Poisson to Wigner-Dyson Transition in the High Temperature QCD Dirac Spectrum

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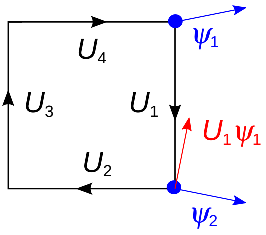
Lattice QCD



- $\psi_i \in \mathbb{C}^{N_c}$ (quark fields on sites)
- Different bases at different sites
- $U_i \in SU(N_c)$ complex rotation
  - $1 \rightarrow 2$ basis transformation (ψ)
  - U's promoted to be genuine dynamical variables

Discretization

Derivative:	$\partial_\mu \psi$	$\rightarrow$	$\frac{1}{a}(\psi_2 - \psi_1)$
Covariant derivative:	$D_\mu \psi$	$\rightarrow$	$\frac{1}{a}(\psi_2 - U_1\psi_1)$
Gluon action:	$\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$	$\rightarrow$	$-\frac{1}{g^2}\mathrm{tr}(U_1U_2U_3U_4)$

The QCD lattice Dirac operator

- Partition function (integrating out quarks):

$$Z = \int \mathscr{D}\psi \mathscr{D}\bar{\psi} \mathscr{D}U \; e^{-S[U] \; - \; \bar{\psi}\{D[U]+M\}\psi}$$

$$= \int \mathscr{D}U \; \det\{D[U]+M\} \cdot e^{-S[U]}$$

- 4d statistical physics system
- degrees of freedom: $U_i \in SU(N_c)$ on links

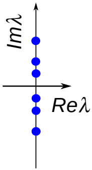
Dirac operator: $D[U]$

- Discretized differential operator depending on U's
- $4N_cV \times 4N_cV$ sparse matrix (V–volume: $\approx 10^3 - 10^7$)
- $(D[U]+M)^{-1}$ appears in physical quantities
- Small eigenvalues (and eigenvectors) physically important

Symmetries and consequences

- Symmetries:

$$\left.\begin{array}{l}\{\gamma_5, D\} = 0 \\ D^\dagger = -D\end{array}\right\} \Rightarrow D = \begin{pmatrix} 0 & iC \\ iC^\dagger & 0 \end{pmatrix}$$



- $\Rightarrow$ Spectrum imaginary, symmetric w.r.t. real axis.

- probability distribution of $[U]$ $\Rightarrow$ random $D[U]$ matrices
- prob. distribution of $D[U]$ $\Rightarrow$ physical quantities
- spectral statistics of $D[U]$ $\Rightarrow$ thermodynamics

What are the statistical properties of the spectrum of $D[U]$?

QCD dynamics important? Symmetries alone determine statistics?

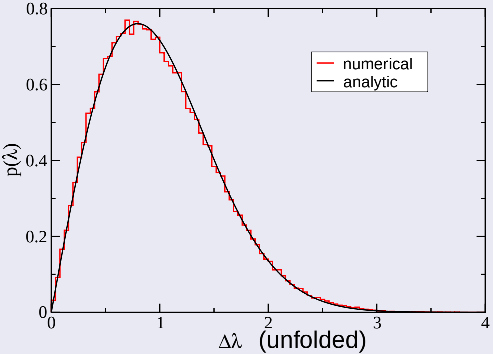
Random matrices

- $N \times N$ matrices with i.i.d. random elements
- For $N \gg 1$ certain eigenvalue statistics are universal!
- Depend on symmetries of the matrices:
  A few symmetry classes (elements real, complex, etc.)
- Within given class eigenvalue statistics barely depend on
  - Distribution of matrix elements
  - Structure of the matrix (which elements $\neq 0$)
- If matrix elements have a Gaussian distribution,
  analytically calculable.
- No preferred basis $\Rightarrow$ typical eigenvectors "delocalized"

Example: chiral orthogonal ensemble

- $\begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix}$ matrices, $W_{ij} \in \mathbb{R}$
- Elements indepently and uniformly distributed in $[-1,1]$

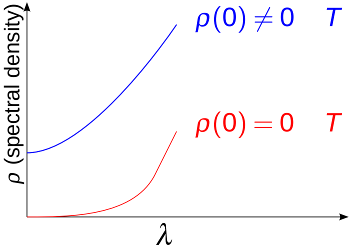
Level spacing distribution



5000
100×100-as matrices

Is the Dirac operator a random matrix?

