

Aspects of the Functional Renormalisation Group

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We discuss structural aspects of the functional renormalisation group. Flows for a general class of correlation functions are derived, and it is shown how symmetry relations of the underlying theory are lifted to the regularised theory. A simple equation for the flow of these relations is provided. The setting includes general flows in the presence of composite operators and their relation to standard flows, an important example being NPI quantities. We discuss optimisation and derive a functional optimisation criterion.

Applications deal with the interrelation between functional flows and the quantum equations of motion, general Dyson-Schwinger equations. We discuss the combined use of these functional equations as well as outlining the construction of practical renormalisation schemes, also valid in the presence of composite operators. Furthermore, the formalism is used to derive various representations of modified symmetry relations in gauge theories, as well as to discuss gauge-invariant flows. We close with the construction and analysis of truncation schemes in view of practical optimisation.

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Contents			
		2. Optimisation and effective cut-off scale	23
		3. Optimisation criterion	25
I. Introduction	1		
II. Preliminaries	2	VI. Applications to functional methods	27
III. Flows	5	A. Functional RG and DS equations	28
A. Setting	5	1. DSEs as integrated flows	28
B. One-parameter flows	7	2. Renormalisation	28
1. Derivation	7	B. Composite operators and NPI flows	29
2. Flow of the Schwinger functional	9	1. Linear flows	30
3. Standard flow	10	2. 2PI flows	31
4. Flow of amputated correlation functions	10	3. Renormalisation	32
C. Flows in terms of mean fields	10	VII. Applications to gauge theories	34
1. Derivation	10	A. Parameterisation	34
2. Flow of the effective action	12	B. Modified Slavnov-Taylor identities	35
3. Standard flow	13	1. STI	36
4. Initial condition for general flows	14	2. mSTI	37
D. General variations	15	3. Flows and alternative representations	39
IV. Renormalisation group flows	15	C. Gauge-invariant flows	41
A. RG flows of general correlation functions	15	1. Background field flows	41
B. RG flows in terms of mean fields	17	2. Geometrical effective action	43
V. Optimisation	18	D. Chiral symmetry and anomalies	44
A. Setting	18	VIII. Truncation schemes and optimisation	45
B. Principle of Minimum Sensitivity	20	A. Field reparameterisations	45
C. Stability criterion	21	B. RG scaling and optimisation	46
D. Functional optimisation	22	C. Integrated flows and fixed points	47
1. Local optimisation	22	D. Optimisation in LPA	48
		E. Optimisation in general truncation schemes	50

IX. Conclusions	51
APPENDIX	52
A. Metric	52
B. Derivatives	53
C. Definition of ΔS_n	53
D. Standard 1PI flows	53
References	54

I. INTRODUCTION

The Functional Renormalisation Group (FRG) in its continuum formulation [1–14] has proven itself as a powerful tool for studying both perturbative and non-perturbative effects in quantum field theory and statistical physics, for reviews see [16–27, 29]. In this approach a regularisation of a quantum theory is achieved by suppressing part of the propagating degrees of freedom related to a cut-off scale k . This results in regularised generating functionals such as the effective action Γ_k where part of the modes have been integrated out. The flow equation describes the response of the generating functional to an infinitesimal variation of k , and can be used to successively integrate-out modes. Hence, a generating functional at some initial scale Λ together with its flow serve as a definition of the quantum theory. For example, the flow equation allows us to calculate the full effective action Γ from an initial effective action Γ_Λ if the latter is well under control. For an infrared momentum cut-off and sufficiently large Λ we have a good grip on Γ_Λ as it can be computed perturbatively.

The main advantages of such a formulation are its flexibility when it comes to truncations of the full theory, as well as its numerical accessibility. Both properties originate in the same structural aspects of such flows. Quite generally functional flows are differential equations that relate an infinitesimal k -variation of a generating functional Z with some functional of Z , its derivatives and the regulator. The quantum theory, and hence the physics, is solely specified by the boundary condition of such a flow. Due to this structure truncations are introduced on the level of the generating functional itself which leads to self-consistent truncated flows. Moreover, a change of degrees of freedom also is done on the level of the generating functional, and the structure of the flow stays the same. Last but not least, numerical stability of the flow for a given problem and truncation is governed by the choice of the specific regularisation procedure.

In other words, the advantages are carried by the structural aspects of the functional RG, whose understanding and further development is the main purpose of the present work. It is not meant as a review and for a more complete list of references we refer the reader to the reviews already cited above, [16–27]. We close the introduction with an overview over the work.

In section II we evaluate functional equations of quantum field theories, such as Dyson-Schwinger equations,

symmetry identities, such as Slavnov-Taylor identities (STIs), and introduce some notation.

In section III flows are derived for general correlation functions including those for the effective action and the Schwinger functional. We present a derivation of the flow equation which emphasises the subtleties of renormalisation. Moreover, no use of the path integral representation is made, the derivation solely relies on the existence of a finite effective action or Schwinger functional for the full theory. First we introduce the setting and notion of regularisation. This is used to derive the general flows (3.28) and (3.60) which comprise the main results of this part. The flows discussed here include those for N -particle irreducible (NPI) quantities as well as relations between the different formulations. For general flows one has to carefully study the boundary conditions. A comparison of results obtained for different regularisations, in particular in view of optimisation, requires the study of variations of the regulator.

In section IV we discuss the fate of RG equations of the full theory displaying reparameterisation invariance in the presence of a general regularisation. This is important when matching the scale dependence of quantities in the presence of the regularisation to that in the full theory without cut-off. The key RG flows are (4.8),(4.20) and are basically generalisations of (3.28) and (3.60).

The important aspect of optimisation is investigated in section V. In most situations one has to rely on truncations to the full theory. Optimised flows should lead to results as close as possible to the full theory within each order of a given systematic truncation scheme. We develop a functional approach to optimisation of general flows which allows us to systematically access and develop optimisation criteria. We discuss the relation between different optimisation ideas used in the literature. The definition of an effective cut-off scale is introduced and a constructive optimisation criterion is put forward in section V D. Roughly speaking, optimal regulators are those, that lead to correlation functions as close as possible to that in the full theory for a given effective cut-off scale.

The rest of the present paper deals with structural applications of these findings. In section VI we relate flows to other functional methods such as Dyson-Schwinger equations or the use of NPI effective actions. To that end we consider flows in the presence of composite operators. In particular we construct practical renormalisation schemes, the latter being of importance for the

renormalisation of Dyson-Schwinger equations and NPI effective actions.

A main motivation for the development of the present approach resides in its application to gauge theories. In section VII various structural aspects of gauge theories are investigated. We discuss the formulation of gauge theories using appropriate degrees of freedom. The modification of symmetry identities in the presence of the regularisation and their different representations are evaluated. The latter allow for a purely algebraic representation of the symmetry identities. We also outline the construction of gauge-invariant flows and discuss the fate of gauge symmetry constraints in these formulations. We close with a brief evaluation of anomalous symmetries in the presence of a regulator.

