Boundary conditions and consistency of effective theories

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Outline

Consistency of Effective Theories - Motivation

Effective theories :

$\Rightarrow$ elimination of degrees of freedom - heavy particles

Consequences

- long range correlations
  $\Rightarrow$ higher order derivative terms in the effective action
- low energy - truncation of the gradient expansion
  Two issues :
  $\Rightarrow$ specification of states - boundary conditions
  $\Rightarrow$ unitarity of the effective theory

Real Scalar Field - Example

Scalar field governed by the action

$$S[\phi] = \int dx \left[\phi \left(\sum_{n=0}^{n_d} c_n \Box^n\right) \phi(x) - V(\phi(x))\right]$$

- time reversal invariant model
- coefficients c_n and potential $V(\phi)$ real and $(-1)^{n_d} c_{n_d} > 0$

Free propagator in momentum space

$$D(p) = \left(\sum_{n=0}^{n_d} (-1)^n c_n (p^2)^n\right)^{-1} = \sum_{j=1}^{n_d} \frac{Z_j}{p^2 - m_j^2}$$

$\Rightarrow$ at least one negative Z factor

- negative norm states

Linear Space with Indefinite Norm

Linear space H with non-definite metric

1. $\langle u|v\rangle = \langle v|u\rangle^*$
2. $\langle u|(a|v\rangle + b|w\rangle) = a\langle u|v\rangle + b\langle u|w\rangle$
3. $H = H_+ + H_-$ where $H_\pm = \{|u\rangle | \langle u|u\rangle \gtrless 0\}$ and $\langle H_+|H_-\rangle = 0$
4. $|u\rangle = |u_+\rangle + |u_-\rangle$, $\langle u_\pm|u_\pm\rangle \gtrless 0$

- basis $\{|n\rangle\}$, non-definite metric $\eta_{mn} = \langle m|n\rangle$ where $\eta^\dagger = \eta$
- matrix elements A_{jk} of an operator A defined by $\langle m|A|n\rangle = \sum_k \eta_{mk} A_{kn}$

Self-adjoint and Skew-adjoint Operators

$\Rightarrow$ adjoint $\bar{A}$ and Hermitian adjoint $A^\dagger$

$$\langle u|\bar{A}|v\rangle = \langle v|A|u\rangle^* \text{ so } \bar{A} = \eta^{-1}A^\dagger\eta \neq A^\dagger$$

- Condition : $\bar{A} = \sigma_A A$

  $\Rightarrow$ valid for self-adjoint operators, $\sigma_A = +1$, and skew-adjoint operators, $\sigma_A = -1$.

- Eigenvectors : $A|\lambda\rangle = \lambda|\lambda\rangle$, $A|\rho\rangle = \rho|\rho\rangle$

  $\Rightarrow$ relation for the spectrum

$$(\lambda - \sigma_A\rho^*)\langle\rho|\lambda\rangle = 0$$

Free Particle Dynamics

Canonical pair of operators $\hat{q}_\sigma$ and $\hat{p}_\sigma$

$\Rightarrow$ either self- or skew-adjoint

$\Rightarrow$ commutation relation $[\hat{q}_\sigma, \hat{p}_\sigma] = i$

Real spectrum

$\Rightarrow \eta(q, q') = \delta(q - \sigma q')$ and $\eta(p, p') = \delta(p - \sigma p')$

Closing relations in coordinate and momentum space

$\mathbb{1} = \int dq |\sigma q\rangle\langle q| = \int dp |\sigma p\rangle\langle p|$

Hamiltonian of harmonic oscillator

$\hat{H}_\sigma = \frac{\sigma}{2}(\hat{p}_\sigma^2 + \hat{q}_\sigma^2) = \sigma \bar{a}_\sigma a_\sigma$

where operator $a_\sigma = (\hat{q}_\sigma + i\hat{p}_\sigma)/\sqrt{2}$ and $[a_\sigma, \bar{a}_\sigma] = \sigma$

Bounded Hamiltonian Conditions

Case of $\sigma = +1$ (for $\sigma = -1$ $a \leftrightarrow \bar{a}$)

