QCD with chemical potential in a small hyperspherical box

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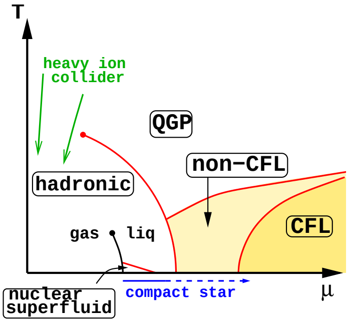
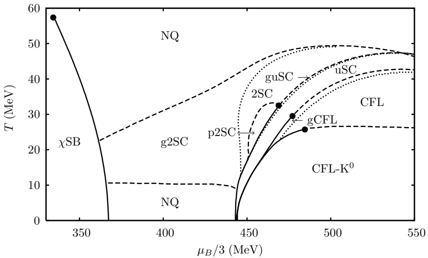
Outline

- Overview of QCD at non-zero quark chemical potential
- Formulation of QCD on $S^1 \times S^3$ using perturbation theory
- Results for various observables for $N = 3$ and $N = \infty$, preliminary $N = 2$ lattice results.

Conjectured phase diagram of QCD

Progress has been made towards obtaining the phase diagram of QCD at non-zero density using lattice simulations as well as models of QCD.





(Left) Conjectured phase diagram of QCD in the μ - T plane. (Right) Detailed possible phase diagram from an NJL model (from Alford arXiv:0907.0200).

Partition Function of QCD

The partition function of QCD at finite temperature $T = 1/\beta$, for N_f quark flavors, each with a mass m_f and coupled to a chemical potential μ_f is:

$$Z_{QCD} = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\int_0^\beta d\tau \int d^3\mathbf{x}\mathcal{L}_{QCD}}$$

where ψ and $\bar{\psi}$ are the fundamental and anti-fundamental fermion fields, respectively, and A is the $SU(N)$ gauge field, $A_\mu = A_\mu^a T^a$.

The Lagrangian is

$$\mathcal{L}_{QCD} = \frac{1}{4g^2}\mathrm{Tr}_F\left(F_{\mu\nu}F_{\mu\nu}\right) + \sum_{f=1}^{N_f} \bar{\psi}_f \left(\not{D}_F(A) - \gamma_0\mu_f + m_f\right)\psi_f,$$

with covariant derivative

$$D_\mu(A) \equiv \partial_\mu - A_\mu.$$

What makes QCD at non-zero μ so difficult?

The Sign Problem:

QCD at finite quark chemical potential μ has a complex action:

$$e^{S_f} = \exp \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\int_0^\beta d\tau \int d^3\mathbf{x}\bar{\psi}\left(\not{D}_F(A)-\gamma_0\mu+m\right)\psi}$$

$$= \log\det\left(\not{D}_F(A) - \gamma_0\mu + m\right) \sim \sum_{n=1}^{\infty}\left[e^{n\beta\mu}e^{i\theta_i n} + e^{-n\beta\mu}e^{-i\theta_i n}\right]$$

- The boltzmann weight e^{-S} is complex so it is not possible to perform lattice simulations which use importance sampling.
- The sign problem also complicates large N analysis: In the large N limit the saddle point approximation becomes valid, but the stationary point of a complex action with respect to the angles of the Polyakov line $P = \mathscr{P}e^{\int_0^\beta \mathrm{d}t\, A_0(x)} = \mathrm{diag}\{e^{i\theta_1}, ..., e^{i\theta_N}\}$ lies in the space where the angles are complex. Therefore the eigenvalues of the Polyakov line lie off the unit circle on an arc in the complex plane.

$\Longrightarrow$ Need to generalize our techniques to handle a complex action.

Region of validity of 1-loop calculations

Properties of $SU(N)$ gauge theories on $S^1 \times S^3$

- Valid for $\min[R_{S^1}, R_{S^3}] \ll \Lambda_{QCD}^{-1}$
  - $\mathbb{R}^3 \times S^1$, small S^1:
    - ★ Good: Allows study at any N and in the limit of large 3-volume. YM/QCD: $m = 0, \mu = 0$: Gross, Pisarski, Yaffe (Rev.Mod.Phys.53:43,1981),
    - ★ Bad: Have to be in the limit of high temperatures (or small S^1)
  - $S^3 \times S^1$, small S^3:
    - ★ Good: Allows study at any temperature (or any S^1). YM: Aharony et al (hep-th/0310285 (JHEP)),
    - ★ Bad: Must be in small 3-volume. Finite N studies are more complicated.

1-loop Lagrangian

Introduce fluctuations around a background field: $A_0 = \alpha + g\mathscr{A}_0$, then gauge fix and retain the one-loop contributions:

$$
\begin{aligned}
\mathcal{L}_{QCD} = &-\frac{1}{2}\mathscr{A}_0^a(D_0^2(\alpha) + \Delta^{(s)})\mathscr{A}_0^a - \frac{1}{2}B_i^a(D_0^2(\alpha) + \Delta^{(v,T)})B_i^a \\
&-\frac{1}{2}C_i^a(D_0^2(\alpha) + \Delta^{(v,L)})C_i^a - \bar{c}(D_0^2(\alpha) + \Delta^{(s)})c \\
&+\sum_{f=1}^{N_f}\bar{\psi}_f(\not{D}_F(\alpha) - \gamma_0\mu_f + m_f)\psi_f
\end{aligned}
$$

where $$A_i = B_i + C_i.$$

- B_i = transverse: $\nabla_i B_i = 0$
- C_i = longitudinal: $C_i = \nabla_i f$

1-loop partition function

Performing the Gaussian integrals the almost-cancellation of the scalar field contributions simplifies the one loop effective partition function:

$$Z(\alpha) =$$

$$\det_{\ell=0}^{1/2}\left(D_0^2(\alpha)+\Delta^{(s)}\right)\det^{-1}\left(-D_0^2(\alpha)+\Delta^{(v,T)}\right)\det^{N_f}\left(\not{D}_F(\alpha)-\gamma_0\mu+m\right)$$