In section VIII we discuss consequences of the functional optimisation criterion and the RG equations for the construction of truncation schemes and optimal regulators. It is shown that a specific class of regulators preserves the RG scalings of the underlying theory. We discuss the use of integrated flows that constitute finite renormalised Dyson-Schwinger equations. These integrated flows can be used in asymptotic regimes or a fixed point analysis within the functional RG setting. The constructive optimisation criterion developed in section V is put to work within a simple example. Further applications are outlined.

II. PRELIMINARIES

We consider the *finite renormalised* Euclidean Schwinger functional $W[J]$ of the theory under investigation, where we do not only allow for source terms for the fundamental fields $\hat{\varphi}$ of the theory, but also for sources for general tensorial composite operators $\hat{\phi}(\hat{\varphi})$ with

$$e^{W[J]} = \int d\mu[\hat{\varphi}] \exp\left\{-S[\hat{\varphi}] + \sum_{n=1}^{n_{\max}} \int_{x_1, \dots, x_n} J^{\alpha_1 \dots \alpha_n}(x_1, \dots, x_n) \hat{\phi}_{\alpha_1 \dots \alpha_n}[\hat{\varphi}](x_1, \dots, x_n)\right\}. \quad (2.1)$$

Here α_i comprises possible Lorentz and gauge group indices and species of fields. The measure $d\mu[\hat{\varphi}]$ ensures the finiteness of the Schwinger functional and hence depends on some renormalisation scale μ , as well as on $S[\hat{\varphi}]$. For the sake of simplicity, and for emphasising the structure of the results, we use a condensed notation with indices a, b that stand for an integration over space-time and a

summation over internal indices:

$$J^a \hat{\phi}_a = \int d^d x J^\alpha(x) \hat{\phi}_\alpha(x), \quad (2.2)$$

In (2.2) we have implicitly defined the ultra-local metric $\gamma^{a a'} = \delta(x - x') \gamma^{\alpha \alpha'}$, leaving the internal part $\gamma^{\alpha \alpha'}$ undetermined. In case $\hat{\phi}_a$ involves fermionic variables we have $J^a \hat{\phi}_a \neq \hat{\phi}_a J^a$. The notation as well as some properties of the metric γ^{ab} are detailed in appendix A. In the general case (2.1) we consider the coupling of N tensorial fields with rank $n_i \leq n_{i+1}$ to the theory. We substitute indices a by multi-indices $\mathbf{a} = a_{11} \dots a_{1n_1}, \dots, a_{N1} \dots a_{Nn_N}$ with $n_N = n_{\max}$. In the general case, different a_{ij} can carry different internal indices, e.g. different representations of a gauge group relating to different species of fields. This is implicitly understood and we identify $a_{ij} = a_j$ from now on in a slight abuse of notation. Contractions read

$$T_1^{\mathbf{a}} T_2^{\mathbf{a}} = \sum_{i=1}^N T_1^{a_1 \dots a_{n_i}} T_2^{a_1 \dots a_{n_i}}, \quad (2.3)$$

and the generalised metric $\gamma^{\mathbf{ab}}$ is defined with (2.3) and

$$T_1^{\mathbf{a}} T_2^{\mathbf{a}} = T_1^{\mathbf{b}} \gamma^{\mathbf{ab}} T_2^{\mathbf{a}} \quad (2.4)$$

The definitions in (2.3),(2.4) are nothing but the extension of the field space to include composite operators $\hat{\phi}_{a_1 \dots a_n}$. The interest in such a general setting is twofold: firstly, it allows us to formulate, at all scales, the theory in terms of physically relevant degrees of freedom. Secondly, it naturally includes the coupling to composite operators and related flows. The source term in the Schwinger functional (2.1) reads

$$J^{\mathbf{a}} \hat{\phi}_{\mathbf{a}} = \sum_{i=1}^N J^{a_1 \dots a_{n_i}} \hat{\phi}_{a_1 \dots a_{n_i}}. \quad (2.5)$$

For $n_i = 1$ for all i the general source term (2.5) boils down to the standard source (2.2). A simple tensorial example is given by $\mathbf{a} = a, a_1 a_2$ and $\hat{\phi}_{\mathbf{a}} = (\hat{\phi}_a, \hat{\phi}_{a_1 a_2}) = (\hat{\varphi}_a, \hat{\varphi}_{a_1} \hat{\varphi}_{a_2})$ with $a = a_1 = a_2 = x$, a scalar field and its two-point function. This leads to a source term

$$J^{\mathbf{a}} \hat{\phi}_{\mathbf{a}} = \int d^d x J(x) \hat{\varphi}(x) + \int d^d x d^d y J(x, y) \hat{\varphi}(x) \hat{\varphi}(y). \quad (2.6)$$

The above example also emphasises that the sources $J^{\mathbf{a}}$ should be restricted to those sharing the (index-) symmetries of the fields $\hat{\phi}_{\mathbf{a}}$. We illustrate this within the

above example of a scalar field. The source term for $\hat{\phi}_{a_1 a_2} = \hat{\phi}_{a_1} \hat{\phi}_{a_2}$ satisfies $J^{a_1 a_2} \phi_{a_1} \phi_{a_2} = J^{(a_1 a_2)} \phi_{a_1} \phi_{a_2}$, where $J^{(a_1 a_2)} = \frac{1}{2}(J^{a_1 a_2} + J^{a_2 a_1})$ is the symmetric part of J . The anti-symmetric part $J^{[a_1 a_2]} = \frac{1}{2}(J^{a_1 a_2} - J^{a_2 a_1})$ does not couple to the field, $J^{[a_1 a_2]} \phi_{a_1} \phi_{a_2} = 0$. Consequently we restrict the sources to the symmetric ones. The symmetry properties of a function $J^{\mathbf{a}}$ or $\phi^{\mathbf{a}}$ are also carried by its derivatives. Again we illustrate this by the example introduced above: derivatives w.r.t. the function $J^{(\mathbf{a})}$ carry its symmetry properties. This entails that

$$\frac{\delta F[J]}{\delta J^{(\mathbf{a})}} = F_{,(\mathbf{a})} = \left(F_{,a}, \frac{1}{2} (F_{,a_1 a_2} + F_{,a_2 a_1}) \right), \quad (2.7)$$

where $J^{(a)} = J^a$. The basic example is the derivative of J w.r.t. J . It reads

$$\frac{\delta J^{(\mathbf{b})}}{\delta J^{(\mathbf{a})}} = \delta_{(\mathbf{a})}^{(\mathbf{b})} = \left(\delta_a^b, \frac{1}{2} (\delta_{a_1}^{b_1} \delta_{a_2}^{b_2} + \delta_{a_2}^{b_1} \delta_{a_1}^{b_2}) \right), \quad (2.8)$$

the second entry on the rhs is the identity kernel in the symmetric subspace. We also have $J_{,[\mathbf{b}]}^{(\mathbf{a})} = 0$ with $J^{[\mathbf{a}]} = 0$, and get $J_{,[\mathbf{b}]}^{[\mathbf{a}]} = (0, \frac{1}{2} (\delta_{a_1}^{b_1} \delta_{a_2}^{b_2} - \delta_{a_2}^{b_1} \delta_{a_1}^{b_2}))$. From now on we suppress this detail. Derivatives are always taken within the appropriate spaces defined by the corresponding projections, and carry the related symmetry properties.