- operators $b = a_+$, $\bar{b} = \bar{a}_+$
- eigenstate of self-adjoint operator $\bar{b}b$: $\bar{b}b|\lambda\rangle = \lambda|\lambda\rangle$
- double infinite series of states
  $\cdots, b^2|\lambda\rangle, b|\lambda\rangle, |\lambda\rangle, \bar{b}|\lambda\rangle, \bar{b}^2|\lambda\rangle, \cdots$
  with corresponding eigenvalues
  $\cdots, \lambda - 2, \lambda - 1, \lambda, \lambda + 1, \lambda + 2, \cdots$ of $\bar{b}b$

Bounded Hamiltonian

Series limited on the left or on the right $\rightarrow$ λ integer

$\langle\lambda|\bar{b}b|\lambda\rangle = \lambda\langle\lambda|\lambda\rangle$ and $\langle\lambda|b\bar{b}|\lambda\rangle = (\lambda + 1)\langle\lambda|\lambda\rangle$

$\Rightarrow$ either $\lambda \geq 0$ or $\lambda \leq -1$

$\Rightarrow$ either $\mathrm{sign}(\langle\lambda + 1|\lambda + 1\rangle) = \mathrm{sign}(\langle\lambda|\lambda\rangle)$

or $\mathrm{sign}(\langle\lambda - 1|\lambda - 1\rangle) = -\mathrm{sign}(\langle\lambda|\lambda\rangle)$

Time Reversal Properties of Operators

Time reversal Θ (way to trace negative norm states)

$\sigma = +1,\ \Theta : \hat{q} \to \hat{q},\ \ \hat{p} \to -\hat{p},\ \ c \to c^*$

$\hat{q}$-type operators

- operators with well-defined time reversal parity : $\Theta A = \tau_A A$, with $\tau_A = \pm 1$
- Heisenberg representation $\Theta : A(t) \to \bar{A}(-t)$
  $\Rightarrow$ for the time derivative $\tau_{\partial_0 A} = -\tau_A$
  $\Rightarrow$ self- or skew-adjoint operators $\Theta : A \to \bar{A} = \sigma_A A$

Time parity and the signature of the linear space $\Rightarrow$

$$\tau_A = \sigma_A$$

Yang-Mills-Higgs Model in Euclidean Space-time

Model with higher order derivatives
$\Rightarrow$ Yang-Mills and scalar fields, imaginary time

$$S[\phi, \phi^\dagger, A] = \int d^d x \left[K(D) - \phi^\dagger L(D^2) D^2 \phi + V(\phi^\dagger \phi) \right]$$

- $D_\mu = \partial_\mu - iA_\mu$ - covariant derivative
- $A_\mu = A_\mu^a \tau^a$ - gauge field
- τ^a - generators of the gauge group
- K, L - bounded from below, polynomials of D of order at most $2n_d$ and $2n_d - 2$, respectively
- $V(\phi^\dagger \phi)$ - scalar field potential

Treatment of Higher Orders of Derivatives

Appearance of higher order time derivatives

$\Rightarrow$ Solution in classical systems :

- introduction of new coordinates for each higher order derivative except the last one

  $A_{j\mu}(x) = D_0^j A_\mu(x)$ and $\phi_j(x) = \partial_0^j \phi(x)$

  for $j = 0, \ldots, n_d - 1$
- time reversal parity of the coordinates

  $A_{j\mu}(x) : \tau = (-1)^{j+\delta_{\mu,0}}$ and $\phi_j(x) : \tau = (-1)^j$
- applicable also in case of quantum fields by means of path integral for $A_{j\mu}(x)$ and $\phi_j(x)$

Theory on the Lattice

Aim

Finding fields with Green functions satisfying Wightman's axioms in real time

- reflection positivity $\Rightarrow$ lattice regularization - positivity of the transfer matrix in imaginary time

Lattice

- fields : $\phi(n) = a\phi(x)$, $\phi^\dagger(n) = a\phi^\dagger(x)$,
  $U_\mu(n) = U^\dagger_{-\mu}(n+\hat{\mu}) = e^{igaA_\mu(n)}$,
  a - lattice spacing, $\hat{\mu}$ - unit vector in direction μ
- gauge transformation : $\phi(n) \rightarrow \omega(n)\psi(n)$,
  $\phi^\dagger(n) \rightarrow \omega^\dagger(n)\psi^\dagger(n)$, $U_\mu(n) \rightarrow \omega(n+\hat{\mu})U_\mu(n)\omega^\dagger(n)$

