Baryonic matter in the lattice Gross-Neveu model

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with
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Introduction I

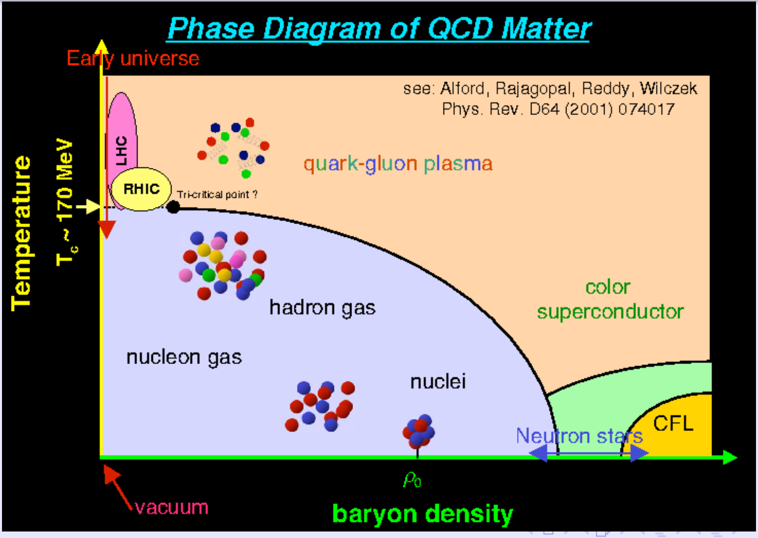
- Chiral phase transition in QCD:

$$\left.\begin{array}{c}\text{high temperature}\\ \text{or}\\ \text{high density}\\ \text{quark-gluon plasma}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\text{low temperature}\\ \text{or}\\ \text{low density}\\ \text{hadronic phase}\end{array}\right.$$

- At T_c or ρ_c hadrons start to overlap, chiral symmetry is restored:

$$T_c \simeq O(m_\pi)$$
$$\rho_c \simeq 3 - 5\rho_0, \quad \rho_0 \simeq 0.15\text{fm}^{-1}$$

QCD Phase diagram



Introduction II

- Understanding the properties of the transition is an intrinsically non-perturbative problem
  - $\Rightarrow$ methods like lattice simulations, effective theories,... are necessary.
- Standard lattice methods fail for finite density QCD
  - $\Rightarrow$ indirect methods have (large, unknown) systematic errors.
- Study QCD related models where failure is absent or under control

$\Rightarrow$ **Gross-Neveu model**

Definition of the model

- Euclidean lagrangian density in 2D [Gross, Neveu '74]

$$\mathcal{L} = \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x) \partial\!\!\!/ \, \psi^\alpha(x) - \frac{g^2}{2} \left(\sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x) \psi^\alpha(x) \right)^2 ,$$

  where $\psi^\alpha(x)$ are 2-component Dirac spinors and α flavour index.

- Introduce a scalar field $\sigma(x)$ conjugate to $\sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x)\psi^\alpha(x)$:

$$\mathcal{L} = \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x) \partial\!\!\!/ \, \psi^\alpha(x) + \frac{1}{2g^2}\sigma(x)^2 + \sigma(x) \sum_{\alpha=1}^{N} \bar{\psi}^\alpha(x)\psi^\alpha(x).$$

Properties

- The Gross-Neveu model
  - is renormalisable and asymptotically free,

$$\beta(g) = -\frac{N-1}{2\pi} g^3 + O(g^5),$$

  - has a $O(2N) \times \Gamma$-symmetry where Γ is the discrete chiral symmetry

$$\Gamma: \qquad \psi \to \gamma_5 \psi, \quad \bar{\psi} \to -\bar{\psi}\gamma_5, \quad \sigma \to -\sigma,$$

  - exhibits spontaneous breaking of the discrete chiral symmetry

    $\Rightarrow$ fermions acquire non-vanishing mass $\sigma_0 = \langle \sigma \rangle$ (dimensional transmutation).

Note: there is no Goldstone boson due to Γ being a discrete symmetry.

Large-N limit

- In the large-N limit with $\lambda = g^2 N$ fixed, the model can be solved analytically:
  - Integrate out the fermions to obtain $Z = \int_{[d\sigma]} \exp\{-S_{\text{eff}}\}$,

$$S_{\text{eff}} = N \left\{ \int_{[dx]} \frac{\sigma(x)^2}{2\lambda} - \operatorname{Tr}\log[\not\partial + \sigma] \right\}.$$

  - The minimum of the effective potential is given by

$$\partial_{\sigma(x)} S_{\text{eff}}/N = \frac{\sigma(x)}{\lambda} - \partial_{\sigma(x)} \operatorname{Tr}\log[\not\partial + \sigma] = 0, \ \forall x.$$

Gap equation

- For constant σ this reduces to a single equation

$$\frac{\sigma}{\lambda} = \partial_\sigma \mathrm{Tr} \log \left[\not{\partial} + \sigma \right],$$

or in momentum space

$$\sigma = 0 \quad \text{or} \quad \frac{1}{\lambda} = \int_{[dk]} \frac{2}{k^2 + \sigma^2}.$$

$\Rightarrow$ *gap equation (self consistency equation)*

- Equivalent equations via Hartree-Fock, Schwinger-Dyson, Bethe-Salpeter approaches.

Gap equation

- To leading order in $1/N$ the spectrum consists of

  [Dashen, Hasslacher, Neveu '75; Feinberg, Zee '97]

  $$m_1 = \sigma_0 \sim \Lambda \exp\left\{-\frac{\pi}{\lambda}\right\}, \qquad \text{single fermion,}$$

  $$m_n = \sigma_0 \cdot \frac{2N}{\pi} \sin\left(\frac{n\pi}{2N}\right), \qquad \text{n-fermion bound state,}$$

  $$m_B = \sigma_0 \cdot \frac{2N}{\pi}, \qquad \text{kink-antikink state ('baryon').}$$

- For chirally twisted spatial boundary conditions the single kink state

  $$\sigma(x) = \sigma_0 \tanh(\sigma_0 x)$$

  - is topologically stable,
  - interpolates between the two vacua related by the discrete γ_5-symmetry.

The phase structure I

- The GN model possesses a rich μ-T phase structure:

  [Dashen, Ma, Rajaraman '75; Wolff '85; Karsch, Kogut, Wyld '87]

  - Mermin-Wagner-Coleman theorems forbid spontaneous breaking of
    - continuous symmetry at $T = 0$,
    - discrete symmetry at $T \neq 0$.
  - fluctuations are expected to destroy any long range order

    $\Rightarrow$ free massless boson propagator is logarithmic in 2D,
  - however, fluctuations are suppressed at large N:

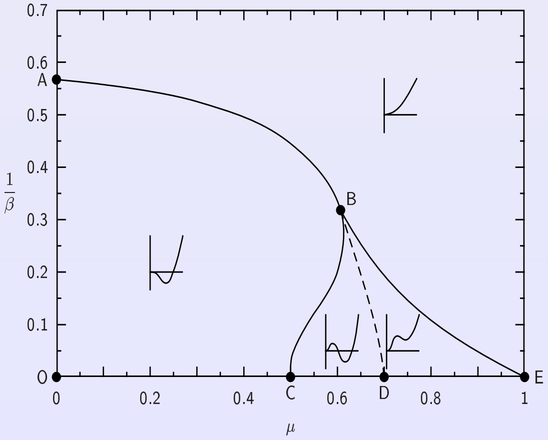
$$\langle\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)\rangle \sim 1 + \frac{1}{N} \ln|x - y| + O(1/N^2)$$

  becomes constant as $N \to \infty$.