Within the above conventions the Schwinger functional (2.1) reads

$$e^{W[J]} = \int d\mu[\hat{\phi}] \exp\{-S[\hat{\phi}] + J^{\mathbf{a}} \hat{\phi}_{\mathbf{a}}(\hat{\phi})\}. \quad (2.9)$$

Many of the structural results presented here can be already understood within a scalar theory with a single field. There we have $\mathbf{a} = a = x$ with the ultra-local metric $\gamma^a_{a'} = \delta(x - x')$. In these cases one can simply ignore the additional notational subtleties in the presence of fermions and tensorial fields.

The definition (2.9) is rather formal. For most interacting theories it is impossible to strictly prove the non-perturbative existence of $d\mu[\hat{\phi}] \exp\{-S[\hat{\phi}]\}$, not to mention determining it in a closed form. Here we follow a bootstrap approach in simply assuming that a finite $W[J]$ exists. This assumption is less bold than it seems at first sight. It is merely the statement that the classical action $S[\hat{\phi}]$ admits a well-defined quantum field theory in terms of appropriately chosen fields $\hat{\phi}(\hat{\phi})$. Then quite general normalised expectation values $I[J] = \langle \hat{I}[J, \hat{\phi}] \rangle$ are defined by

$$I[J] = e^{-W[J]} \hat{I}[J, \frac{\delta}{\delta J}] e^{W[J]}. \quad (2.10)$$

The I include correlation functions that relate to one particle irreducible (1PI) as well as connected and disconnected Green functions in ϕ . Subject to the definition of $\hat{\phi}$ this may include NPI Green functions in the fundamental fields $\hat{\phi}$. As an important sub-class included in (2.10) we present normalised N -point functions

$$I_{\mathbf{a}_1 \dots \mathbf{a}_N}^{(N)} = \langle \prod_{i=1}^N \hat{\phi}_{\mathbf{a}_i} \rangle, \quad (2.11a)$$

with

$$\hat{I}_{\mathbf{a}_1 \dots \mathbf{a}_N}^{(N)} = \prod_{i=1}^N \frac{\delta}{\delta J^{\mathbf{a}_i}}. \quad (2.11b)$$

The correlation functions (2.11) include all moments of the Schwinger functional and their knowledge allows the construction of the latter. A simple example for (2.11) is $\langle \hat{\phi} \rangle$, the expectation value of the operator $\hat{\phi}$ coupled to the current with $\hat{I}^{(1)} = \frac{\delta}{\delta J}$. We briefly illustrate the construction of connected or 1PI Green functions by an important example, the full propagator. With the 1-point function $\langle \hat{\phi} \rangle$, the propagator $W_{,\mathbf{a}_1 \mathbf{a}_2}[J] = \langle \hat{\phi}_{\mathbf{a}_1} \hat{\phi}_{\mathbf{a}_2} \rangle_{1\text{PI}}$ follows as $I_{\mathbf{a}_1 \mathbf{a}_2}^{(2)} - I_{\mathbf{a}_1}^{(1)} I_{\mathbf{a}_2}^{(1)}$.

Further important examples are correlation functions I where $\hat{I}[J, \frac{\delta}{\delta J}]$ generates a symmetry of the theory at hand. Let us first consider general Dyson-Schwinger (DS) equations, formally given by

$$\int \mathcal{G}[\hat{\phi}] \left(d\mu[\hat{\phi}] \Psi[\hat{\phi}] e^{-S[\hat{\phi}] + J^{\mathbf{a}} \hat{\phi}_{\mathbf{a}}[\hat{\phi}]} \right) = 0, \quad (2.12)$$

with bosonic operator \mathcal{G} . For (2.12) to hold the operator $\mathcal{G}[\hat{\phi}]$ has to generate a symmetry of the path integral. For infinitesimal transformations \mathcal{G} , (2.12) translates into

$$I[J] = 0, \quad (2.13a)$$

with

$$\hat{I} = \left[(\mathcal{G}\Psi) - \Psi(\mathcal{G}S) + \Psi J^{\mathbf{a}} (\mathcal{G}\hat{\phi})_{\mathbf{a}} \right] \left(\hat{\phi} = \frac{\delta}{\delta J} \right). \quad (2.13b)$$

In (2.13) we have assumed $(\mathcal{G}d\mu) = 0$. We emphasise that this can be easily achieved by reducing $d\mu$ to the flat measure with $d\mu = d\hat{\phi} \Psi_1$. With $d\mu' = d\hat{\phi}$ and $\Psi' = \Psi_1 \Psi$ we substitute $d\mu \Psi = d\mu' \Psi'$. The simplest relevant example for (2.13) is provided by the standard DS equations. They encode translation invariance of the flat measure $d\hat{\phi}$. Accordingly, the standard DS equations are obtained with $\hat{\phi} = \hat{\phi}$, $\mathcal{G}[\hat{\phi}] = \frac{\delta}{\delta \hat{\phi}}$, $d\mu = d\hat{\phi}$ and $\Psi = 1$. Within this choice we arrive at

$$I_{\text{DSE}}[J] = J - \left\langle \frac{\delta S}{\delta \hat{\phi}} \right\rangle = 0, \quad (2.14a)$$

with

$$\hat{I}_{\text{DSE}} = J - \frac{\delta S}{\delta \hat{\phi}}(\hat{\phi} = \frac{\delta}{\delta J}). \quad (2.14b)$$

Eq. (2.14) is the well-known functional Dyson-Schwinger equation. It assumes a multiplicative renormalisation procedure preserving all symmetries ($d\mu = d\hat{\phi}$, $\Psi = 1$). When additive renormalisation is required, or when we study a renormalisation procedure breaking the symmetries of the classical action, this can be captured in a non-trivial Ψ ¹.