Partition Function of the Model

Partition function

$Z = \int D[U]D[\phi^\dagger]D[\phi]e^{-S_L}$

Lattice action

$$S_L = \sum_n \sum_{\gamma'} a_{\gamma'} \mathrm{tr} U_{\gamma'}(n) + \sum_n \phi^\dagger(n) \sum_\gamma U_\gamma^\dagger(n)\phi(n+\gamma)b_\gamma + \sum_n V(\phi^\dagger(n)\phi(n))$$

γ', γ - closed and open paths respectively, up to n_d
$U_\gamma(n)$ - the path ordered product of the link variables along this path

Action S_L real
$\Rightarrow$ for each γ, $\Theta\gamma$ included in the sums with $a_{\Theta\gamma'} = a^*_{\gamma'}$, $b_{\Theta\gamma} = b^*_\gamma$

Identification of Lattice Field Varibles

Construction of n_d lattice field variables

- regroupment of n_d consecutive lattice sites in the time direction with their field variables into a single n_d-component field
  $\Rightarrow$ a single blocked time slice
- fields :
  $\phi_j(t, \mathbf{n}) = \phi(n_d t + j, \mathbf{n})$,
  $U_{j,\mu}(t, \mathbf{n}) = U_\mu(n_d t + j, \mathbf{n})$
  with t integer, $j = 1, \cdots, n_d$

Lattice Action in Terms of New Variables

Lattice action

$S_L = \sum_t [L_s(t) + L_{km}(t) + L_{kg}(t)]$

where :

$L_s(t) = S_s[U(t), \phi^\dagger(t), \phi(t)]$

$L_{kg}(t) = S_{kg}[U(t), U(t+1)]$

$$L_{km}(t) = \sum_{t,\mathbf{m},\mathbf{n}} \phi_j^\dagger(t+1,\mathbf{m})\Delta_{j,k}(\mathbf{m},\mathbf{n};U(t),U(t+1))\phi_k(t,\mathbf{n}) + c.c.$$

Positivity of transfer matrix condition

For any functional $\mathcal{F}$ of physical fields for positive t

$$\langle 0|\mathcal{F}\Theta[\mathcal{F}]|0\rangle \geq 0$$

Time Reversal of Field Variables

Time reversal of the functional $\mathcal{F}$

$\Theta\mathcal{F}[\phi, \phi^\dagger, U] = \mathcal{F}[\Theta\phi, \Theta\phi^\dagger, \Theta U]$

with fields

$$
\begin{aligned}
\Theta\phi_j(n) &= \phi^\dagger_{\Theta j}(\Theta n), \\
\Theta\phi^\dagger_j(n) &= \phi_{\Theta j}(\Theta n), \\
\Theta U_{j,\mu}(n) &= \begin{cases} U^\dagger_{\Theta j,\mu}(\Theta n) & \mu = 1, 2, 3, \\ U_{\Theta j-1,\mu}(\Theta n) & \mu = 0, j < n_d, \\ U_{j,\mu}(\Theta n - \hat{0}) & \mu = 0, j = n_d, \end{cases}
\end{aligned}
$$

and space-time coordinate $n = (t, \mathbf{n})$

$$\Theta(t, \mathbf{n}) = (-t, \mathbf{n})$$

$\Rightarrow$ site-inversion realization of time reversal $t \to -t$

Functionals with Well Defined Time Inversion Parity

Functionals of the type

$$\Theta\mathcal{F}[\phi(n),\phi^\dagger(n),U(n)]=\tau_{\mathcal{F}}\mathcal{F}[\phi(\Theta n),\phi^\dagger(\Theta n),U(\Theta n)]$$