$\Rightarrow$ take large-N limit before thermodynamic limit!

The 'old' phase diagram I

From the homogeneous mean field approximation [Wolff '85]:



- AB $\rightarrow$ 2nd order
- BD $\rightarrow$ 1st order
- B $\rightarrow$ tricritical point
- BCE $\rightarrow$ metastability region
- $T_c^A = e^{C/\pi} = 0.5669$
- $\mu_c^D = 1/\sqrt{2} = 0.7071$
- $T_c^B = 0.3183, \mu_c^B = 0.6082$

The 'old' phase diagram II

- On general grounds one expects from widely separated baryons
$$-\frac{\partial}{\partial\rho}\ln Z\Big|_{\rho=0,T=0} \equiv \mu_c = m_B .$$
  - mean field approximation is in conflict with this,
  - ad-hoc reconciliation via a droplet model of baryons, yielding a modified baryon mass $m_B = 1/\sqrt{2}$.
  - Something wrong with the mean field approach? No, but. . .

$\Rightarrow$ Assumption of translational invariance of σ is invalid.

The revised phase diagram I

- Thies et al. recently clarified the structure of cold baryonic matter in the GN model:

  [Schön, Thies '00; Thies, Brzoska '02; Thies, Urlichs '03; Thies '03; Schnetz, Thies, Urlichs '05]

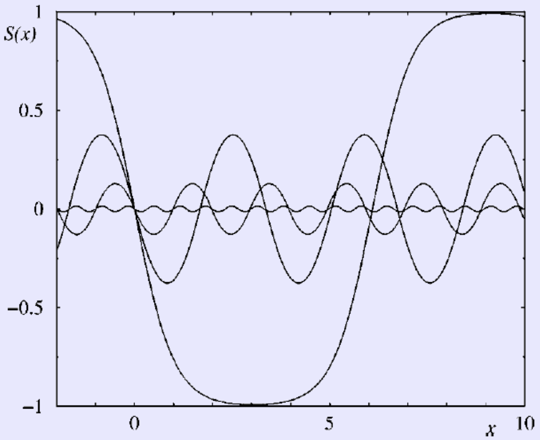
  - they use a Hartree-Fock approach with a spatially varying scalar potential,
  - the gap equation becomes a set of non-linear self-consistency equations,
  - potential ansatz inspired by the scalar potential for a single baryon:

$$\sigma(x) = 1 + y\left[\tanh(yx - c_0) - \tanh(yx + c_0)\right],$$

where $c_0 = \frac{1}{2}\text{arctanh}(y)$ and $y = y(\sigma_0)$.

The revised phase diagram II

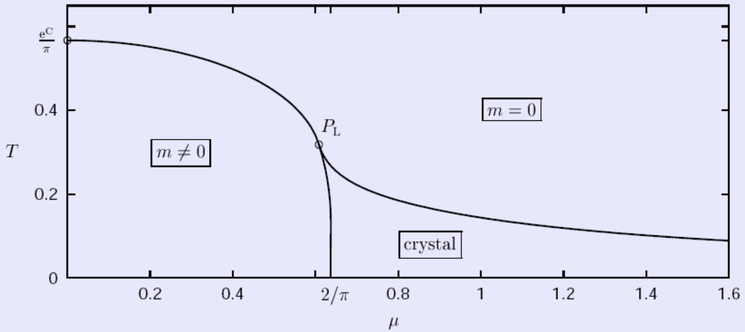
Scalar potential ansatz:



- motivated by matter at low density forming isolated baryons,
- Pöschl-Teller potential wells can be periodically extended,
- leads to a general ansatz satisfying self-consistency equation.

The revised phase diagram III

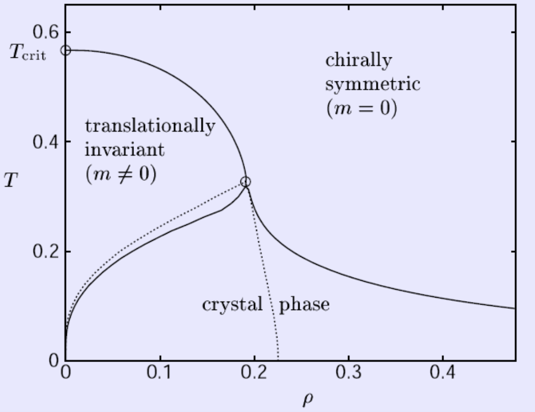
- In addition to the massive and massless Fermi gas, there is a new baryonic crystal phase at low temperature:



- $\mu_c = \frac{2}{\pi}$ now consistent with m_B, no first order transition at $\mu \neq 0$.

The revised phase diagram IV

(T, ρ)-phase diagram:



- $T_{\rm crit}$ unchanged,
- tricritical point turns into multi-critical point at the same location.

These findings motivate to look for the new phase in lattice models.

GN model with staggered fermions I

- Consider the staggered GN action:

$$S = N \sum_x \frac{\sigma(x)^2}{2\lambda} + \sum_{x,y} \sum_{\alpha=1}^{N} \bar{\chi}^\alpha(x) \left[D_{xy} + \Sigma_{xy}\right] \chi^\alpha(y)$$

  where the Dirac operator

$$D_{xy} = \frac{1}{2}\left[\delta_{x,y+\hat{1}} - \delta_{x,y-\hat{1}}\right] + \frac{1}{2}(-1)^{x_1}\left[\delta_{x,y+\hat{2}} - \delta_{x,y-\hat{2}}\right]$$

  describes 2 flavours and

$$\Sigma_{xy} = \frac{1}{4}\delta_{xy}\left(\sigma(x) + \sigma(x-\hat{1}) + \sigma(x-\hat{2}) + \sigma(x-\hat{1}-\hat{2})\right).$$

- Modification $\sigma \to \Sigma$ is necessary to ensure correct continuum limit [Cohen, Elitzur, Rabinovici '83].

GN model with staggered fermions II

- Discrete chiral symmetry is preserved:

$$\chi(x) \to (-1)^{x_1+x_2}\chi(x),\ \bar{\chi}(x) \to -(-1)^{x_1+x_2}\bar{\chi}(x),\ \sigma(x) \to -\sigma(x).$$

- A finite chemical potential $\Leftrightarrow$ time component of an imaginary external constant Abelian vector potential [Hasenfratz, Karsch '83]:

  $\Rightarrow$ weighting the temporal derivatives with factors $\exp(\pm\mu)$.

- In momentum space this amounts to the replacement

$$k_t \Rightarrow k_t - i\mu.$$

- Imaginary chemical potential corresponds to a non-trivial magnetic flux [Huang, Schreiber '94].

