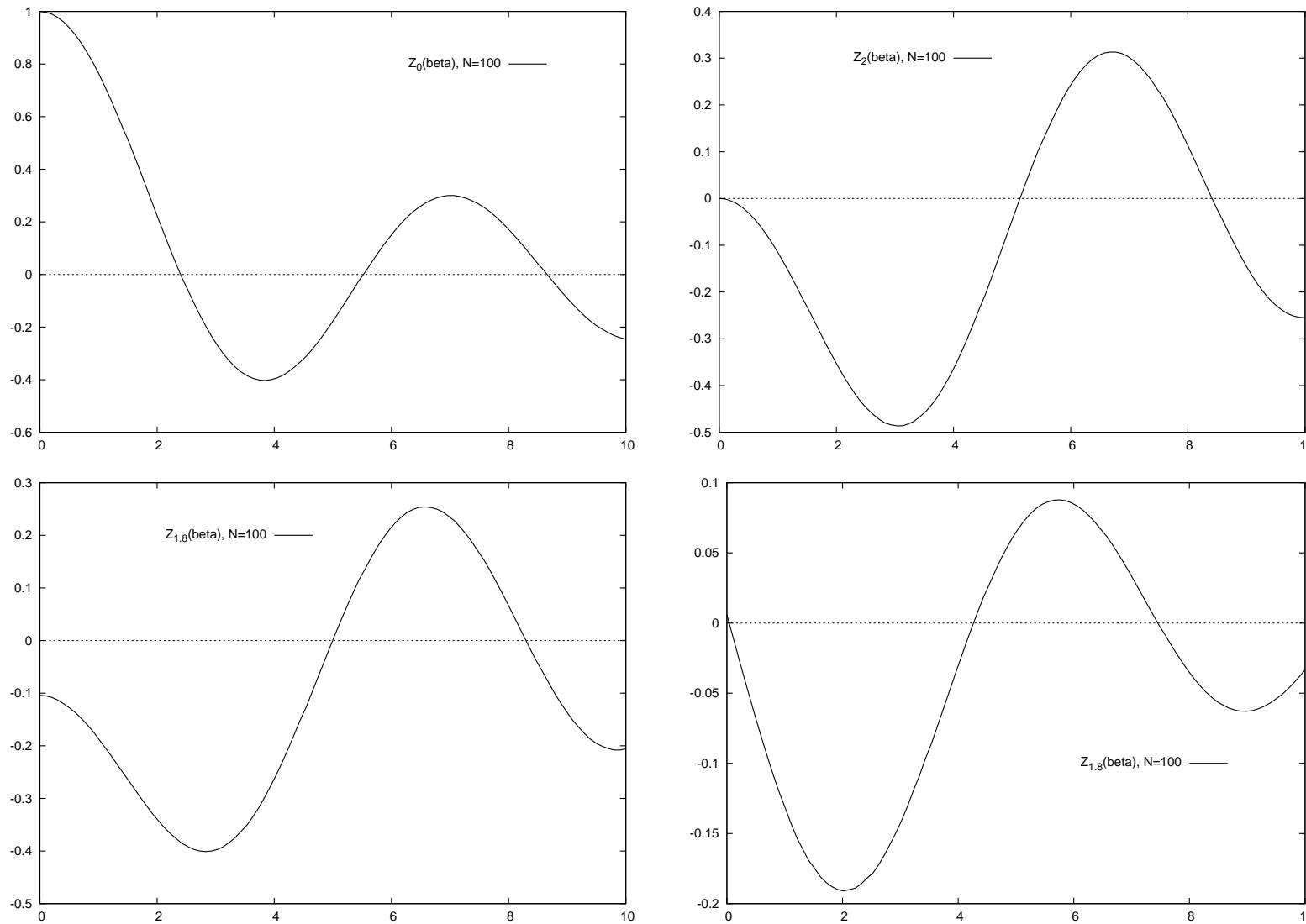


Figure 6: $Z_p(\beta)$ for $p = 0$ and $p = 2$ and for $p = 1.8$, with $N = 100$ x-discretization. Notice that the imaginary part vanishes for even p .

Real FP Equation

$$\begin{aligned} \dot{P}_p(x, y, t) = & \\ (\lambda \partial_x^2 + \beta \partial_x \cos x \sinh y - \partial_y(p + \beta \sin x \cosh y)) P_p(x, y, t) \end{aligned} \quad (29)$$

We again we discretize x (k) with p.b.c. With partial FT we get:

$$\begin{aligned} \dot{\tilde{P}}_p(k, y, t) = & -\lambda k^2 \tilde{P}_p(k, y, t) - p \partial_y \tilde{P}_p(k, y, t) \\ - & i \frac{\beta}{2} \sinh(y) k \left(\tilde{P}_p(k+1, y, t) + \tilde{P}_p(k-1, y, t) \right) \\ + & i \frac{\beta}{2} \partial_y \left\{ \cosh(y) \left[\tilde{P}_p(k+1, y, t) - \tilde{P}_p(k-1, y, t) \right] \right\}. \end{aligned} \quad (30)$$

The reality of $P(x, y, t)$ implies

$$\tilde{P}(k, y, t) = \tilde{P}^*(-k, y, t) \quad (31)$$

y is discretized with pbc and $\delta y = c\sqrt{\delta t}$.