In case \mathcal{G} generates a symmetry of the action, $\mathcal{G}S = 0$, the above relation simplifies. Restricting ourselves also to invariant functionals Ψ with $\mathcal{G}\Psi = 0$ we are led to

$$I^\alpha[J] = 0, \quad (2.15a)$$

with

$$\hat{I}^\alpha = \Psi J^\alpha \mathcal{G}^\alpha \hat{\phi}_\alpha, \quad (2.15b)$$

where α carries the group structure of the symmetry. In (2.15) we have used the bosonic nature of \mathcal{G} as well as assuming that the symmetry is maintained within the quantisation: $(\mathcal{G}^\alpha d\mu) = 0$. It is often possible and helpful to rewrite symmetries in terms of derivative operators \mathcal{G} with $\mathcal{G}^2 \hat{\phi} = 0$. This might necessitate the introduction of auxiliary fields. For example, in a gauge theory we deal with the BRST symmetry with $\mathcal{G} = \mathfrak{s}$, the BRST derivative. We add source terms for $\mathcal{G}\hat{\phi}$ with $J^\alpha \hat{\phi}_\alpha \rightarrow J^\alpha \hat{\phi}_\alpha + Q^\alpha (\mathcal{G}\hat{\phi})_\alpha$. The Schwinger functional $W = W[J, Q]$ is a functional of both, J and Q , and we are led to

$$\hat{I}_\mathfrak{s} = J^\alpha \frac{\delta}{\delta Q^\alpha} \quad \text{and} \quad I_\mathfrak{s} = J^\alpha \frac{\delta W[J, Q]}{\delta Q^\alpha} = 0. \quad (2.16)$$

We conclude that the set of I defined in (2.10) provides the full information about the quantum theory as it spans the set of all correlations functions $\{\mathcal{O}\}$. In this context we emphasise again that not all correlation functions of interest are directly given by the correlation functions I , a simple example being the propagator $W_{,\mathbf{ab}} = I_{\mathbf{a}_1 \mathbf{a}_2}^{(2)} - I_{\mathbf{a}_1}^{(1)} I_{\mathbf{a}_2}^{(1)}$.

The key object in the present approach is the Schwinger functional of the theory, or some related generating functional. Often one concentrates on the Wilsonian effective action $S_{\text{eff}}[\Phi]$, the generating functional for amputated connected Green functions. It is defined by

$$S_{\text{eff}}[\Phi] := -W[S^{(2)}[0] \Phi], \quad (2.17)$$

where $S^{(2)}[0] = \delta^2 S / (\delta \Phi)^2 [\Phi = 0]$. The advantage of working with the Schwinger functional W or S_{eff} is that it allows for the most straightforward derivation of functional identities. However, a more tractable object is the effective action Γ , the generating function of 1PI Green functions of $\phi = \langle \hat{\phi} \rangle$. It is obtained as the Legendre transform of W ,

$$\Gamma[\phi] = \sup_J (J^\mathbf{a} \phi_\mathbf{a} - W[J]). \quad (2.18)$$

Eq. (2.18) includes NPI effective actions [194–196] for an appropriate choice of $\phi_\mathbf{a}[\varphi]$. The definition (2.18) leads to

$$\Gamma^{,\mathbf{a}}[\phi] = \gamma^{,\mathbf{a}}_\mathbf{b} J^\mathbf{b}(\phi), \quad (2.19a)$$

$$W_{,\mathbf{a}}[J] = \phi_\mathbf{a}(J), \quad (2.19b)$$

implying that the field ϕ is the mean field, $\phi = \langle \hat{\phi} \rangle$. In (2.19) we have used that $J^\mathbf{a} \phi_\mathbf{a} = \phi^\mathbf{a} J_\mathbf{a} = \phi_\mathbf{a} \gamma^{,\mathbf{a}}_\mathbf{b} J^\mathbf{b}$. The derivatives in (2.19) are taken with respect to the variables of Γ and W respectively, that is $\Gamma^{,\mathbf{a}} = \frac{\delta \Gamma}{\delta \phi_\mathbf{a}}$ and $W_{,\mathbf{a}}[J] = \frac{\delta W}{\delta J^\mathbf{a}}$. Furthermore it follows that

$$W_{,\mathbf{ac}} \Gamma^{,\mathbf{cb}} = \gamma^{,\mathbf{b}}_\mathbf{a}, \quad (2.20)$$

The definition (2.10) and the relation (2.13) translate into the corresponding equations in terms of 1PI quantities by using (2.19),(2.20) as well as

$$W[J(\phi)] = \phi_\mathbf{a} \Gamma^{,\mathbf{a}} - \Gamma[\phi], \quad (2.21)$$

and

$$\frac{\delta}{\delta J^\mathbf{a}} = W_{,\mathbf{ab}} \frac{\delta}{\delta \phi_\mathbf{b}}, \quad (2.22)$$

For composite fields one usually splits up the irreducible part of $\langle \hat{\phi}_\mathbf{a} \rangle$. As an example we study 2PI scalar fields $\hat{\phi}_\mathbf{a} = (\hat{\varphi}_\mathbf{a}, \hat{\varphi}_{a_1} \hat{\varphi}_{a_2})$. There we have $\phi_{a_1 a_2} = \langle \hat{\varphi}_{a_1} \hat{\varphi}_{a_2} \rangle = \phi_{a_1 a_2}^{\text{ir}} + \phi_{a_1} \phi_{a_2}$ with $\phi_\mathbf{a} = \langle \hat{\varphi}_\mathbf{a} \rangle$. Here $\phi_{a_1 a_2}^{\text{ir}}$ is the 1PI part of $\phi_{a_1 a_2}$. This extends to general composite operators and we parameterise $\Gamma^{\text{ir}}[\phi^{\text{ir}}] := \Gamma[\phi(\phi^{\text{ir}})]$. The ϕ^{ir} -derivative of Γ^{ir} reads

$$\Gamma^{\text{ir},\mathbf{a}}[\phi^{\text{ir}}] = \phi_{\mathbf{c},\mathbf{a}}(\phi^{\text{ir}}) \gamma^{\mathbf{c}}_\mathbf{b} J^\mathbf{b}(\phi^{\text{ir}}), \quad (2.23)$$

where $\phi_{\mathbf{c},\mathbf{a}}(\phi^{\text{ir}})$ stands for the derivative of ϕ w.r.t. ϕ^{ir} . Within the above 2PI example (2.23) boils down to $(\Gamma^{\text{ir},\mathbf{a}}[\phi^{\text{ir}}]) = (J^{a_1 a_2}, J^\mathbf{a} + 2J^{ab} \phi_b)$, where we have used that $J^{ab} = J^{(ab)}$. We close with the remark that it does not make a difference in the relations of this section whether we have tensorial multi-indices \mathbf{a} or a vector index a .

¹ More precisely it is captured in a non-trivial $d\mu$ absorbed in Ψ' .