- $\lambda = 0$ special point of the spectrum (symmetry).
- Transition around $T_c \approx 200\text{MeV}$



$\rho(0) \neq 0$ $T < T_c$ hadronic phase

$\rho(0) = 0$ $T > T_c$ quark-gluon plasma

$\rho(0) \neq 0 \quad \Rightarrow$ statistics of low eigenvalues of $D[U]$ is described by a random matrix model with the symmetries of $D[U]$

analytically (σ-model) $+$ numerically (lattice QCD)

What about $T > T_c$?

- No analytic results.
- Numerically (lattice QCD) $\Rightarrow$ random matrix description OK
  (Pullirsch et al. Phys. Lett. **B427**, 119, 1998)
- Random matrix description $\Rightarrow$ delocalized eigenvectors.
- But! $T > T_c$ $\Rightarrow$ Low eigenmodes of $D[U]$ are strongly localized.

Spectrum: random matrix statistics $\Rightarrow$ $\Leftarrow$ Eigenvectors: localized

???

Localization of eigenmodes

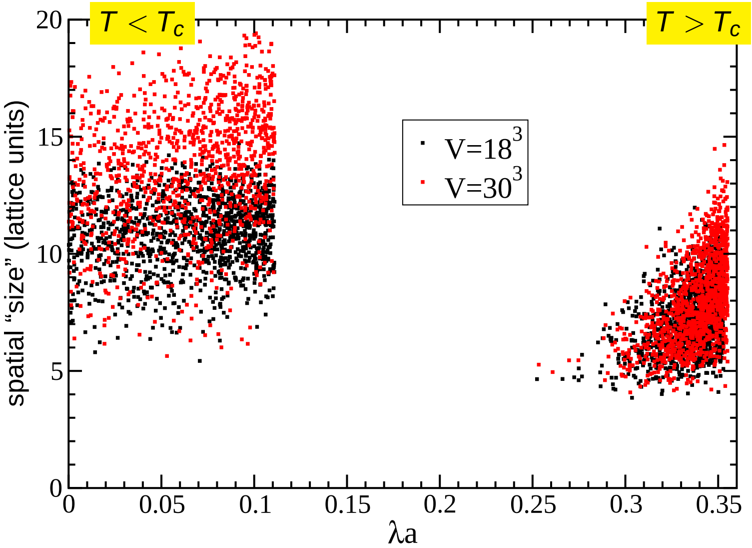
- How to measure spatial "extension" of (normalized) eigenmodes?
- Participation ratio:

$$\mathscr{V} = \left[\sum_x (\psi^\dagger \psi(x))^2 \right]^{-1}.$$

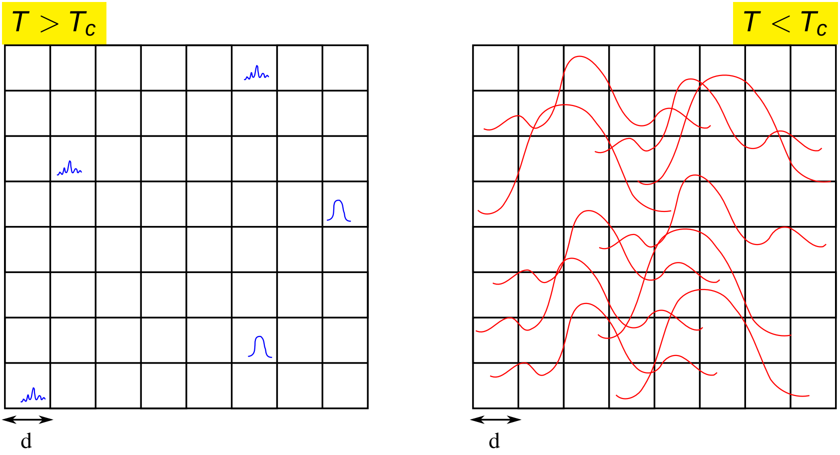
- If ψ is constant in volume V and zero elsewhere $\Rightarrow \mathscr{V} = V$.
- $\mathscr{V}$ measures "volume" occupied by eigenmode.
- $\Rightarrow$ Define 3-dimensional "size":

$$d = \left[\frac{\mathscr{V}}{N_t} \right]^{1/3}.$$

Localization: comparison of $T > T_c$ and $T < T_c$



Typical lowest eigenvectors for $T > T_c$ and $T < T_c$



Simple picture for $T > T_c$

- There is a scale d such that:
  - Small modes are localized in volumes $\leq d^3$.
  - Average number of small modes in volume d^3 is $\ll 1$.
- $\Rightarrow$ Small eigenvalues occur independently.

Independence of small eigenvalues

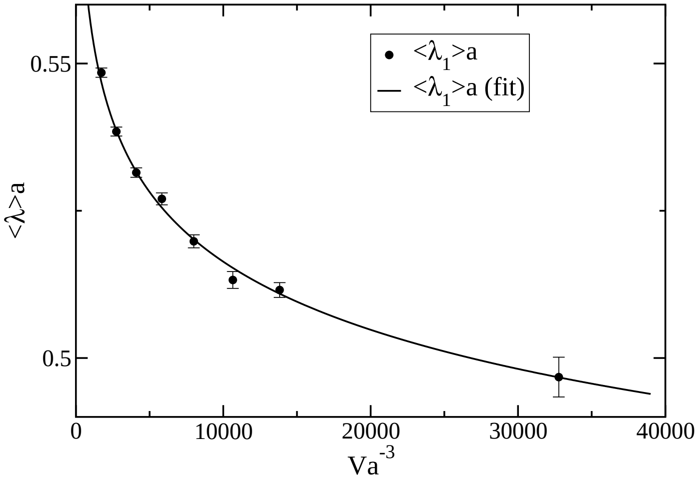
- Assumptions
  - The number of eigenvalues occurring in disjoint intervals are independent random variables. (Generalized Poisson distribution).
  - Ansatz for the spectral density (for small λ's): $\rho(\lambda) \;\approx\; C \cdot \lambda^{\alpha}$ (Poisson: $\alpha = 0$).
- $\Rightarrow$ Analytic predictions for the distribution of small eigenvalues, including volume dependence.

  (TGK, Phys. Rev. Lett. **104** 031601, 2010)
- Two free parameters: $C, \alpha \;\; (\mu = \frac{1}{\alpha+1})$

Lattice: 3-volume dependence of $\langle\lambda_1\rangle$ (T fixed)

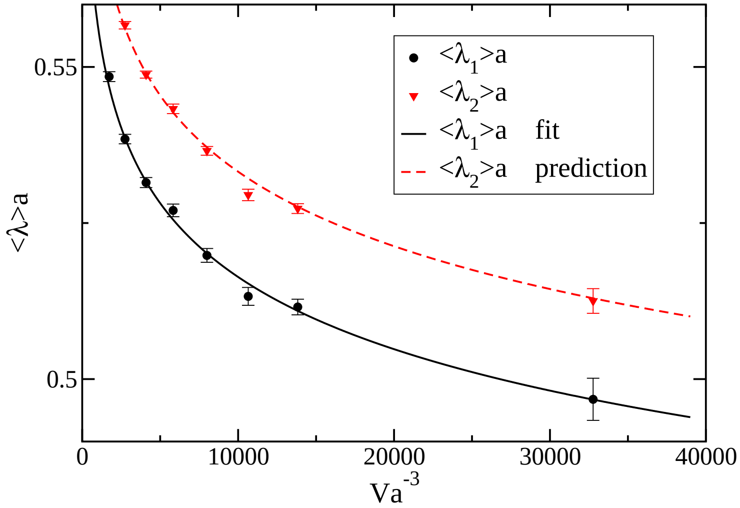
Analytic: $\langle\lambda_1\rangle = (CV\mu)^{-\mu} \cdot \Gamma(1+\mu)$