Combinations of local fields satisfying :

$$\begin{aligned}
\phi_{\tau,j}(t,\mathbf{n}) &= \frac{1}{2}[\phi_j(t,\mathbf{n})+\tau\phi^\dagger_{\Theta j}(t,\mathbf{n})],\\
\phi^\dagger_{\tau,j}(t,\mathbf{n}) &= \frac{1}{2}[\phi^\dagger_j(t,\mathbf{n})+\tau\phi_{\Theta j}(t,\mathbf{n})],\\
U_{\tau,j,\mu}(t,\mathbf{n}) &= \frac{1}{2}[U_{j,\mu}(t,\mathbf{n})+\tau U^\dagger_{\Theta j,\mu}(t,\mathbf{n})], \quad \mu=1,2,3
\end{aligned}$$

with $j=1,\ldots,(n_d-1)/2$

Gauge Invariance

Gauge fixing

- gauge transformation

  $$\omega(n_d t+j,\mathbf{n}) = [U_0(n_d t+j-1,\mathbf{n})\cdots U_0(n_d t+1,\mathbf{n})]^\dagger$$

  for $2 \leq j \leq n_d$ (on the original lattice, with simple time slices)
- cancellation of the time component of the gauge field within the block time slices and $U_{j,0}(t,\mathbf{n}) \to \mathbb{1}$ for $j < n_d$

Proof of Positivity of Transfer Matrix

Lattice action split into three pieces

$$S_L = S_0 + S_+ + S_-$$

Time reversal invariance of microscopic dynammics

$$S_\pm[\Psi(t)] = \Theta[S_\mp[\Psi(t)]] = S_\mp[\Psi(\Theta t)]$$

$$S_0[\Psi] = S_0[\Psi(\Theta t)]$$

Introducing $\Psi = (U, \phi^\dagger, \phi)$

$$\langle 0|\mathcal{F}\Theta\mathcal{F}|0\rangle = \int D[\Psi] e^{-S_0[\Psi]} e^{-S_+[\Psi]} \mathcal{F}[\Psi] e^{-S_-[\Psi]} \Theta[\mathcal{F}[\Psi]]$$

Proof of Positivity of Transfer Matrix - Continued

$\Rightarrow$ Time reversal invariance of the vacuum state $\Theta|0\rangle = |0\rangle$

$$\langle 0|\mathcal{F}\Theta\mathcal{F}|0\rangle = \int D[\Psi]e^{-S_0[\Psi]}e^{-S_+[\Psi]}\mathcal{F}[\Psi]\Theta\left[e^{-S_+[\Psi]}\mathcal{F}[\Psi]\right]$$

$$= \int D_{t=0}[\Psi]\int D_{t>0}[\Psi]e^{-\frac{S_0}{2}-S_+}\mathcal{F}\Theta\mathcal{F}\int D_{\Theta t>0}[\Psi]e^{-\frac{S_0}{2}-S_+}$$

Assuming $\tau_{\mathcal{F}}$ - time-reversal parity of $\mathcal{F}[\Psi]$

$$\langle 0|\mathcal{F}\Theta\mathcal{F}|0\rangle = \tau_{\mathcal{F}}\left(\int D_{t\geq 0}[\Psi]e^{-\frac{1}{2}S_0[\Psi]-S_+[\Psi]}\mathcal{F}[\Psi]\right)^2$$

For $\tau_{\mathcal{F}} = 1$

$$\langle 0|\mathcal{F}\Theta\mathcal{F}|0\rangle \geq 0$$

Boundary Conditions

Positivity of the transfer matrix

$\Rightarrow$ equations valid for each trajectory in the path integral

Boundary conditions :

$$\Psi(t_f) = \tau_\Psi \Psi(t_i)$$

- periodic (antiperiodic) boundary conditions for time-reversal even (odd) variables
- generalized KMS conditions

Example :

$$\phi_j(x) = \partial_0^j \phi(x), \;\; \tau = (-1)^j$$
$$\phi_j(t_f, \mathbf{x}) = (-1)^j \phi_j(t_i, \mathbf{x})$$

Conclusion

- Truncation of the gradient expansion of the effective theory (after elimination of heavy particle modes) $\Rightarrow$ question of unitarity
- Reflection positivity demonstrated for the Yang-Mills-Higgs model with higher order derivatives, within the Fock space span by local, time-reversal invariant functionals of the fields acting on the time-reversal invariant vacuum
- Restriction : generalized KMS boundary conditions

Thank you for your attention !