Eigenvalues ε_l and degeneracies d_l of Laplacians on S^3:

$$\Delta^{(type)}\Omega_{j,l,m_1,m_2}(\theta_1,...,\theta_3) = -\varepsilon_l^{(type)2}\Omega_{j,l,m_1,m_2}(\theta_1,...,\theta_3)$$

Example: scalars

$$\varepsilon_l^{(s)2} = l(l+2)/R^2$$

$$d_l^{(s)} = (l+1)^2$$

where $l = 0, 1, ...,$ and R is the radius of S^3.

1-loop partition function: S^1 contribution

The eigenvalues of the Dirac operator can be computed in frequency space in terms of the Matsubara frequencies:

$$D_0(\alpha) \to i\omega_n^- - \alpha,$$

where the Matsubara frequencies, for antiperiodic (thermal) boundary conditions are

$$\omega_n^- = (2n+1)\pi/\beta.$$

We define the Polyakov loop:

$$P = \mathscr{P} e^{\int_0^\beta \mathrm{d}t\, A_0(x)} = e^{\beta\alpha} = \mathrm{diag}\{e^{i\theta_1}, ..., e^{i\theta_N}\}$$

1-loop effective action

Simplification of the effective partition function leads gives the effective action

$$\begin{aligned} S(P) &= -\log Z(P) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - z_b(n\beta/R)\right) \mathrm{Tr}_A P^n \\ &\qquad + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} N_f \, z_f(n\beta/R, mR) \left[e^{n\beta\mu} \mathrm{Tr}_F P^n + e^{-n\beta\mu} \mathrm{Tr}_F P^{\dagger n} \right], \end{aligned}$$

where

$$z_b(\beta/R) = \sum_{\ell=1}^{\infty} d_\ell^{(v,T)} e^{-\beta \varepsilon_\ell^{(v,T)}} = 2 \sum_{\ell=1}^{\infty} \ell(\ell+2) e^{-n\beta(\ell+1)/R}$$

$$z_f(\beta/R, mR) = \sum_{\ell=1}^{\infty} d_\ell^{(f)} e^{-\beta \varepsilon_\ell^{(f,m)}} = 2 \sum_{\ell=1}^{\infty} \ell(\ell+1) e^{-\beta \sqrt{(\ell+\frac{1}{2})^2 + m^2 R^2}/R}$$

For the pure Yang-Mills theory the weak-coupling analogue of the deconfinement transition temperature can be calculated in the large N limit: $T_d R \simeq 0.759$ or $\beta_d/R \simeq 1.317$ [Aharony et al (hep-th/0310285)].

Important observables

At finite N we can calculate observables by numerically performing the integrals over the gauge field angles θ_i.

$$\langle \mathcal{O} \rangle \equiv \frac{\int [\mathrm{d}\theta] e^{-S} \mathcal{O}}{Z}, \qquad Z = \int [\mathrm{d}\theta] e^{-S}.$$

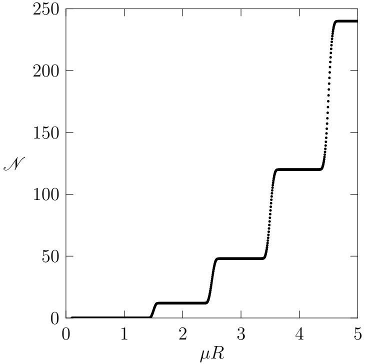
- Polyakov loop order parameters for phase transitions: $\langle \mathrm{Tr} P \rangle$, $\langle \mathrm{Tr} P^\dagger \rangle$.

$$\mathscr{P}_1 = \langle \mathrm{Tr} P \rangle$$

$$\mathscr{P}_{-1} = \langle \mathrm{Tr} P^\dagger \rangle$$

  - These should differ for $N > 2$.
- Average Phase: $\langle e^{i\phi} \rangle_{pq} \equiv Z / Z_{pq}$.
  - This tells us where the sign problem is severe.
- Average number $\mathcal{N} = \langle N_{quarks} - N_{antiquarks} \rangle$.
  - Gives the net number of quarks allowed at a given value of μ.
- Pressure: $\mathcal{P}$
- Energy: E

Average fermion number $\mathcal{N}$ ($N = 3$, $N_f = 1$, $mR = 0$)



- Average fermion number for QCD on $S^1 \times S^3$ with $m = 0$, $\beta/R = 30$ (low T).