GN model with overlap fermions I

- Consider the massless overlap Dirac operator
  [Narayanan, Neuberger '94; Neuberger '98]

$$D = m\left\{1 + D_W(-m)\left[D_W^\dagger(-m)D_W(-m)\right]^{-1/2}\right\}$$

satisfying the Ginsparg-Wilson relation $D^\dagger + D = \frac{1}{m}D^\dagger D$.

- The coupling to the scalar field is introduced like

$$\mathcal{L} = \bar{\psi}(x)\left[\left(D_{x,y} - \frac{\sigma(x)}{4m}D_{x,y} - D_{x,y}\frac{\sigma(y)}{4m}\right) + \sigma(x)\delta_{x,y}\right]\psi(y)$$

consistent with a covariant scalar density.

- For $\sigma \to$ const. it is just the usual mass term

$$\left(1 - \frac{\sigma}{2m}\right)D + \sigma.$$

GN model with overlap fermions II

- For constant σ we can work in momentum space; we have (for $m = 1$):

$$D = \left\{ 1 + \left(i\gamma_\mu \mathring{p}_\mu + \frac{1}{2}\hat{p}_\mu^2 - 1 \right) \left[\left(\frac{1}{2}\hat{p}_\mu^2 - 1 \right)^2 + \mathring{p}_\mu^2 \right]^{-1/2} \right\}$$

where $\mathring{p}_\mu = \sin(k_\mu), \hat{p}_\mu = 2\sin(\frac{k_\mu}{2})$ with appropriate b.c.

- Chemical potential as before, replacing everywhere

$$k_t \Rightarrow k_t - i\mu.$$

Thermodynamic limit for homogeneous condensate

- We need to calculate the (real) fermion determinant

$$\ln\det D = 2\sum_{t=0}^{L_t/2-1}\sum_{k=0}^{L_x/2-1}\ln\left[\hat{p}_t^2+\hat{p}_k^2+\sigma^2\right]$$

  where $\hat{p}_t, \hat{p}_k$ are lattice momenta, L_t, L_x lattice extensions.

- For a homogeneous condensate one can perform the thermodynamic limit analytically,

$$\lambda = \frac{L_t}{2}\left(\sum_{t=0}^{L_t/2-1}\frac{1}{\left(\sigma^2+\hat{p}_t^2\right)\sqrt{1+\frac{1}{\sigma^2+\hat{p}_t^2}}}\right)^{-1}.$$

Thermodynamic limit for inhomogeneous condensate I

- For an inhomogeneouscondensate we have

$$\det D = \prod_{t=0}^{L_t-1} 2^{L_x} \det\left(P_t - \left(\frac{1}{2}\right)^{L_x}\right)$$

with the reduced matrices [Gibbs '86; Hasenfratz, Toussaint '91]

$$P_t = \prod_{x=0}^{L_x/2-1} \left(\Omega_t(2x)\Omega_{L_t/2+t}(2x+1)\right)$$

and

$$\Omega_t(x) = \begin{pmatrix} \hat{p}_t + \sigma(x) & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

- Can be interpreted as a transfer matrix in space at each t.

Thermodynamic limit for inhomogeneous condensate II

- If the condensate is invariant under translation by l_x and $L_x = nl_x$,

$$\det D_t = 2^{nl_x} \det \left(P_t^n - 2^{-nl_x} \right) .$$

- In the thermodynamic limit we then simply have

$$\lim_{n \to \infty} \frac{1}{n} \ln \det D_t = \sum_{t=0}^{L_t - 1} \ln \lambda_t^{(1)}$$

where $\lambda_t^{(1)}$ is the larger of the two eigenvalue of P_t.

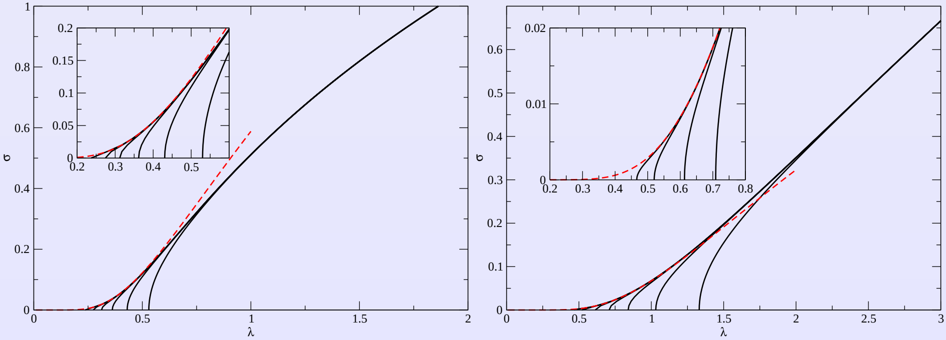
- Length scale L_x of the box size is replaced by l_x, the wave length of the condensate.

Homogeneous mean field results at $\mu = 0, T = 0$

- Gap equation yields σ as a function of λ:

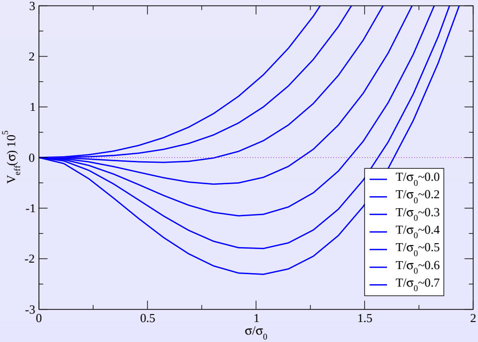
$\Rightarrow$ non-perturbative β-function vs asymptotic scaling

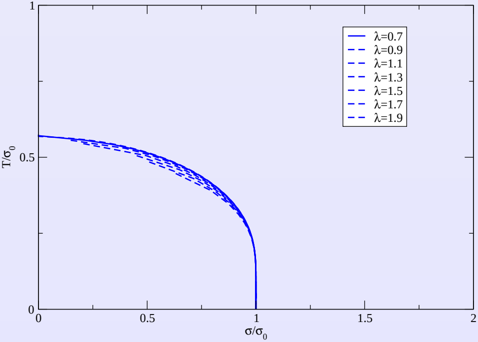
staggered operator, asymptotic scaling $2^{3/2}e^{-\pi/2\lambda}$ — overlap operator, asymptotic scaling $1.5539\ldots e^{-\pi/\lambda}$



Homogeneous mean field results at $\mu = 0$

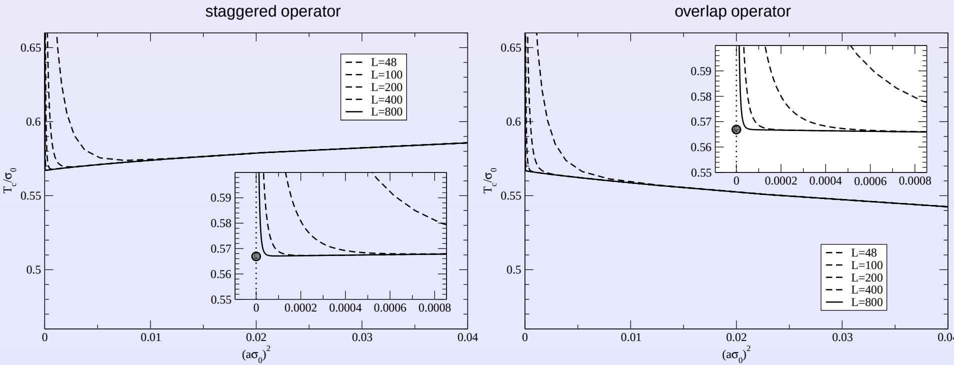
- second order transition at $T_c, \mu = 0$:
  (overlap Dirac operator, $\lambda = 1.0, L = 200$)