III. FLOWS

In interacting quantum theories it is hardly possible to compute generating functionals, such as the Schwinger functional W , in a closed form. In most situations one resorts to systematic expansion schemes like perturbation theory or the $1/N$ -expansion that come with a small expansion parameter. In strongly interacting systems truncations are not supported by a small expansion parameter and have to be used with care. In general either case requires renormalisation [30, 31]. Renormalisation group invariance encodes the independence of physics under general reparameterisations of the theory, or, put differently, the physical equivalence of (UV) cut-off procedures. RG invariance can be used to resolve the momentum dependence of the theory by trading RG scaling for momentum scaling. RG transformations always imply the scaling of all parameters of the theory, e.g. couplings and masses. In turn, the change of a physical parameter is related to an RG rescaling. For example, changing the mass-parameter of the theory leads to the Callan-Symanzik equation [32, 33]. Presented as a differential equation for a generating functional, e.g. the Schwinger functional W or the effective action Γ , it constitutes a functional RG equation [32]. The momentum dependence is more directly resolved by block-spinning on the lattice [34]. In the continuum theory this is implemented with a momentum cut-off [1–14] leading to the Wilsonian RG.

The strong interrelations between the different RG concepts as well as their physical differences become apparent if presented as Functional Renormalisation Group equations for generating functionals. FRG formulations are also suitable for both discussing formal aspects as well as practical applications. The FRG has been introduced with a smooth momentum cut-off for simplifying proofs of perturbative renormalisability and the construction of effective Lagrangians in [6], see also [9, 37–39]. More recently, there has been an increasing interest in FRG methods as a computational tool for accessing both perturbative as well as non-perturbative physics, initiated by [10–14]. The recent success of FRG methods was also triggered by formal advances that led to a deeper understanding of the FRG, and here we aim at further progress in this direction. We close with a brief overview on the literature in view of structural aspects: general formal advances have been made in [40–107]. Progress in the construction of FRG flows in gauge theories has been

achieved in [108–160]. FRG flows in gravity are investigated in [161–170], and dynamical non-equilibrium evolutions have been studied in [171–178]. All these formal advances have been successfully used within applications, see reviews [16–27, 29].

A. Setting

The starting point of our analysis is the finite renormalised Schwinger functional W in (2.9). So far we only assumed its existence without offering a method of how to compute it. We shall turn the problem of computing the path integral (2.9) into the task of successively integrating out modes, each step being well-defined and finite. To that end we modify the Schwinger functional as follows:

$$e^{W[J,R]} = e^{-\Delta S[\frac{\delta}{\delta J},R]} e^{W[J]}, \quad (3.1)$$

where

$$\Delta S[\frac{\delta}{\delta J}, R] = \sum_n R^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta J^{\mathbf{a}_1}} \dots \frac{\delta}{\delta J^{\mathbf{a}_n}}. \quad (3.2)$$

If used as a regulator, the operator $\exp -\Delta S$ in (3.2) should be positive (on $\exp W$), and $\Delta S[\frac{\delta}{\delta J}, 0] = 0$. For example, the standard setting is given by $\mathbf{a} = a$, $\hat{\phi}^a = \hat{\varphi}^a$ and

$$\Delta S[\frac{\delta}{\delta J}, R] = R^{\mathbf{a}\mathbf{b}} \frac{\delta}{\delta J^{\mathbf{a}}} \frac{\delta}{\delta J^{\mathbf{b}}}. \quad (3.3)$$

A factor $1/2$ on the rhs common in the literature is absorbed into R . With the restrictions $\mathbf{a} = a$, $\hat{\phi}^a = \hat{\varphi}^a$, and up to RG subtleties, (3.3) leads to a modification of the kinetic term $S[\hat{\varphi}]$ in (2.9): $S[\hat{\varphi}] \rightarrow S[\hat{\varphi}] + R^{ab} \hat{\varphi}_a \hat{\varphi}_b$. More generally, (3.3) results in a modification of the propagation of the field ϕ which is possibly composite. Such a modification can be used to suppress the propagation of ϕ -modes in the path integral. In particular, it allows for a simple implementation of a smooth momentum cut-off [6, 10–14]. An amplitude regularisation has been put forward in [62–66, 85] and relates to $\Delta S \simeq S$ or parts of S , which ensures positivity. A specifically simple flow of this type is the functional Callan-Symanzik flow [32, 33]. In specific theories, e.g. those with non-linear gauge symmetries, more general regulator terms can prove advantageous. ΔS can also be used to construct boundary RG flows, in particular thermal flows [19, 133–136], and dynamical time evolutions [178].

General regulator terms ΔS according to (3.2) involve higher order derivatives and derivatives w.r.t. currents

coupled to composite operators. In this general setting a different point of view is more fruitful: the operator $\exp -\Delta S$ adds source terms for composite operators to the Schwinger functional. For example, in the standard case with $\mathbf{a} = a$ and (3.3) a source term for $\hat{\phi}_a \hat{\phi}_b$ with current R^{ab} is introduced. For the class of positive regulator terms $\Delta S[\hat{\phi}, R]$ the exponential $\exp -\Delta S$ is a positive operator with spectrum $[0, 1]$ on $\exp W$ and the correlation functions (2.11). Then, under mild assumptions the existence of $W[J, R] \leq W[J, 0]$ follows from that of $W[J, 0] = W[J]$. Consequently $\exp -\Delta S$ can be used for suppressing degrees of freedom, more precisely J -modes, in the Schwinger functional $W[J]$.

We add that $W[J, R]$ is not well-defined for general R . A simple example is a mass-like R with $R^{ab} = \frac{1}{2}m^2\delta^{ab}$ for a scalar theory. Such an insertion leads to an unrenormalised Callan-Symanzik flow [32, 33]. The required renormalisation can be added explicitly via a redefinition of R^{ab} that generates appropriate subtractions. This amounts to an explicit construction of a BPHZ-type renormalisation which is one way to render the Callan-Symanzik flow finite. From now on such a redefinition of R is assumed whenever it is necessary; in most cases, however, the regulators R generate finite $W[J, R]$ from the outset. A necessary condition for the latter is a sufficiently fast decay of R in the ultraviolet.