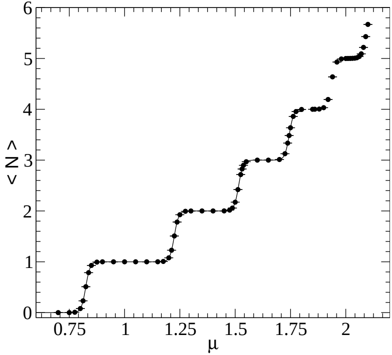
$$\mathcal{N} = \frac{1}{\beta}\left(\frac{\partial \ln Z}{\partial \mu}\right)$$
$$= \frac{-1}{\beta Z}\int [\mathrm{d}\theta]\, e^{-S}\left(\frac{\partial S}{\partial \mu}\right)$$

$$\mathcal{N} \xrightarrow[\beta\to\infty]{} \frac{2N_f}{Z}\int [\mathrm{d}\theta]\, e^{-S}\sum_{l=1}^{\infty}\sum_{i=1}^{N} l(l+1)\left[\frac{e^{\beta\mu}}{e^{\beta\mu}+e^{-i\theta_i+\beta(l+1/2)/R}}\right]$$

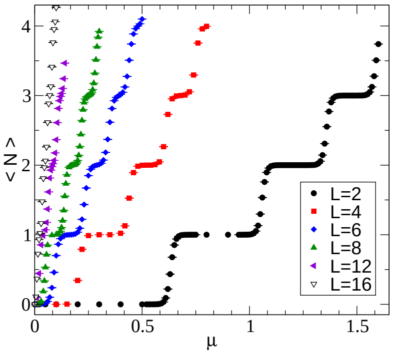
Each level L starts at $(\mu R)_0 = L + 1/2$ and has:

$$\text{height}: \quad h_L = N N_f \sum^{L} 2l(l+1); \qquad\qquad \text{width}: \quad w = \Delta(\mu R) = 1$$

Classical non-linear $O(2)$ sigma-model results from lattice (Banerjee and Chandrasekharan, arXiv:1001.3648)

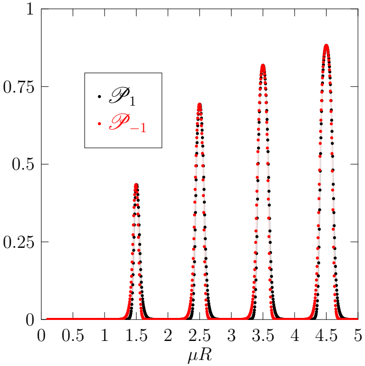


Average charge number



Levels appear to go away with increasing $L_s = L$, but this may not be true if L_t/L_s is kept fixed at a large value.

Polyakov lines: $\mathscr{P}_1 = \langle \mathrm{Tr} P \rangle$ and $\mathscr{P}_{-1} = \langle \mathrm{Tr} P^\dagger \rangle$ for $m = 0$

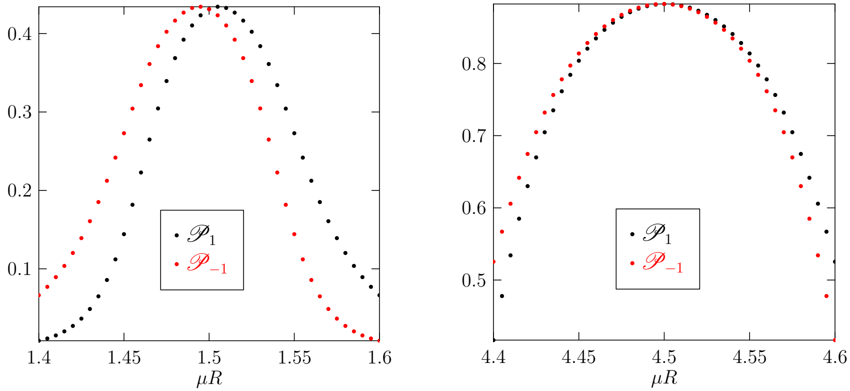


$$\mathscr{P}_1 \equiv \frac{\int [\mathrm{d}\theta] e^{-S} \sum_{i=1}^{N} e^{i\theta_i}}{Z},$$
$$\mathscr{P}_{-1} \equiv \frac{\int [\mathrm{d}\theta] e^{-S} \sum_{i=1}^{N} e^{-i\theta_i}}{Z}.$$

$\mathscr{P}_1 \neq \mathscr{P}_{-1}$ because the non-zero chemical potential led to $S_f \sim \sum_{n=1}^{\infty} \left[e^{n\beta\mu} e^{i\theta_i n} + e^{-n\beta\mu} e^{-i\theta_i n} \right]$

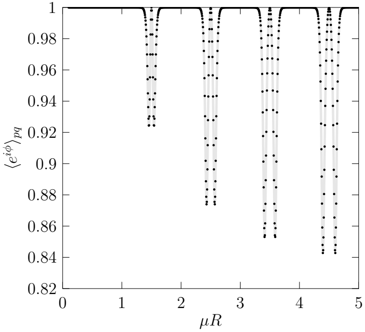
- There is a spike in $\mathscr{P}_1$ and $\mathscr{P}_{-1}$ corresponding to each jump in the average fermion number, $\mathscr{N}$.
- Deconfinement occurs in between the levels, as they are being filled.
- Given our notation, $\mathscr{P}_{-1}$ always preceeds $\mathscr{P}_1$ at each transition.