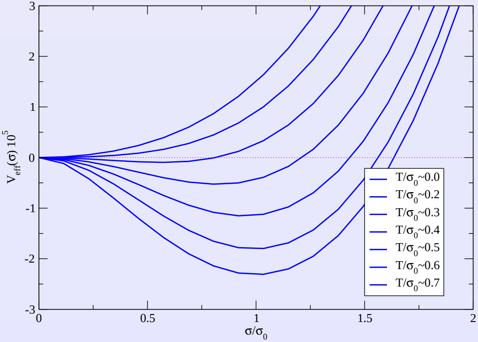
Homogeneous mean field results at $\mu = 0$

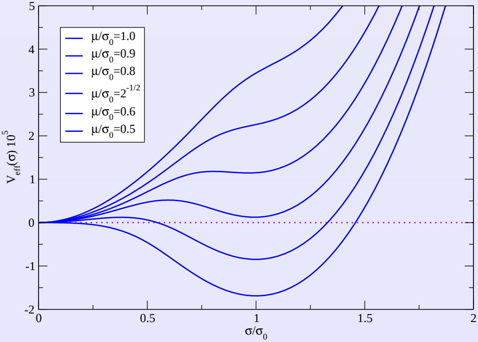
- Scaling of T_c/σ_0 vs $(a\sigma_0)^2$:



Homogeneous mean field results

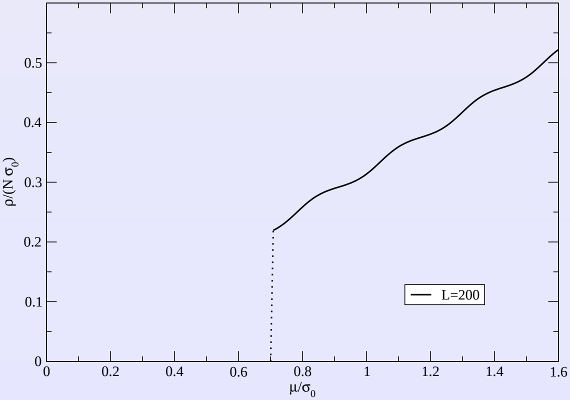
- second order transition at $T_c, \mu = 0$ vs first order at $T = 0, \mu_c$:
  (overlap Dirac operator, $\lambda = 1.0, L = 200$)





Homogeneous mean field results

- Normalised fermion density vs chemical potential at $T \simeq 0$:



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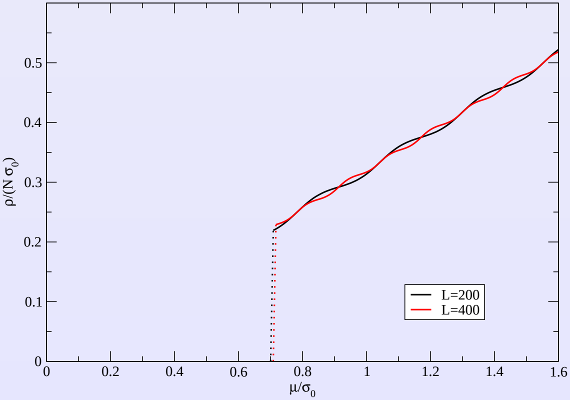

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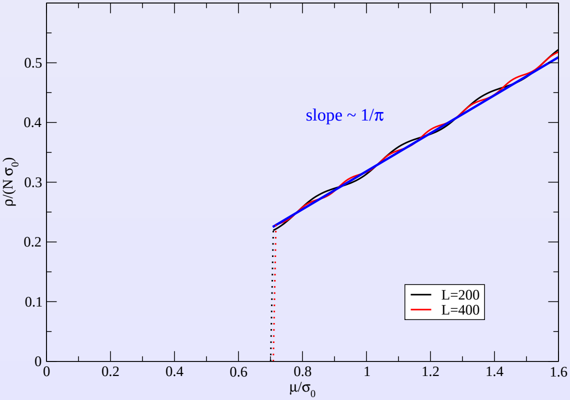

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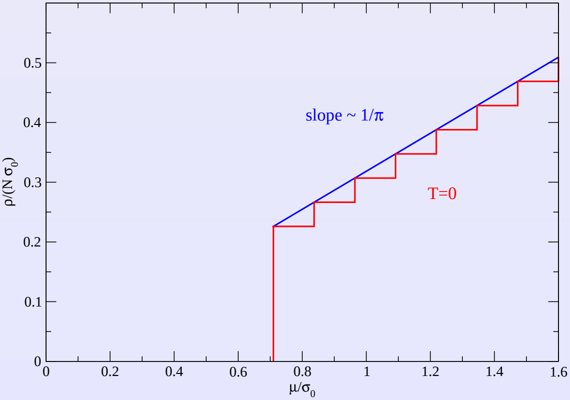

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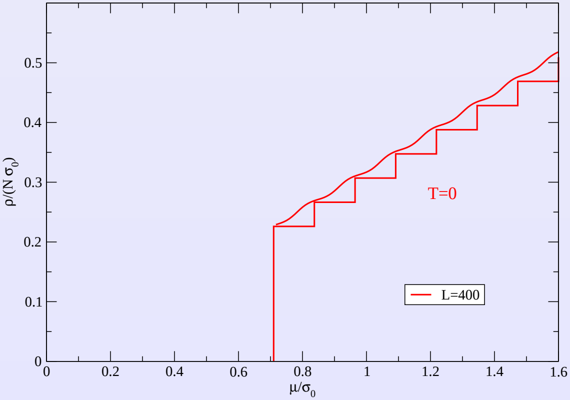
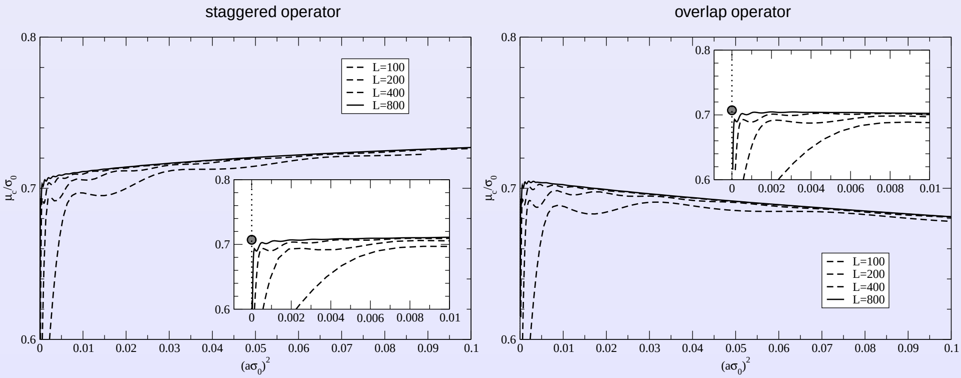