Within this general setting the regulators $R^{\mathbf{a}_1 \cdots \mathbf{a}_n}$ in (3.2) can be (partially) fermionic, even though ΔS should be kept bosonic (even number of fermions involved). A simple example is provided by R^a coupling to a fermion $\hat{\phi}_a$. It is in general not possible to commute J -derivatives and regulators $R^{\mathbf{a}_1 \cdots \mathbf{a}_n}$. Due to the (anti-) commutation relations of the currents $J^{\mathbf{a}}$ only specific tensor structures have to be considered for the R :

$$R^{\mathbf{a}_1 \cdots \mathbf{a}_i \mathbf{a}_{i+1} \cdots \mathbf{a}_n} = (-1)^{\mathbf{a}_i \mathbf{a}_{i+1}} R^{\mathbf{a}_1 \cdots \mathbf{a}_{i+1} \mathbf{a}_i \cdots \mathbf{a}_n}, \quad (3.4)$$

where $(-1)^{\mathbf{a}_i \mathbf{a}_{i+1}}$ is defined in appendix A. Eq. (3.4) expresses the fact that fermionic currents anti-commute, $J^{\mathbf{a}_i} J^{\mathbf{a}_{i+1}} = -J^{\mathbf{a}_{i+1}} J^{\mathbf{a}_i}$, whereas bosonic currents commute with both, bosonic and fermionic currents, leading to $J^{\mathbf{a}_i} J^{\mathbf{a}_{i+1}} = (-1)^{\mathbf{a}_i \mathbf{a}_{i+1}} J^{\mathbf{a}_{i+1}} J^{\mathbf{a}_i}$. This symmetry structure carries over to derivatives of $J^{\mathbf{a}}$. Hence, in (3.2) only that part of R carrying the tensor structure expressed in (3.4) contributes.

For illustration, we again study this setting for the standard regulator (3.3) providing a modification of the propagator. There it follows from (3.4) that for bosonic variables only the symmetric part of the tensor $R^{\mathbf{ab}}$ contributes. For the fermionic part only the anti-symmetric

part is relevant. Here we do not allow for mixed (fermionic-bosonic) parts and (3.4) reduces to

$$R_{\text{bosonic}}^{\mathbf{ab}} = R_{\text{bosonic}}^{\mathbf{ba}}, \quad (3.5a)$$

and

$$R_{\text{fermionic}}^{\mathbf{ab}} = -R_{\text{fermionic}}^{\mathbf{ba}}. \quad (3.5b)$$

The corresponding ΔS are bosonic.

So far we have discussed a modification of the Schwinger functional. The Schwinger functional $W[J, R]$ is only one, if important, correlation function. We seek an extension of (2.10) consistent with (3.1): it should define general normalised expectation values in the regularised theory as well as allowing for a straightforward extension of the symmetry relations $I[J] = 0$ as given in (2.13a). A natural extension is

$$I[J, R] = e^{-W[J, R]} e^{-\Delta S[\frac{\delta}{\delta J}, R]} \hat{I}[J, \frac{\delta}{\delta J}] e^{W[J]}. \quad (3.6)$$

Eq. (3.6) entails that $I[J, 0] = I[J]$ and guarantees well-defined initial conditions $I[J, \infty]$. Moreover, applying the extension (3.6) to a relation $I[J] = 0$ we are led to

$$I[J] = 0 \quad \rightarrow \quad I[J, R] = 0, \quad \forall R. \quad (3.7)$$

Hence a symmetry relation $I[J] = 0$ is lifted to a symmetry relation $I[J, R] = 0$ in the presence of the regulator. Eq. (3.6) can be rewritten solely in terms of $W[J, R]$ as

$$I[J, R] = e^{-W[J, R]} \hat{I}[J, \frac{\delta}{\delta J}, R] e^{W[J, R]}, \quad (3.8a)$$

with

$$\hat{I}[J, \frac{\delta}{\delta J}, R] = e^{-\Delta S[\frac{\delta}{\delta J}, R]} \hat{I}[J, \frac{\delta}{\delta J}] e^{\Delta S[\frac{\delta}{\delta J}, R]}, \quad (3.8b)$$

see also [23]. In case $\hat{I}[J, \frac{\delta}{\delta J}]$ only contains a polynomial in J we can easily determine $\hat{I}[J, \frac{\delta}{\delta J}, R]$ in a closed form. As for $R = 0$, the set of all correlation functions $\{\mathcal{O}[J, R]\}$ can be constructed from the set $\{I[J, R]\}$. A general flow describes the response of the theory to a variation of the source R and, upon integration, resolves the theory. Such flows are provided by derivatives w.r.t. R of correlation functions $\mathcal{O}[J, R]$ in the presence of the regulator

$$\delta R^{\mathbf{a}_1 \cdots \mathbf{a}_n} \frac{\delta \mathcal{O}[J, R]}{\delta R^{\mathbf{a}_1 \cdots \mathbf{a}_n}}. \quad (3.9)$$

Here $\delta R^{\mathbf{a}_1 \cdots \mathbf{a}_n}$ is a small variation. Basic examples for correlation functions \mathcal{O} are the Schwinger functional $W[J, R]$ and the expectation values $I[J, R]$ defined in (3.8).

In case we define one-parameter flows $R(k)$ that are trajectories in the space of regulators R and hence in theory space, the general derivatives (3.9) provide valuable information about the the stability of the chosen one-parameter flows, in particular if these flows are subject to truncations. Stable one-parameter flows can be deduced from the condition

$$\delta R_{\perp}^{\mathbf{a}_1 \dots \mathbf{a}_n} \left. \frac{\delta \mathcal{O}[J, R]}{\delta R_{\mathbf{a}_1 \dots \mathbf{a}_n}} \right|_{R_{\text{stab}}} = 0, \quad (3.10)$$

where $\{R_{\perp}\}$ is the set of operators that provide a regularisation of the theory at some physical cut-off scale k_{eff} , and $R_{\text{stab}} \in \{R_{\perp}\}$. Eq. (3.10) ensures that the flow goes in the direction of steepest descent in case (3.10) describes a minimum. If flows are studied within given approximations schemes, the stability condition (3.10) can be used to optimise the flow. Note that (3.10), in particular in finite approximations, does not necessarily lead to a single R_{stab} . Then (3.10) defines a hypersurface of stable regulators. We also emphasise that (3.10) cannot vanish in all directions δR except at a stable fixed point in theory space. Consequently one has to ensure within an optimisation procedure that the variations δR_{\perp} considered are orthogonal to the direction of the flow. If this is not achieved, no condition is obtained at all. We shall come back to the problem of optimisation in section V.

B. One-parameter flows

1. Derivation

In most cases we are primarily interested in the underlying theory at $R = 0$, that is $\mathcal{O}[J] = \mathcal{O}[J, 0]$, e.g. in $W[J] = W[J, 0]$, the Schwinger functional of the full theory and its moments. Total functional derivatives (3.9) with arbitrary δR^{ab} scan the space of theories given by $W[J, R]$. For computing $W[J]$ it is sufficient to study one-parameter flows with regulators R depending on a parameter $k \in [\Lambda, 0]$ with $R(k = 0) \equiv 0$ and $W[J, R(\Lambda)]$, $\mathcal{O}[J, R(\Lambda)]$ well under control. These one-parameter flows derive from (3.9) as partial derivatives due to variations

$$\delta R = dt \partial_t R, \quad (3.11)$$

where $t = \ln(k/k_0)$ is the logarithmic cut-off scale. The normalisation k_0 is at our disposal, and a standard choice is $k_0 = \Lambda$ leading to $t_{\text{in}} = 0$. In the following we shall drop the normalisation. The flows with (3.11) lead to

correlation functions \mathcal{O}_k that connect a well-known initial condition at Λ with correlations functions $\mathcal{O} = \mathcal{O}_0$ in the full theory. In most cases a well-defined initial condition is obtained for large regulator $R \rightarrow \infty$. This is discussed in section III C 4.