Polyakov lines: $\mathscr{P}_1$ and $\mathscr{P}_{-1}$ with increasing μ



- As μ increases the peaks of $\mathscr{P}_1$ and $\mathscr{P}_{-1}$ get wider indicating that the regions of deconfinement become larger with increasing μ.

Average phase $\langle e^{i\phi} \rangle_{pq}$ for $m = 0$



$$\langle e^{i\phi} \rangle_{pq} \equiv \frac{Z}{Z_{pq}},$$

where the denominator is the "phase quenched" (real action) partition function:

$$Z_{pq} = \int [\mathrm{d}\theta] \left| e^{-S} \right| = \int [\mathrm{d}\theta] e^{Re[-S]}$$

- The average phase shows where the sign problem is severe.
- $\langle e^{i\phi} \rangle_{pq}$ is smallest (largest) when $|\mathscr{P}_1 - \mathscr{P}_{-1}|$ is largest (smallest).

Large N theory at low T

In the large N limit the saddle point method is valid and it is possible to solve for several observables analytically. Considering a single level transition and performing the sum over n the action reduces to

$$S(\theta_i) = -\frac{1}{2}\sum_{i,j=1}^{N} \log \sin^2\left(\frac{\theta_i - \theta_j}{2}\right) + N\sum_{i=1}^{N} V(\theta_i)$$

$$V(\theta) = i\mathcal{N}\theta - \sigma \log\left(1 + \xi e^{i\theta}\right)$$

- $\mathcal{N}$ is a Lagrange multiplier necessary to satisfy the $\det P = 1$ constraint: $\sum_{i=1}^{N} \theta_i = 0$.
- $\sigma \equiv \sigma_l \equiv 2l(l+1)\frac{N}{N_f}$
- $\xi \equiv \exp\left(\beta(\mu - \varepsilon)\right)$
- $\varepsilon \equiv \varepsilon_l \equiv \sqrt{m^2 + (l+1/2)^2 R^{-2}}$

Equation of Motion

The saddle point solution is found by solving the equation of motion $\partial S/\partial\theta_i = 0$. This becomes:

$$i\mathcal{N} - \frac{i\sigma\xi e^{i\theta_i}}{1+\xi e^{i\theta_i}} = \frac{1}{N}\sum_{j(\neq i)} \cot\left(\frac{\theta_i-\theta_j}{2}\right)$$

Define the eigenvalues of the Polyakov line: $z_i = e^{i\theta_i}$. Then the equation of motion is

$$\mathcal{N} - \frac{\sigma\xi z_i}{1+\xi z_i} = \frac{1}{N}\sum_{j(\neq i)} \frac{z_i+z_j}{z_i-z_j}$$

The trace of the Polyakov line is

$$\mathscr{P}_n = \langle \mathrm{Tr} P^n \rangle = \frac{1}{N}\sum_{i=1}^{N} e^{in\theta_i}$$

$\Longrightarrow\ \mathscr{P}_{-n} \neq \mathscr{P}_n^*$ $(\langle \mathrm{Tr} P^{-n}\rangle \neq \langle \mathrm{Tr} P^n\rangle^*)$ for the saddle point solution.

Fermion number

Adding EOMs for all the θ_is we find that the Lagrange multiplier is

$$\mathcal{N} \xrightarrow[N\to\infty]{} \frac{1}{N}\sum_i \frac{\sigma\xi z_i}{1+\xi z_i} = \frac{T}{N^2}\frac{\partial \log Z}{\partial \mu}$$

which is the effective fermion number, $\mathcal{N} = \mathscr{N}/N^2$, valid in the large N limit.

Limits:

$$\begin{aligned} \text{As } \xi \to 0 \qquad & \mathcal{N} \to 0, \\ \text{As } \xi \to \infty \qquad & \mathcal{N} \to \sigma. \end{aligned}$$

This is in agreement with the $N = 3$ results for a single level transition.

Small ξ confined phase

As ξ (μ) increases from 0 the eigenvalues are continuously distributed along a closed contour $\mathcal{C}$ in the z-plane up to some critical value.

It is useful to consider a map between the theory on the unit circle and the theory in the complex z-plane of the Polyakov line eigenvalues. To this end

$$\frac{1}{N}\sum_i \longrightarrow \int_{-\pi}^{\pi}\frac{ds}{2\pi} = \oint_{\mathcal{C}}\frac{dz}{2\pi i}\varrho(z) \ ,$$

The contour is given by the inverse map $z(s)$, which can be obtained by solving the differential equation

$$i\frac{ds}{dz} = \varrho(z)$$

subject to the initial condition $z = e^{is}$ when $\xi = 0$.

Constraints

The distribution must satisfy the normalization condition

$$\oint_{\mathcal{C}} \frac{dz}{2\pi i} \varrho(z) = 1$$

and the $\det P = 1$ constraint

$$\int_{\mathcal{C}} \frac{dz}{2\pi i} \varrho(z) \log z = 0$$

EOM for eigenvalues on a closed contour

Using

$$\frac{1}{N}\sum_i \longrightarrow \int_{-\pi}^{\pi}\frac{ds}{2\pi} = \oint_{\mathcal{C}}\frac{dz}{2\pi i}\varrho(z)$$

we convert the EOM to an integral form

$$zV'(z) = \mathfrak{P}\oint_{\mathcal{C}}\frac{dz'}{2\pi i}\,\varrho(z')\frac{z+z'}{z-z'}\ , \qquad zV'(z) = \mathcal{N} - \frac{\sigma\xi z}{1+\xi z}\ .$$

where $\mathfrak{P}$ indicates principal value and the integral over the closed contour allows for evaluation of the right-hand side using Cauchy's theorem.