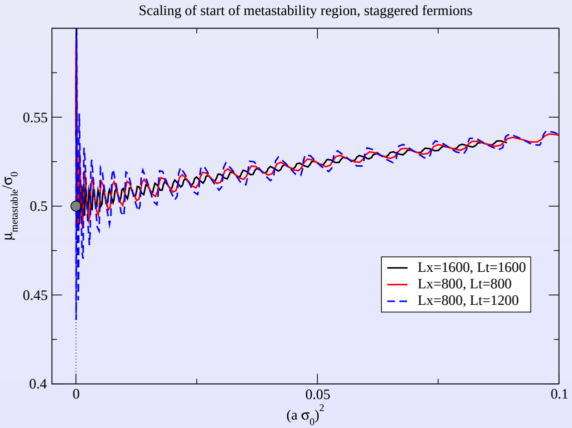
Homogeneous mean field results

- Scaling of μ_c/σ_0 vs $(a\sigma_0)^2$:



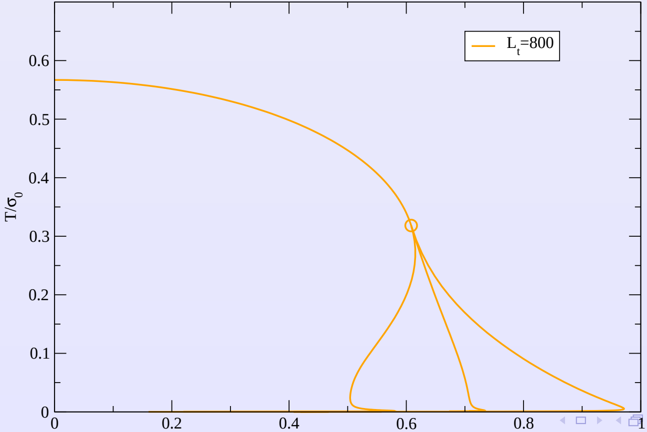
Homogeneous mean field results

- Scaling of the entry into the metastable region at $T \simeq 0$:



Homogeneous mean field results

- Phase diagram from homogeneous mean field, in the thermodynamic limit:



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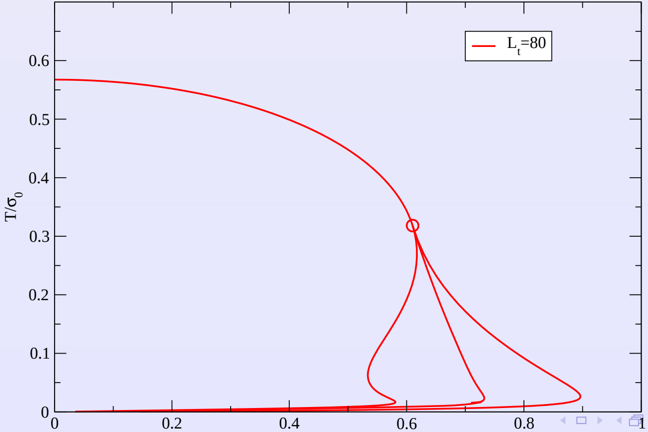

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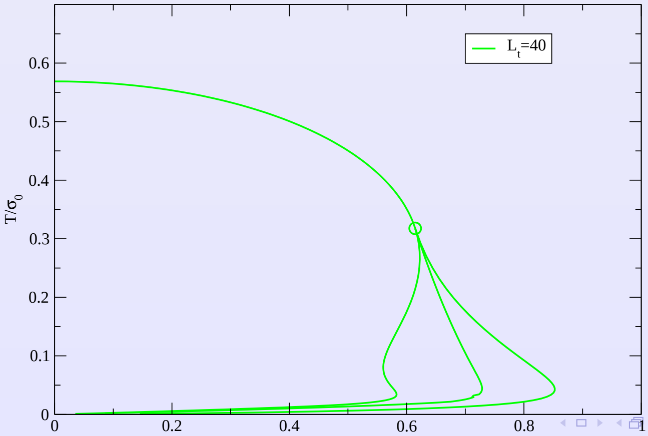

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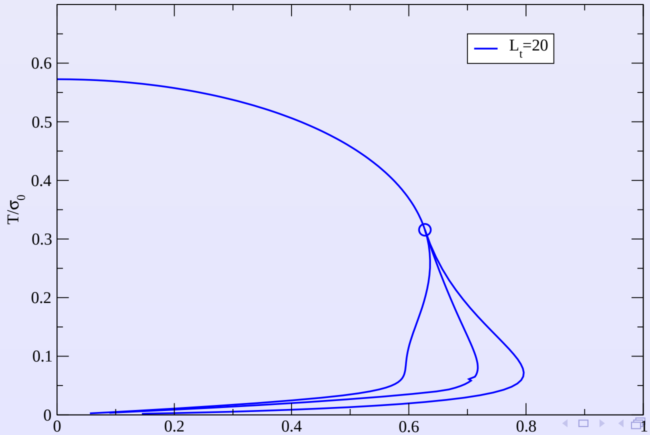

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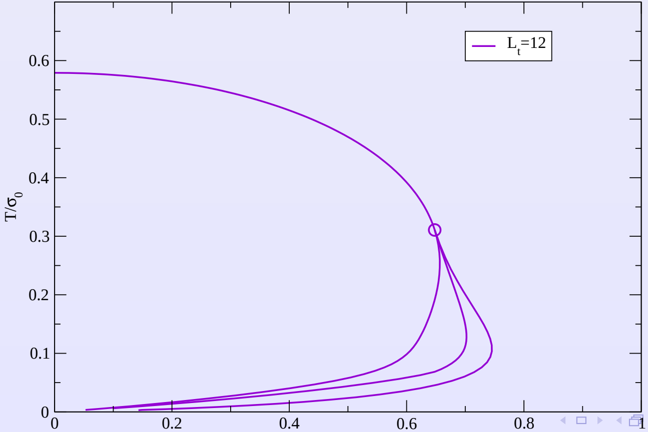

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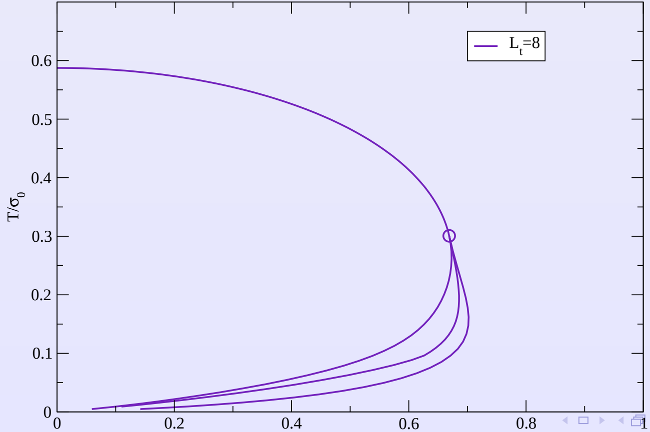