Parameters fitted: μ, C



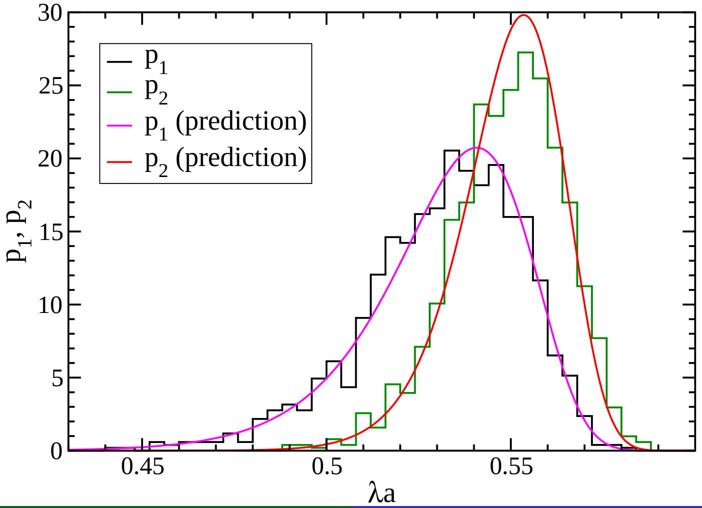
3-volume dependence of $\langle \lambda_2 \rangle$

$\langle \lambda_2 \rangle = (CV\mu)^{-\mu} \cdot \Gamma(2+\mu)$ No free parameter!



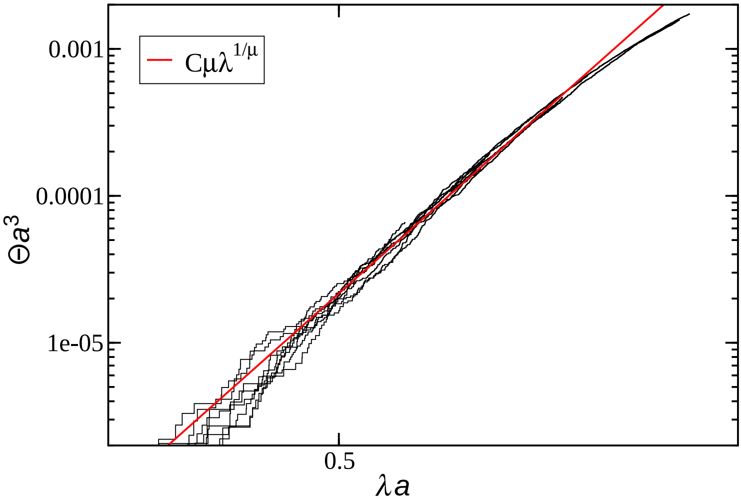
No free parameter!

$p_1(\lambda)$ and $p_2(\lambda)$ contain the already fitted parameters μ, C.