Distribution $\varrho(z)$ for eigenvalues on a closed contour $\mathcal{C}$

We start from the distribution $\varrho(z)$ with $z \equiv re^{i\phi}$ in the form of delta-functions:

$$\frac{1}{N}\sum_{j=1}^{N} = \int \mathrm{d}r\mathrm{d}\phi \frac{1}{N}\sum_{j=1}^{N}\delta(r-r_j)\delta(\phi-\phi_j)$$

$$= \oint \frac{\mathrm{d}z}{iz}\frac{1}{N}\sum_{j=1}^{N}\delta(\phi-\phi_j-i\log(r/r_j))$$

so

$$\varrho(z) = \frac{2\pi}{zN}\sum_{j=1}^{N}\delta(\phi-\phi_j-i\log(r/r_j))$$

then we can solve the EOM assuming using this most general form to obtain the form constrained by the potential:

$$\varrho(z) = \frac{c_1}{c_2 z} - \frac{\mathcal{N}}{c_2 z} + \frac{\sigma\xi/c_2}{1+\xi z} \propto V'(z)$$

Distribution $\varrho(z)$ in the small ξ confined phase

The small ξ confined phase with the pole $-\xi^{-1}$ outside. The EOM and the normalization condition give

$$\varrho(z) = \frac{1}{z} + \frac{\sigma\xi}{1+\xi z} \ .$$

with $\mathcal{N} = 0$ as expected.

Solving the differential equation $i\frac{ds}{dz} = \varrho(z)$ leads to

$$e^{is} = z(1+\xi z)^{\sigma}$$

which we invert to get the contour $z(s)$. The Polyakov lines are

$$\mathscr{P}_1 = \int_{\mathcal{C}} \frac{dz}{2\pi i}\varrho(z) z = 0 \ , \qquad \mathscr{P}_{-1} = \int_{\mathcal{C}} \frac{dz}{2\pi i}\varrho(z)\frac{1}{z} = \sigma\xi \ .$$

where $\mathscr{P}_{-1} \neq \mathscr{P}_1^*$ as advertised.

Extent of the small ξ confined phase

As ξ is increased the condition that the pole $-\xi^{-1}$ lies outside $\mathcal{C}$ must break down. Indeed, as ξ is increased there comes a point when $\varrho(z)$ vanishes, $z = -\frac{1}{\xi(1+\sigma)}$. This happens when

$$\xi = \xi_1 = \frac{\sigma^\sigma}{(1+\sigma)^{1+\sigma}}$$

and a gap opens up on the negative z-axis signaling a phase transition as in the matrix model of Gross and Witten [Phys. Rev. D **21** (1980) 446].

In terms of μ and ε the line of transitions in the $(\mu,\ T)$ plane is

$$\mu = \varepsilon - T\big[(1+\sigma)\log(1+\sigma) - \sigma\log\sigma\big]$$

valid in the low T limit.

Large ξ confined phase

The vanishing of the potential term in the action in the large ξ limit requires that the contour closes here too. The analysis is similar to that of the smal ξ confined phase and we find that

$$\varrho(z) = \frac{1+\sigma+\xi z}{z(1+\xi z)} \ ,$$

from the requirement that $-\xi^{-1}$ lies inside $\mathcal{C}$. This gives $\mathcal{N} = \sigma$ as expected and the level is occupied.

The Polyakov line expectation values are

$$\mathscr{P}_1 = \frac{\sigma}{\xi} \ , \qquad \mathscr{P}_{-1} = 0 \ .$$

where comparing with the small ξ confined phase $\mathscr{P}_{\pm 1}$ swaps over along with the replacement $\xi \to \xi^{-1}$.

Extent of the large ξ confined phase

As ξ is decreased the condition that the pole $-\xi^{-1}$ lies inside $\mathcal{C}$ must break down. Indeed, as ξ is decreased there comes a point when $\varrho(z)$ vanishes, $z = -\frac{1+\sigma}{\xi}$. This happens when

$$\xi = \xi_2 = \frac{(1+\sigma)^{1+\sigma}}{\sigma^\sigma} \ .$$

and a gap opens up again on the negative z-axis.

In terms of μ and ε the line of transitions in the $(\mu,\ T)$ plane is

$$\mu = \varepsilon + T\big[(1+\sigma)\log(1+\sigma) - \sigma\log\sigma\big]$$

valid in the low T limit.