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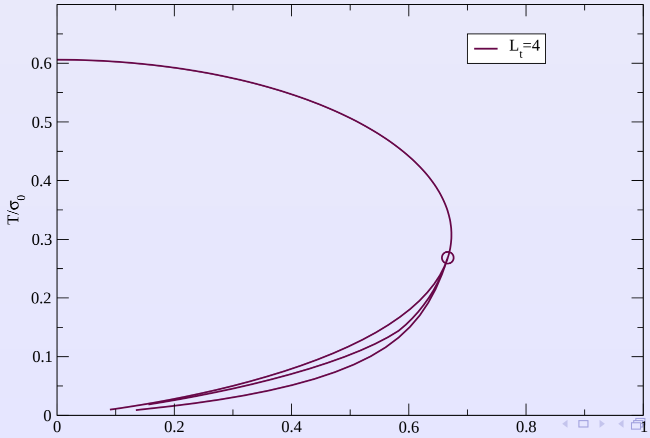

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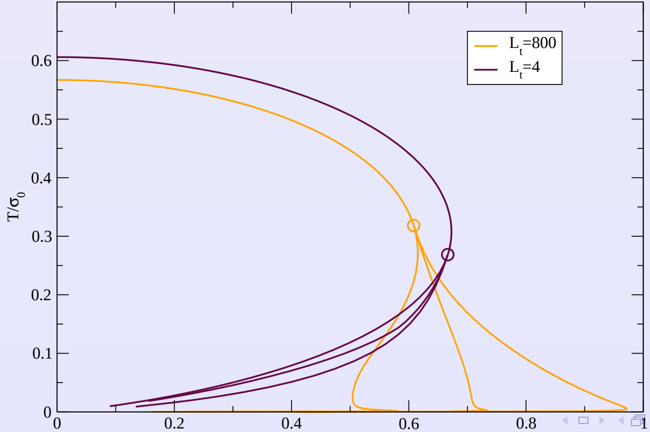
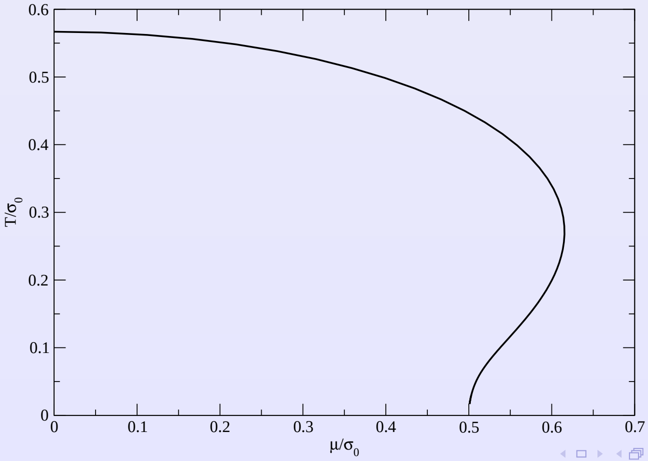


Homogeneous mean field results

- Phase diagram from imaginary chemical potential [Huang, Schreiber '94]:



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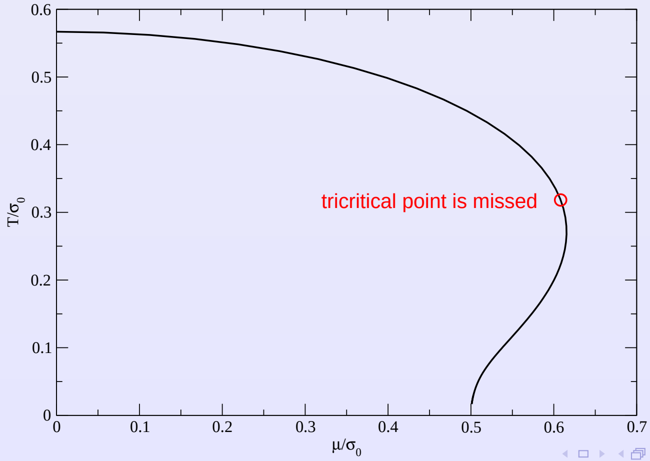
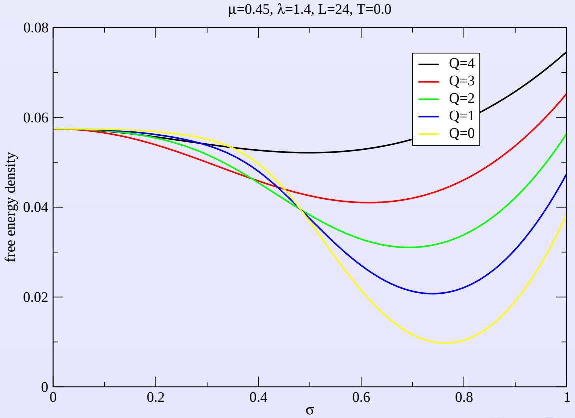


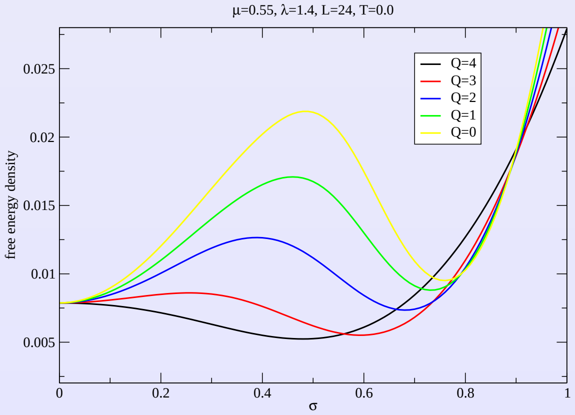
Crystal phase results

- Free energy density for kink-antikink solutions, variational calculation at $\mu = 0.45$:



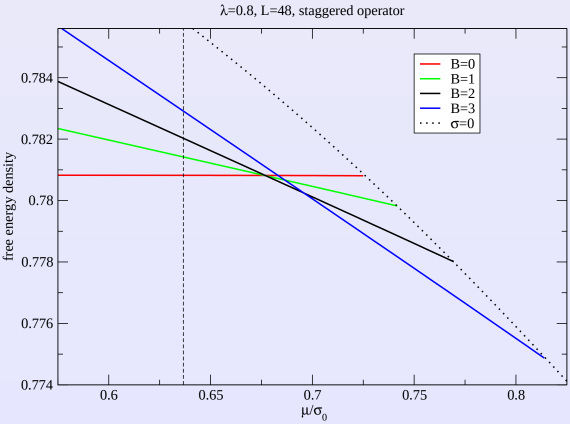
Crystal phase results

- behaviour suggests second order transition:



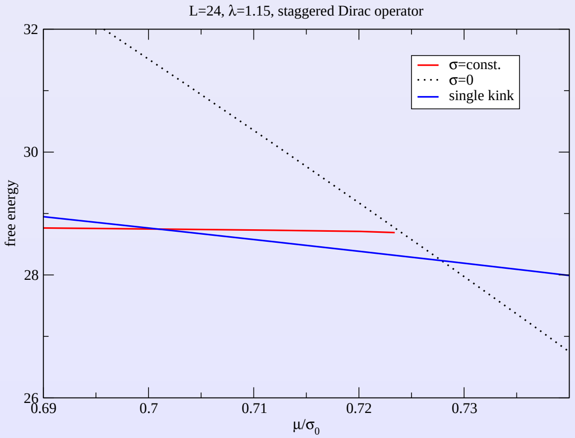
Crystal phase results

- Free energy density vs. different kink-antikink solutions, $\lambda = 0.8$:



Crystal phase results

- Crystal phase towards strong coupling, $\lambda = 1.15$:



Crystal phase results

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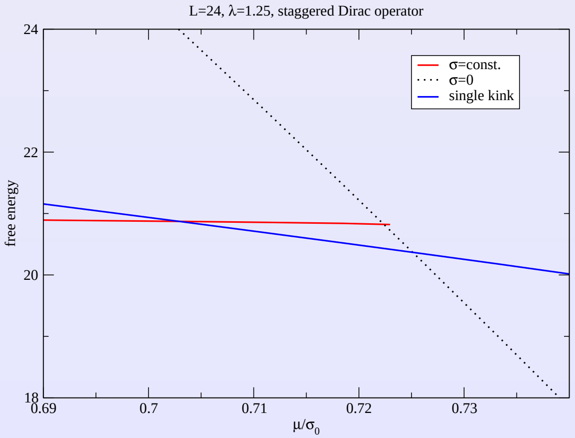
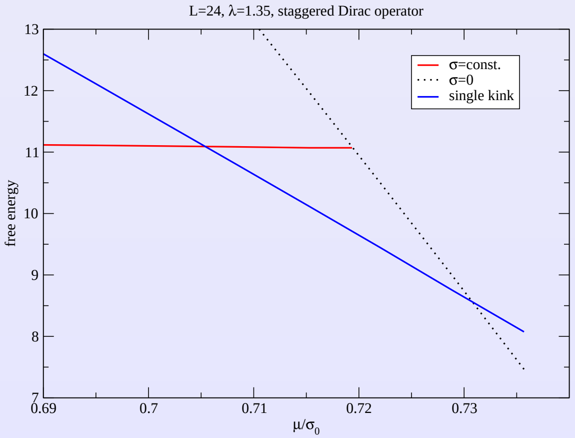


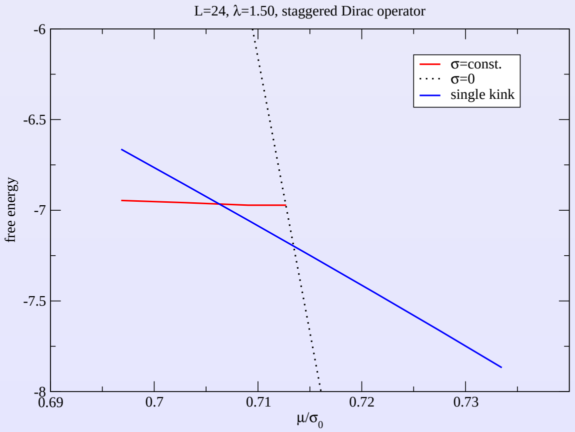
Crystal phase results

- Crystal phase towards strong coupling, $\lambda = 1.35$:



Crystal phase results

- Crystal phase towards strong coupling, $\lambda = 1.50$:



Crystal phase results

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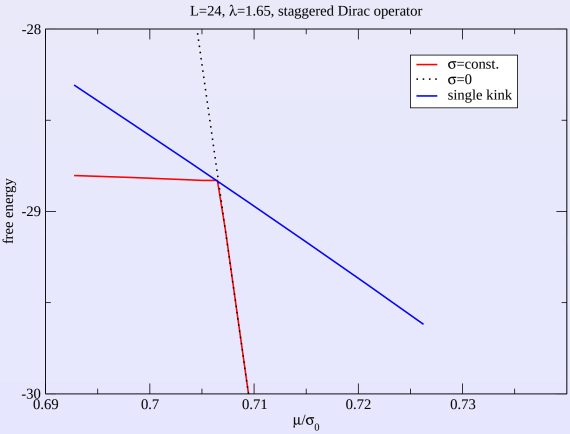
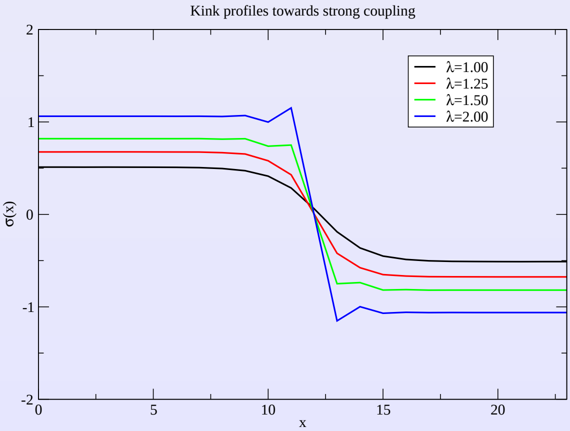


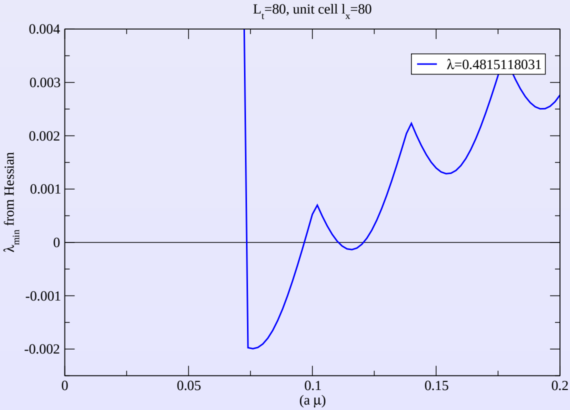
Crystal phase results

- Single kink solutions towards strong coupling:



Crystal phase results

- Instability of the $\sigma = 0$ free energy density wrt spatial variations $\Leftrightarrow$ end of the crystal phase



Crystal phase results

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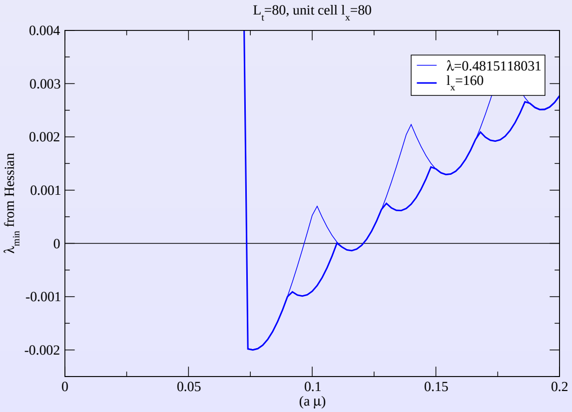
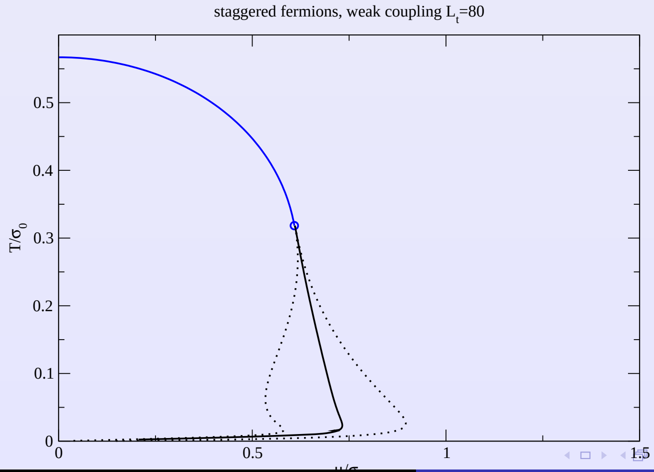