Integrated spectral density, $\Theta(\lambda)$

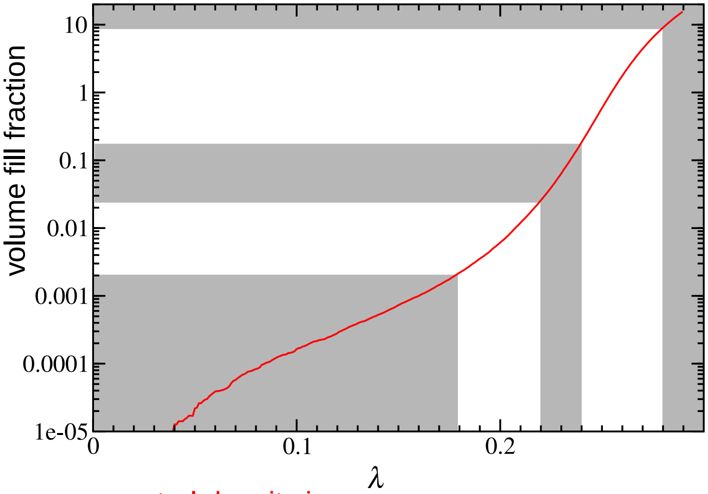
$\Theta(\lambda) = \mu C \lambda^{\frac{1}{\mu}}$ using the fitted parameters μ, C.



$T > T_c$: Poisson or random matrix statistics?

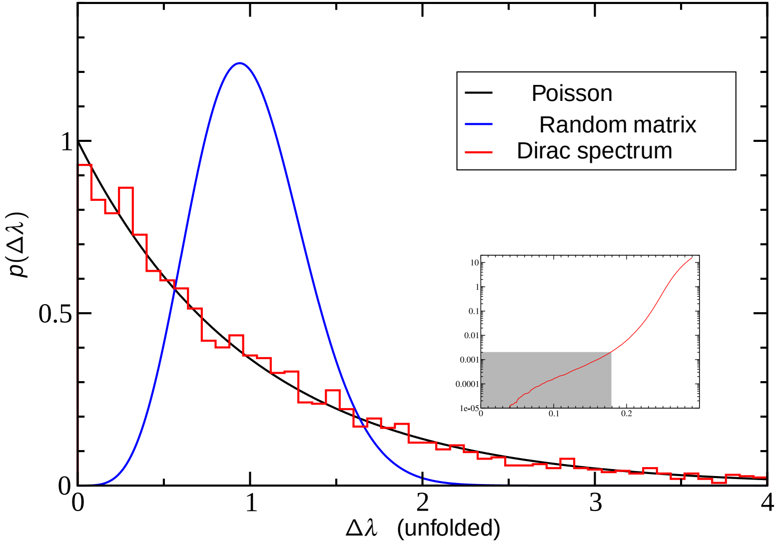
- Smallest eigenvalues follow generalized Poisson statistics.
- How far up does this continue in the spectrum?