The deconfined (open) phase: $\xi_1 \leq \xi \leq \xi_2$

In the deconfined phase the distribution has a gap and the eigenvalues lie on an arc $\mathcal{C}$ in the complex z-plane with endpoints $\tilde{z}$ and $\tilde{z}^*$. To solve for the case where the contour is open it is necessary to use the resolvent / spectral curve method. In anaolgy with the Gross-Witten-Wadia model the resolvent is

$$\omega(z) = -\frac{1}{N}\sum_j \frac{z+z_j}{z-z_j} = -\int_{\mathcal{C}} \frac{dz'}{2\pi i}\varrho(z')\frac{z+z'}{z-z'}$$

which is continuous everywhere except on the contour which which lies on a (square root) branch cut in the z-plane. It is clear that

$$\lim_{|z|\to 0}\omega(z) = 1 \ , \qquad \lim_{|z|\to\infty}\omega(z) = -1 \ .$$

We take the resolvent to be everywhere continuous except over the branch cut. Then from the Plemelj formulae the EOM is

$$zV'(z) = -\frac{1}{2}\left[\omega(z+\epsilon)+\omega(z-\epsilon)\right] \ , \qquad z \in \mathcal{C}$$

The distribution of the eigenvalues in the deconfined phase

The spectral density of eigenvalues is obtained from

$$z\varrho(z) = \frac{1}{2}\left[\omega(z+\epsilon) - \omega(z-\epsilon)\right] \ , \qquad z \in \mathcal{C} \ .$$

which implies that we can solve for various observables using an average of the form

$$\int_{\mathcal{C}} \frac{dz}{2\pi i} \varrho(z) F(z) = \oint_{\tilde{\mathcal{C}}} \frac{dz}{4\pi i z} \omega(z) F(z)$$

Following the technique of Wadia [EFI-79/44-Chicago] we solve the EOM for the resolvent and density of eigenvalues

$$\omega(z) = -zV'(z) + f(z)\sqrt{(z-\tilde{z})(z-\tilde{z}^*)} \, , \quad z\varrho(z) = f(z)\sqrt{(z-\tilde{z})(z-\tilde{z}^*)} \, .$$

where

$$f(z) = \frac{\sigma}{(1+\xi z)\left|\frac{1}{\xi} + \tilde{z}\right|},$$

$$\tilde{z} = \frac{-1}{\xi\left(1+\sigma-\mathcal{N}\right)^2}\left[\mathcal{N}^2 + 1 + \sigma - \mathcal{N}\sigma + 2i\sqrt{\mathcal{N}\left(\sigma-\mathcal{N}\right)\left(1+\sigma\right)}\right].$$

Fermion number and Polyakov lines

We impose the $SU(N)$ condition

$$\oint_{\tilde{\mathcal{C}}} \frac{dz}{4\pi i z}\,\omega(z)\log z = 0 \ .$$

to obtain the effective fermion number $\mathcal{N}$ from

$$\xi = \frac{(\sigma-\mathcal{N})^{\sigma-\mathcal{N}}(1+\mathcal{N})^{1+\mathcal{N}}}{\mathcal{N}^{\mathcal{N}}(1+\sigma-\mathcal{N})^{1+\sigma-\mathcal{N}}} \ .$$

The Polyakov lines are obtained from an expansion of the resolvent

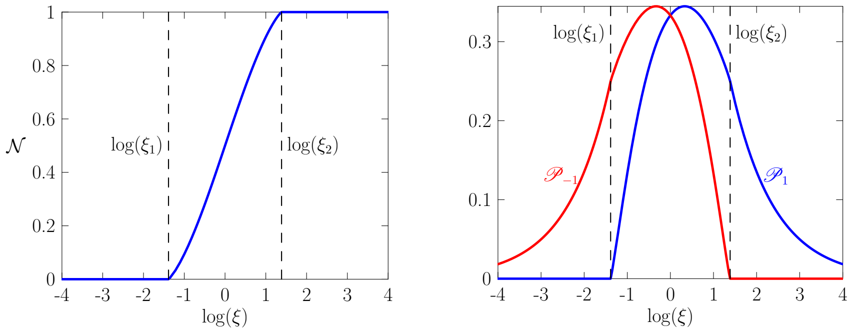
$$\omega(z) = -1 - 2\sum_{n=1}^{\infty}\frac{1}{z^n}\mathscr{P}_n$$

$$\omega(z) = 1 + 2\sum_{n=1}^{\infty} z^n \mathscr{P}_{-n}$$

For a single winding

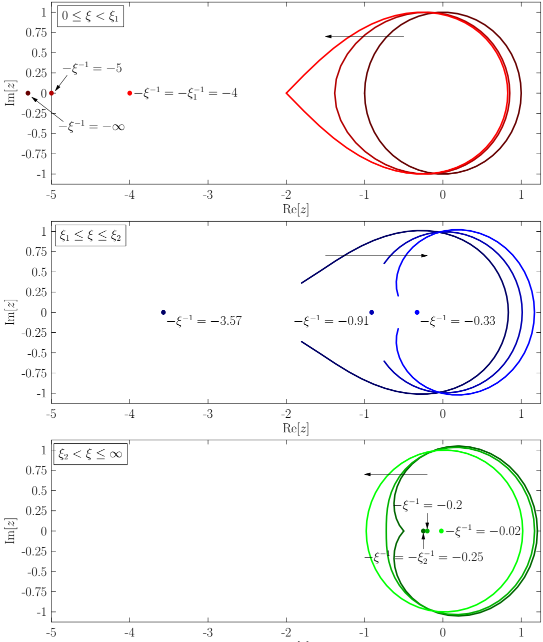
$$\mathscr{P}_1 = \frac{\mathcal{N}}{\sigma+1-\mathcal{N}}\frac{1}{\xi} \ , \qquad \mathscr{P}_{-1} = \frac{\sigma-\mathcal{N}}{1+\mathcal{N}}\xi$$

Large N theory at low T



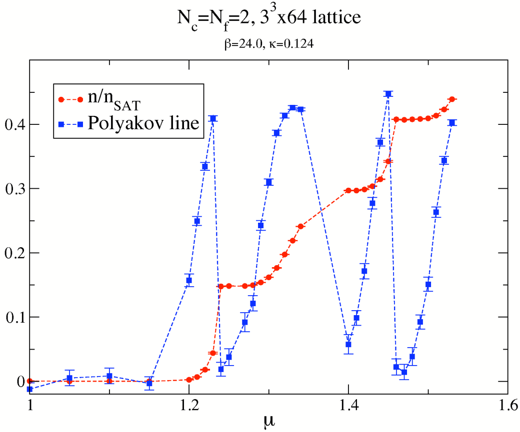
- The discontinuities in the effective fermion number and the Polyakov lines mark the third-order Gross-Witten-Wadia transitions.