incommensurability effects

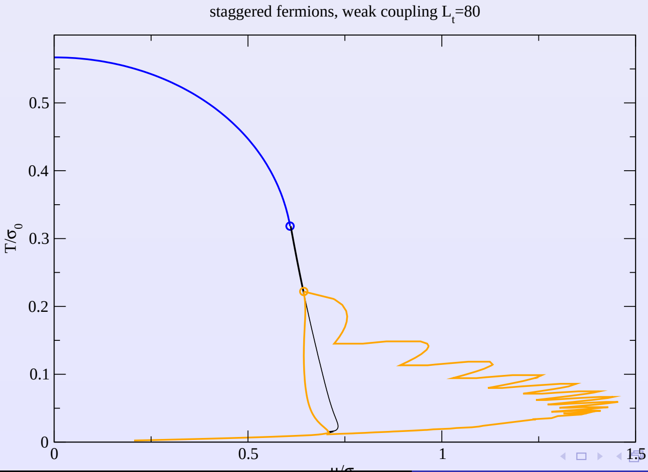
Crystal phase results

- Phase diagram:



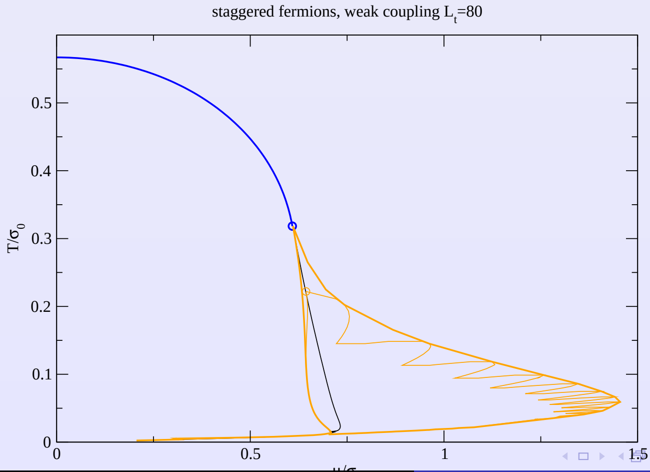
Crystal phase results

- Phase diagram with crystal phase, unit cell $l_x = 80$:



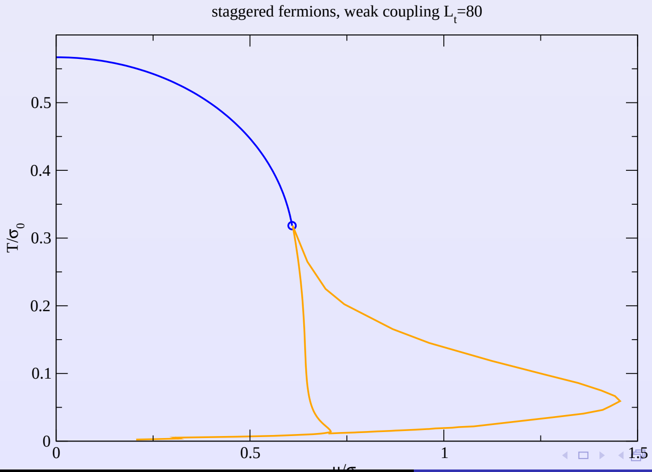
Crystal phase results

- Phase diagram with crystal phase, thermodynamic limit:



Crystal phase results

- Phase diagram with crystal phase:



Summary

- The breakdown of translational invariance of the ground state requires a revision of the GN model phase diagram.
- Besides the massive and massless Fermi gas phase, a new phase of baryonic matter emerges:
  $\Rightarrow$it forms a baryon crystal
- The transition to the new phase is always second order.
- We are investigating the new phase on the lattice:
  - crystal phase disappears at strong coupling, topological excitations fall through the lattice,
  - large volumes are necessary,
  - poses potential obstacle for simulations at finite density.

Outlook

- Crystal phase is caused by topological excitations $\Rightarrow$ look for effect in other models
  - 't Hooft model in 1+1 dimensions (chiral spiral)
  - NJL model in 2+1 dimensions (with continuous chiral symmetry)
  - QCD wit $N_f = 2$ in 3+1 dimensions:
    $\Rightarrow$SU($N_f = 2$) symmetry allows topological Skyrmion solutions
- Other related work:
  - large-N_c QCD in 3+1 dimension [Deryagin, Grigoriev, Rubakov '92]

Gross-Neveu model at finite N

- Most natural formulation in terms of Majorana fermions.
- For the Wilson lattice discretisation:

$$\mathcal{L} = \frac{1}{2}\xi^T \mathcal{C}(\gamma_\mu \tilde{\partial}_\mu + m - \frac{1}{2}\partial^* \partial)\xi - \frac{g^2}{4}\left(\xi^T \mathcal{C}\xi\right)^2 .$$

- For even N each pair of Majorana fermions may be considered as on Dirac fermion

$$\psi = \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2), \quad \bar{\psi} = \frac{1}{\sqrt{2}}(\xi_1^T - i\xi_2^T)\mathcal{C}.$$

- Integrating the fermions yields the Pfaffian

$$Z = \mathrm{Pf}[\mathcal{C}(\gamma_\mu \tilde{\partial}_\mu + m - \frac{1}{2}\partial^* \partial)].$$

Gross-Neveu model at finite N

- Expanding the Grassmannian Boltzmann factor one obtains a loop representation in terms of monomers and dimers.
- Partition function sum is over all non-oriented, self-avoiding loops

$$Z = \sum_{\{k(x,\mu)\}\in\mathcal{L}} \rho[k], \quad \mathcal{L} \in \{\mathcal{L}_{00}, \mathcal{L}_{10}, \mathcal{L}_{01}, \mathcal{L}_{11}\}.$$

- This is equivalent to a special case of the 8-vertex model

$$Z_{8-vertex} = \sum_{l\in\mathcal{L}} \prod_{x\in l} w(x).$$

Examples of 8-vertex models

- Ising model on the dual lattice,
- Ising model in high-temperature expansion,
- close packed dimer problem,
- QED_2 at $\beta = 0$ with Wilson fermions,
- GN model with Majorana Wilson fermions.

Numerical simulation of 8-vertex models

- Very powerful 'Worm'-type algorithms can be applied:
  - amounts to enlarging the configuration space by open loops,
  - corresponds to sampling directly the correlation function.
- Critical slowing-down much suppressed.
- I see no objections to adapt this to $d > 2$.