With arbitrary analytic $f(z)$ we have

$$\int dx dy P_p(x, y, t) f(x + iy) = \int dx \rho_p(x, t) f(x) \quad (32)$$

hence, in particular

$$\int dx dy P_p(x, y, t) e^{i k x - k y} = \int dy e^{-k y} \tilde{P}_p(k, y, t) \quad (33)$$

$$= \int dx \rho_p(x, t) e^{i k x} = \tilde{\rho}_p(k, t) \quad (34)$$

If we assume the validity of (28) as established from the numerical evidence we then have:

$$\mathcal{E}\{e^{i q x - q y}\}_p = \overline{\{e^{i q z}\}_p} \quad (35)$$

which can be seen as justification for the definitions (14,15).

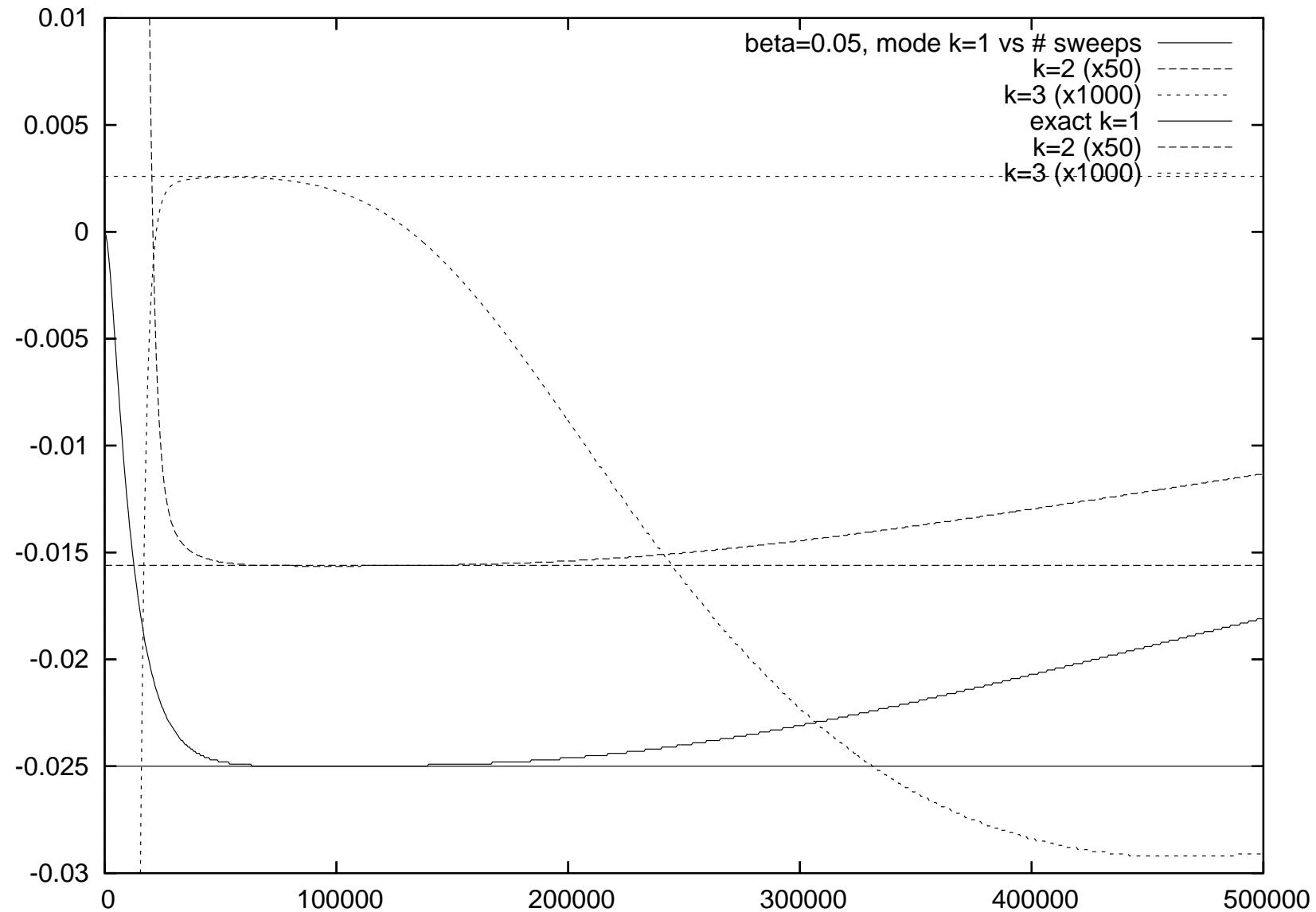


Figure 7:

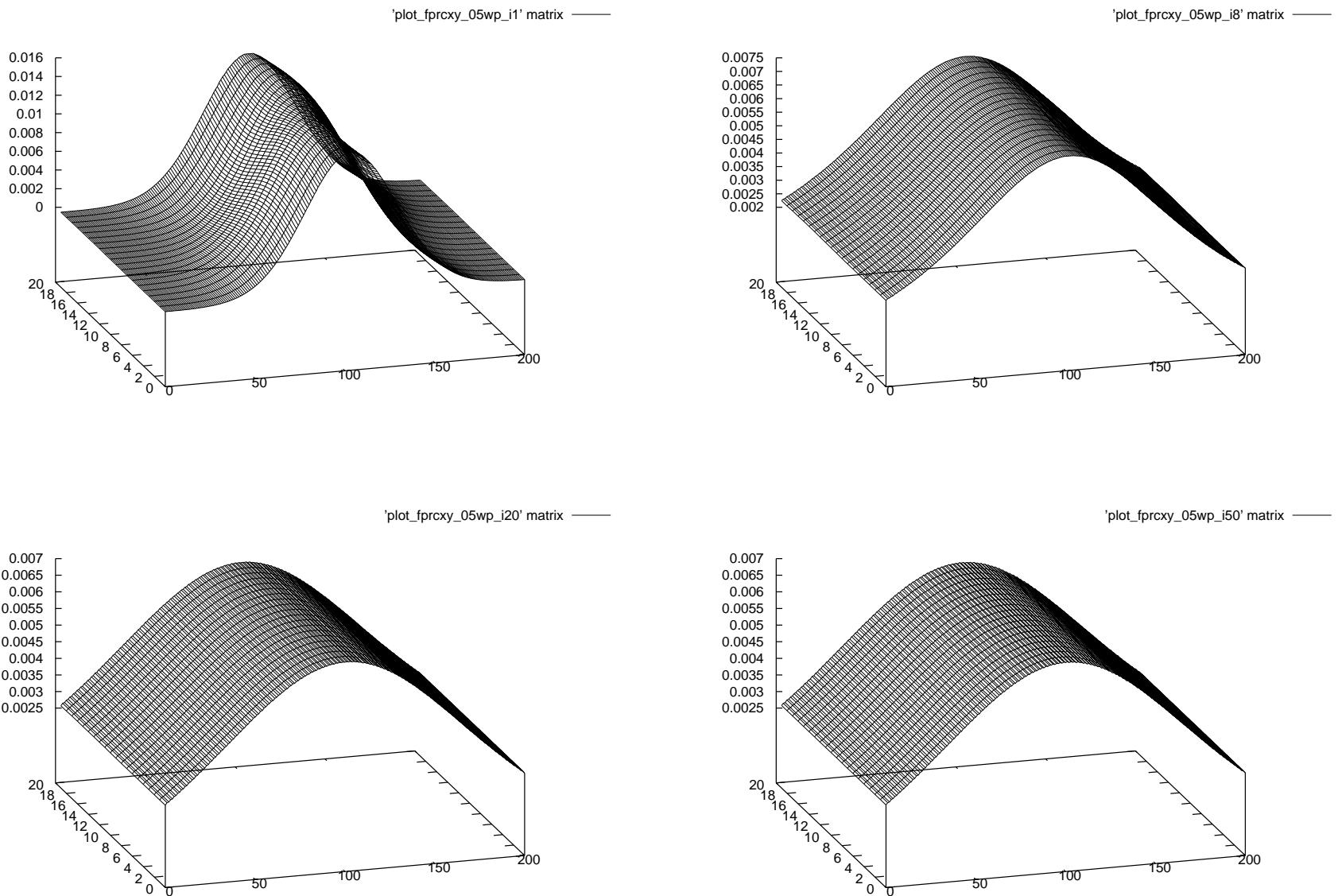


Figure 8:

Discussion

- There is high motivation to try to define numerical simulations for real time problems, where mainly approximate methods have been used so far:
 - High energy physics and QCD Plasma
 - Cosmology
 - Ultra-Cold quantum gases ...
- In our study we encountered a number of problems,
 - some of them easily manageable (run away's, $\delta\vartheta$ dependence),
 - some demanding more work and understanding (fixed point structure)
- A reasonable way to proceed might be not to try to arrive at a universal algorithm, but to start from the particular physical problem and define a method which can then be shown to work reliably.