Distribution in the deconfined phase



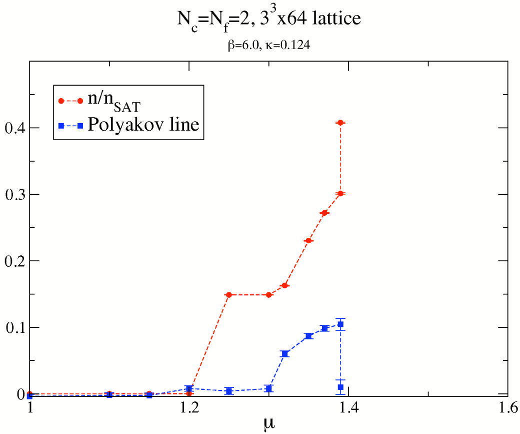
- The contour $\mathcal{C}$, which gives the distribution of the eigenvalues of the Polyakov line, showing the transition from the small ξ closed phase (in red), the open phase (in blue) and the large ξ closed phase (green).

Preliminary lattice results from 2-color QCD



Simulation results for $N = 2$ QCD confirm the level structure of the fermion number and the associated spikes in the Polyakov line at each level transition. The curious smooth $\rightarrow$ sharp feature of the observables at the transitions needs study to determine if it is a result of larger coupling, or perhaps resulting from working on the 4-torus.

Strong coupling lattice results from 2-color QCD



Simulation results from $N = 2$ QCD considering larger coupling strength (smaller $\beta = 2N/g^2$) show that the spikes in the Polyakov line are scaled down. A small spike in $\mathscr{P}_1$ is expected around $\mu = 1.2$ but more data is needed to determine if it is there.

Conclusions

- QCD at finite chemical potential on $S^1 \times S^3$ has a complex action which results the stationary solution lying in the configuration space of complexified gauge fields.
- Expectation values for observables can be obtained at finite N by numerically integrating over the gauge fields.
- Observables and the distribution of the gauge field eigenvalues can be calculated analytically in the large N limit using the saddle point method of Gross, Witten, and Wadia generalized to deal with a complex action.
- For small mR, the fermion number as a function of the chemical potential suggests a level-structure where the level transitions correspond to spikes in the Polyakov line.
- For large mR, a continuum limit is obtained and the observables exhibit the "Silver blaze" feature, remaining zero until onset is reached at $\mu = m$. The confinement-deconfinement transitions return for sufficiently large μ.

Outlook

- Add more flavors and look for color-superconducting phases through calculation of observables like $\psi\psi$, $(\bar{\psi}\psi)^2$
- Make a connection with Complex Langevin which is a non-perturbative technique
- Consider higher-loop corrections and go beyond the Gaussian approximation to obtain effects from increased coupling strength
- Formulate a related theory from the gravity side (eg. $\mathcal{N} = 4$ SYM + fundamental flavor branes and chemical potential)
- Calculate the phase diagram for imaginary chemical potential and compare with lattice simulations.

Continuum results (large mR)

Since all of our observables are a function only of β/R, mR, or μR, then we can obtain a continuum limit by taking:

- β/R small (high T perturbation theory),
- μR large (high density perturbation theory),
- mR large (heavy quarks).

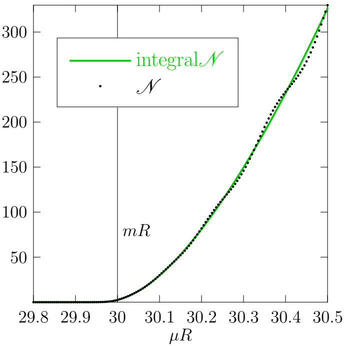
We take mR large. Then, in the vicinity of $\mu = m$:

$$
\begin{aligned}
z_f(n\beta/R, mR) &= 2\sum_{l=0}^{\infty} l(l+1) e^{-n\beta\sqrt{(l+1/2)^2 R^{-2}+m^2}} \\
&= 2\int_0^\infty \mathrm{d}y\, \left(y^2 - \frac{1}{4}\right) e^{-\frac{n\beta}{R}\sqrt{y^2+m^2R^2}} \\
&\qquad + 4\int_{mR}^\infty dy\, \frac{y^2+\frac{1}{4}}{e^{2\pi y}+1} \sin\left(\frac{n\beta}{R}\sqrt{y^2-m^2R^2}\right)
\end{aligned}
$$

$$
\xrightarrow[mR\to\infty]{} 2\int_0^\infty \mathrm{d}y\, \left(y^2 - \frac{1}{4}\right) e^{-n(\beta/R)\sqrt{y^2+m^2R^2}}
$$

$\mathcal{N}$ for $m \to \infty$

- For non-zero quark mass the expectation value $\mathcal{N}$ exhibits "Silver Blaze" behavior: Bulk observables are zero until onset.
- Onset occurs at the mass of the lightest particle $\mu \simeq m$.