The most-studied one-parameter flow relates to a successive integration of momentum modes of the fields φ , that is k is a momentum scale. More specifically, we discuss regulators leading to an infrared regularisation with IR scale k of the theory under investigation, the scale k providing the parameter $k \in [k_{\text{in}}, 0]$. To that end we choose regulator terms $\Delta S[\varphi] = R^{ab} \varphi_a \varphi_b$ for a scalar theory with

$$R = R(p^2) \delta(p - p'), \quad (3.12)$$

with the properties

- (i) it has a non-vanishing infrared limit, $p^2/k^2 \rightarrow 0$, typically $R \rightarrow k^2/2$ for bosonic fields.
- (ii) it vanishes for momenta p^2 larger than the cut-off scale, for $p^2/k^2 \rightarrow \infty$ at least with $(p^2)^{(d-1)/2} R \rightarrow 0$ for bosonic fields.
- (ii)' (ii) implies that it vanishes in the limit $k \rightarrow 0$. In this limit, any dependence on R drops out and all correlation functions \mathcal{O}_k reduce to the correlation functions in the full theory $\mathcal{O} = \mathcal{O}_0$, in particular the Schwinger functional W_k and the Legendre effective action Γ_k .
- (iii) for $k \rightarrow \infty$ (or $k \rightarrow \Lambda$ with Λ being some UV scale much larger than the relevant physical scales), R diverges. Thus, the saddle point approximation to the path integral becomes exact and correlation functions \mathcal{O}_k tend towards their classical values, e.g. $\Gamma_{k \rightarrow \Lambda}$ reduces to the classical action S .

Property (i) guarantees an infrared regularisation of the theory at hand: for small momenta the regulator generates a mass. Property (ii) guarantees the (ultraviolet) definiteness of $W[J, R]$. The insertion ΔS vanishes in the ultraviolet: no further ultraviolet renormalisation is required, though it might be convenient. It is property (ii) that facilitates perturbative proofs or renormalisability. Properties (ii)' and (iii) guarantee well defined initial conditions, and ensure that the full theory is the end-point of the flow. In most cases the regulator $R = p^2 r(p^2/k^2)$ is a function of $x = p^2/k^2$, up to the prefactor carrying the dimension. For such regulators the condition (iii) follows

already from (i). For regulators (3.12) with the properties (i)-(iii) we can study flows from a well-known initial condition, the classical theory or perturbation theory, to the full theory. Integrating the flow resolves the quantum theory. The properties (i),(ii) guarantee that the flow is local in momentum space leading to well-controlled limits $x \rightarrow 0, \infty$. In turn, mass-like regulators violate condition (ii): additional renormalisation is required. Moreover, the flow spreads over all momenta which requires some care if taking the limits $k^2 \rightarrow 0, \infty$, see e.g. [19].

General one-parameter flows are deduced from (3.1), (3.8) by inserting regulators $R(k)$ where $k \in [\Lambda, 0]$. The condition $R(0) \equiv 0$ guarantees that the endpoint of such a flow is the full theory. For one-parameter flows, (3.1) reads

$$e^{W_k[J]} = e^{-\Delta S_k[\frac{\delta}{\delta J}]} e^{W[J]} \quad (3.13)$$

with

$$\Delta S_k[\frac{\delta}{\delta J}] = \Delta S[\frac{\delta}{\delta J}, R(k)],$$

and ΔS is defined in (3.2). Similarly we rewrite (3.8) as

$$I_k[J] = e^{-W_k[J]} \hat{I}_k[J, \frac{\delta}{\delta J}] e^{W_k[J]} \quad (3.14a)$$

with

$$\hat{I}_k[J, \frac{\delta}{\delta J}] = e^{-\Delta S_k[\frac{\delta}{\delta J}]} \hat{I}[J, \frac{\delta}{\delta J}] e^{\Delta S_k[\frac{\delta}{\delta J}]}. \quad (3.14b)$$

We also recall that (3.14) entails that $I_0[J] = I[J]$ and

$$I[J] = 0 \quad \rightarrow \quad I_k[J] = 0 \quad \forall k, \quad (3.15)$$

that is a symmetry relation $I[J] = 0$ is lifted to a relation $I_k[J] = 0$ in the presence of the cut-off. The flow of k -dependent quantities $I_k, \partial_t I_k$ with $t = \ln k$ at fixed current J allows us to compute $I[J]$, if the initial condition I_Λ is under control. For momentum flows, this input is the high momentum part of I at some large initial scale Λ . Perturbation theory is applicable for large scales, and hence $I_\Lambda[J]$ is well under control. The flow equation $\partial_t I_k$ can be evaluated with (3.6) for $R(k)$. However, for later purpose it is more convenient to approach this question as follows. Let us study the operators

$$\hat{F}[J, \frac{\delta}{\delta J}] = \partial_t \hat{I}[J, \frac{\delta}{\delta J}], \quad (3.16)$$

and

$$\Delta \hat{I} = [\partial_t, \hat{I}]. \quad (3.17)$$

Here the t -derivative acts on everything to the right, i.e. $\partial_t \hat{I}G[J] = (\partial_t \hat{I})G[J] + \hat{I}\partial_t G[J]$, and is taken at

fixed J . The notation for partial derivatives is explained in appendix B. The functionals I, F and ΔI fall into the class of functionals (2.10) and can be lifted to their R -dependent analogues (3.8), and in particular to $F_k, I_k, \Delta I_k$ as defined in (3.14). The full Schwinger functional $W[J] = W_0[J]$ is independent of t , $\partial_t W = 0$, and we derive from (3.6) that $F = \Delta I$ and consequently

$$F_k = \Delta I_k. \quad (3.18)$$

Moreover, the most interesting I are expectation values in the full theory and do not depend on t . For this class we have $\Delta \hat{I} = 0$ leading to $F_k = 0$. Still, the consideration of more general F_k will also prove useful so we do not restrict ourselves to $F_k = 0$. The general \hat{F}_k is derived from (3.14b) with help of

$$[\partial_t, R^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta J^{\mathbf{a}_1}} \dots \frac{\delta}{\delta J^{\mathbf{a}_n}}] = \dot{R}^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta J^{\mathbf{a}_1}} \dots \frac{\delta}{\delta J^{\mathbf{a}_n}}. \quad (3.19)$$