Regimes of the spectrum

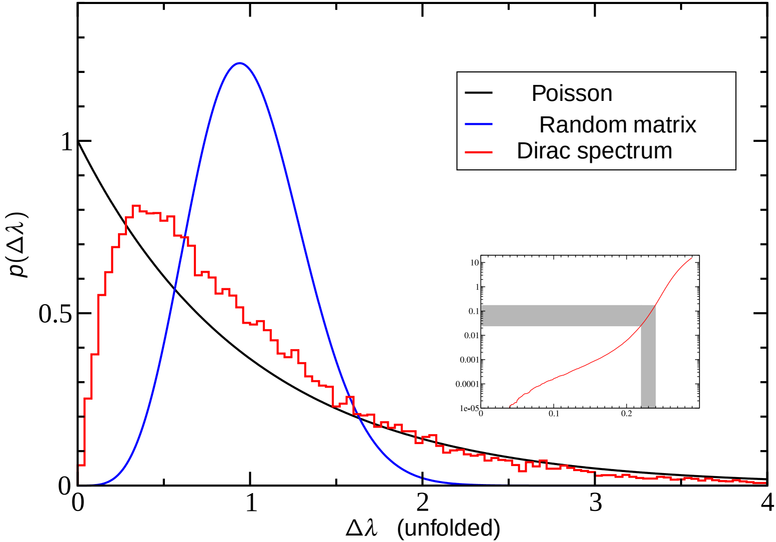


spectral density increases
eigenvectors get delocalized $\longrightarrow$

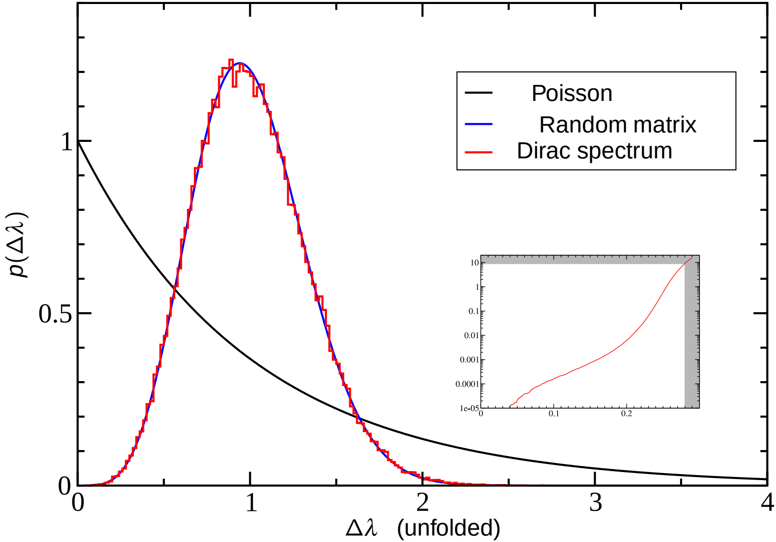
Level spacing distribution (lowest eigenvalues)



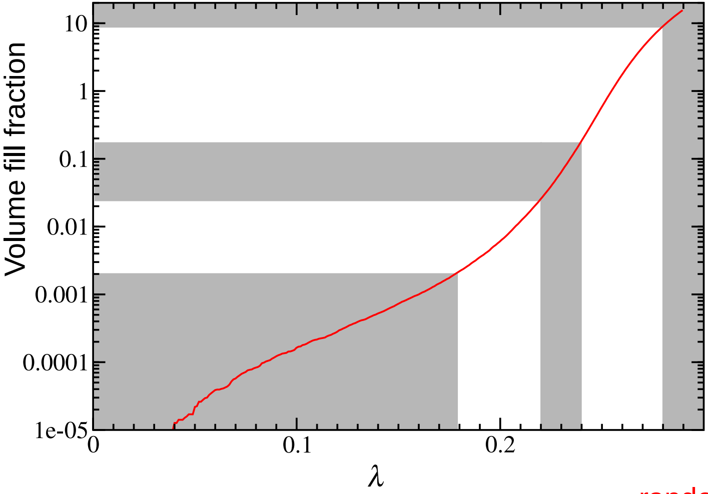
Transition regime



“Bulk” of the spectrum



Statistics of eigenvalues above T_c



Poisson transition random matrix →

Analogy: Anderson localization

Is the hadron $\rightarrow$ quark-gluon plasma transition an Anderson transition? (Garcia-Garcia, Osborn, Phys. Rev. **D75**, 034503 2007)

Anderson localization:

- Perfect periodic crystal $\rightarrow$ Delocalized electron states: bands
- Defects (disorder) $\rightarrow$ 1-electron H-operator "random matrix".
- Strong disorder $\rightarrow$ localized states appear at the band edge.

Further questions

- Does this survive the continuum limit? (It seems so!)
- How universal?
  (Here: $N_c = 2$, overlap, staggered, different N_T's.)
- What happens at $T \approx T_c$?
  (Here: $T \gg T_c$.)
- Physical mechanism?
  - $d \approx \frac{1}{T}$
  - Role of fermion boundary condition.
- Physical consequences?
  (What to look for in heavy ion colliders?)