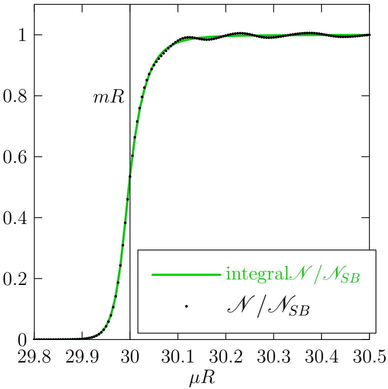
$$\mathcal{N} \xrightarrow[\beta\to\infty]{} \frac{2N_f}{Z}\int [\mathrm{d}\theta]\, e^{-S}\int_0^\infty \mathrm{d}y(y^2-1/4)\sum_{i=1}^{N}\left[\frac{e^{\beta\mu}}{e^{\beta\mu}+e^{-i\theta_i+(\beta/R)\sqrt{y^2+m^2R^2}}}\right]$$

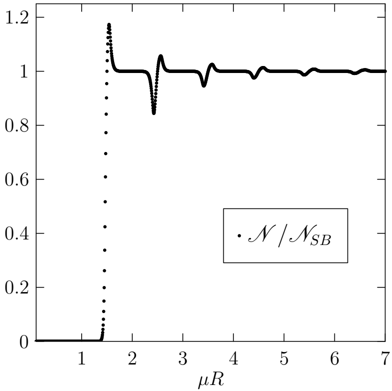
Each level L has:

$$\text{height}: \quad h_L = NN_f\sum_{l=1}^{L}2l(l+1) \to NN_f\int_0^L \mathrm{d}y\, 2(y^2-1/4)$$

$$\text{width}: \quad \Delta(\mu R) \to \left(\sqrt{(y+\mathrm{d}y)^2+m^2R^2}-\sqrt{y^2+m^2R^2}\right) \to 0$$

$\mathcal{N}$, $\mathcal{P}$, E approach the Stefan-Boltzmann limit

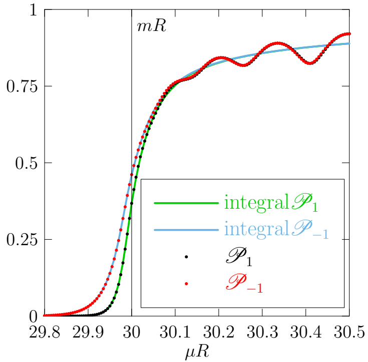




- The Stefan-Boltzmann limit is the zero interaction free fermion limit. On $S^1 \times S^3$ we obtain it from the one-loop result taking all the $\theta_i = 0$, corresponding to the "deconfined" phase, e. g.

$$\mathcal{N}_{SB} \xrightarrow[\beta\to\infty]{} 2NN_f \int_0^\infty \mathrm{d}y(y^2 - 1/4) \left[\frac{e^{\beta\mu}}{e^{\beta\mu} + e^{(\beta/R)\sqrt{y^2+m^2R^2}}} \right]$$

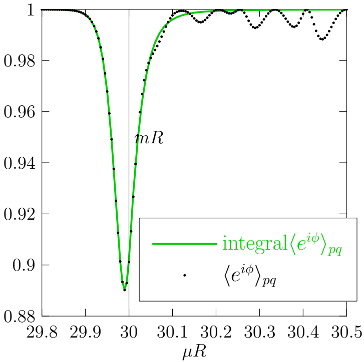
Polyakov liness: $\mathscr{P}_1$ and $\mathscr{P}_{-1}$ for $m \to \infty$



$\mathscr{P}_1$ and $\mathscr{P}_{-1}$ as a function of chemical potential for large quark mass near onset at $\mu R = mR = 30$. $N = 3$, $N_f = 1$ $\beta/R = 30$ (low T).

- At low but non-zero temperatures the confinement-deconfinement oscillations can be delayed by taking $mR \to \infty$.
- The transition in μR occurs around onset at mR and becomes sharper with increasing mR.
- The integral approximation to z_f (curves) breaks down shortly after the onset transition and the oscillations return. The larger we take mR, the farther in μR we can go before breakdown.

Average phase $\langle e^{i\phi} \rangle_{pq}$



- In the limit of large mR, spike in the average phase as a function of μR marks the onset transition. This is followed by a brief respite from large phase fluctuations.
- Again we find that $\langle e^{i\phi} \rangle_{pq}$ is smallest (largest) when $|\mathscr{P}_1 - \mathscr{P}_{-1}|$ is largest (smallest).

Successful techniques that deal with or avoid the sign problem

Lattice techniques valid for $\mu/T < 1$

- Taylor expansion
- Reweighting
- Imaginary μ + analytic continuation

Infinite volume perturbation theory

- chiral perturbation theory
- large μ perturbation theory

Using models

- 2-color QCD
- Random Matrix Theory
- Nambu-Jona-Lasinio Models
- AdS/CFT

Other (New)

- Complex Langevin
- Finite spatial volume perturbation theory (this talk)