Distribution of smallest eigenvalue

Let's first compute $\mathscr{P}_\emptyset(\lambda_1,\lambda_2)$, the probability that there is no eigenvalue in the $[\lambda_1,\lambda_2]$ interval:

Average number of eigenvalues in a small $\Delta x (\to 0)$ interval: $V\rho(x)\Delta x$

- Probability of ≥ 2 eigenvalue: $\to 0$
- Probability of 0 eigenvalue: $1 - V\rho(x)\Delta x$
- Probability of 1 eigenvalue: $V\rho(x)\Delta x$

No correlation $\Rightarrow$ Probabilities factorize:

$$\mathscr{P}_\emptyset(\lambda, x+\Delta x) = \mathscr{P}_\emptyset(\lambda, x)\cdot[1 - V\rho(x)\Delta x]$$

$$\Rightarrow \quad \frac{d\mathscr{P}_\emptyset(\lambda,x)}{dx} = -V\rho(x)\cdot\mathscr{P}_\emptyset(\lambda,x) = -VCx^\alpha\cdot\mathscr{P}_\emptyset(\lambda,x)$$

Distribution of the smallest eigenvalue contd.

$\mathscr{P}_\emptyset(\lambda_1, \lambda_2)$ contd.:

Solution with initial value

$$\mathscr{P}_\emptyset(\lambda,\lambda) = 1 :$$

$$\mathscr{P}_\emptyset(\lambda,x) = \exp\left[-\frac{VC}{\alpha+1}\left(x^{\alpha+1}-\lambda^{\alpha+1}\right)\right]$$

Probability density of smallest eigenvalue, $p_1(\lambda)$:

$$\text{Probability of}\begin{cases} \text{no eigenvalue in } [0,\lambda] : & \mathscr{P}_\emptyset(0,\lambda) \\ \text{one eigenvalue in } [\lambda,\lambda+d\lambda] : & V\rho(\lambda)\,d\lambda \end{cases}$$

$$\mathscr{P}_\emptyset(0,\lambda)\cdot V\rho(\lambda)\,d\lambda = p_1(\lambda)\,d\lambda$$

$$\Rightarrow \quad p_1(\lambda) = \exp\left(-\frac{CV}{\alpha+1}\lambda^{\alpha+1}\right)CV\lambda^{\alpha}$$

Further analytic predictions

Distribution of the second smallest eigenvalue, $p_2(\lambda)$:

$$\text{Probability of} \begin{cases} \text{Smallest eigenvalue in } [x, x+\Delta x]: & p_1(x)\Delta x \\ \text{No eigenvalue in } [x+\Delta x, \lambda]: & \mathscr{P}_\emptyset(x,\lambda) \\ \text{One eigenvalue in } [\lambda, \lambda + d\lambda]: & V\rho(\lambda) d\lambda \end{cases}$$

$$\Rightarrow \quad p_2(\lambda) = \int_0^\lambda dx\, p_1(x) \cdot \mathscr{P}_\emptyset(x,\lambda) \cdot V\rho(\lambda)$$

$$= \frac{C^2 V^2}{\alpha+1} \exp\left(-\frac{CV}{\alpha+1}\lambda^{\alpha+1}\right) \lambda^{2\alpha+1}.$$

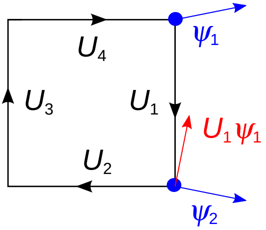
Average smallest eigenvalues

$$\langle\lambda_1\rangle = (CV\mu)^{-\mu} \cdot \Gamma(1+\mu)$$

$$\langle\lambda_2\rangle = (CV\mu)^{-\mu} \cdot \Gamma(2+\mu)$$

$$\mu = \frac{1}{\alpha+1}$$

Lattice QCD



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