In (3.19) we have used that $[\partial_t, \frac{\delta}{\delta J}] = 0$ as $\partial_t = \partial_t|_J$. The rhs of (3.19) commutes with $\Delta S_k[\frac{\delta}{\delta J}]$ and we conclude that $(\partial_t + \Delta S[\frac{\delta}{\delta J}, \dot{R}]) \exp -\Delta S_k = (\exp -\Delta S_k) \partial_t$. Inserting \hat{F} into (3.14b) and using (3.19) we are led to \hat{F}_k with

$$\hat{F}_k = \left(\partial_t + \Delta S[\frac{\delta}{\delta J}, \dot{R}] \right) \hat{I}_k. \quad (3.20)$$

The expression in the parenthesis in (3.20) is an operator acting on everything to the right. Inserting (3.20) into (3.14a) we arrive at

$$e^{-W_k} \left(\partial_t + \Delta S[\frac{\delta}{\delta J}, \dot{R}] \right) e^{W_k} I_k = \Delta I_k, \quad (3.21)$$

valid for general I_k given by (3.14). ΔI_k on the right hand side carries the explicit t -scaling of the operator \hat{I} and vanishes for t -independent \hat{I} . In order to get rid of the exponentials in (3.21) we use that $\frac{\delta}{\delta J} e^{W_k} = e^{W_k} \left(\frac{\delta}{\delta J} + \frac{\delta W_k}{\delta J} \right)$. With this relation (3.21) turns into

$$\left(\partial_t + \dot{W}_k + \Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}] \right) I_k = \Delta I_k, \quad (3.22)$$

where we have introduced the expectation value $\phi = \langle \hat{\phi} \rangle_J$ of the operator coupled to the current

$$\phi_{\mathbf{a}}[J] := W_{k, \mathbf{a}}[J]. \quad (3.23)$$

Eq. (3.22) involves the flow of the Schwinger functional, \dot{W}_k , reflecting the normalisation of I_k . Independent flows of I_k are achieved by dividing out the flow of the Schwinger functional. The flow \dot{W}_k is extracted from (3.22) for the choice $I_k = 1$ with $\Delta I_k = 0$, following from $\hat{I} = 1$ and $\Delta \hat{I} = [\partial_t, \hat{I}] = 0$. Then, (3.22) boils down to

$\dot{W}_k + (\Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]) = 0$, where both expressions are functionals and not operators. More explicitly it reads

$$\left(\partial_t + \sum_n \dot{R}^{\mathbf{a}_1 \dots \mathbf{a}_n} \right. \quad (3.24)$$

$$\left. \times \left(\frac{\delta}{\delta J} + \phi \right)_{\mathbf{a}_1} \dots \left(\frac{\delta}{\delta J} + \phi \right)_{\mathbf{a}_{n-1}} \frac{\delta}{\delta J^{\mathbf{a}_n}} \right) W_k[J] = 0.$$

Eq. (3.24) is the flow equation for the Schwinger functional. It links the flow of the Schwinger functional, \dot{W}_k , to a combination of connected Green functions $W_{k, \mathbf{a}_1 \dots \mathbf{a}_n}$. For quadratic regulators (3.3) we obtain the standard flow equation for the Schwinger functional,

$$\left(\partial_t + \dot{R}^{\mathbf{ab}} \frac{\delta}{\delta J^{\mathbf{a}}} \frac{\delta}{\delta J^{\mathbf{b}}} + \dot{R}^{\mathbf{ab}} \phi_{\mathbf{a}} \frac{\delta}{\delta J^{\mathbf{b}}} \right) W_k[J] = 0. \quad (3.25)$$

We remark for comparison that the standard notation involves a factor $\frac{1}{2}$ in the \dot{R} -terms. It has been shown in [54] that (3.25) is the most general form of a one loop equation. Eq. (3.24) makes this explicit in a more general setting as the one considered in [54]. Only flows depending on $W_{k, \mathbf{a}_1 \dots \mathbf{a}_n}$ with $n \leq 2$ contain one loop diagrams in the full propagator. Note in this context that J couples to a general operator ϕ , not necessarily to the field.

Eq. (3.24) is the statement that the flow operator $\Delta S_1[J, \dot{R}] = \dot{W}_k + \Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]$ with

$$\Delta S_1[J, \dot{R}] = \Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}] - (\Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]), \quad (3.26)$$

is given by all terms in $\Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]$ with at least one derivative $\frac{\delta}{\delta J}$ acting to the right. For later use we also define $\Delta S_n[J, \dot{R}]$ as the part of ΔS with at least n J -derivatives. Their definitions and properties are detailed in appendix C. The operator of interest here, ΔS_1 , can be written with an explicit J -derivative as

$$\Delta S_1[J, \dot{R}] = \Delta S^{\mathbf{a}}[J, \dot{R}] \frac{\delta}{\delta J^{\mathbf{a}}}. \quad (3.27)$$

The operator $\Delta S^{\mathbf{a}}[J, \dot{R}]$ is defined in (C.1). Using (3.24) and the definition (3.27) in (3.22) we arrive at

$$\left(\partial_t + \Delta S^{\mathbf{a}}[J, \dot{R}] \frac{\delta}{\delta J^{\mathbf{a}}} \right) I_k = \Delta I_k, \quad (3.28)$$

valid for general $I_k, \Delta I_k$ given by (3.14). ΔI_k carries the explicit t -scaling of \hat{I} and is derived from (3.17). The partial t -derivative is taken at fixed current J . The flow of a general functional I_k requires the knowledge of $\phi_{\mathbf{a}}[J] = W_{k, \mathbf{a}}[J]$ and ΔI_k . Only for those I_k that entail this information in a closed form, $\phi = \phi[I_k]$ and $\Delta I_k = \Delta I_k[I_k]$, the flow equation (3.28) can be used without further input except that of I_{Λ} .

2. Flow of the Schwinger functional

We proceed by describing the flow (3.28) for correlation functions (3.14) within basic examples. To begin with, we study the flow of the Schwinger functional W_k . First we note that its flow (3.24) was derived from (3.22) with $I = 1$. The final representation (3.28) was indeed achieved by dividing out (3.24). Nonetheless, the latter should follow from the general flow equation (3.28). Naively one would assume that $I_k = W_k$ can be obtained from a t -independent operator \hat{I} , that is $\Delta \hat{I} = 0$. However, inserting the assumption $I_k = W_k$ into the flow (3.28) and using (3.24) we are led to

$$\Delta I_k = \Delta S^{\mathbf{a}}[J, \dot{R}] \phi_{\mathbf{a}} - (\Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]). \quad (3.29)$$

which does not vanish for all J , e.g. for quadratic regulators it reads $\Delta I_k = \dot{R}^{\mathbf{ab}} \phi_{\mathbf{a}} \phi_{\mathbf{b}}$. Hence (3.29) proves that $I_k = W_k$ implies $\Delta \hat{I} \neq 0$. Indeed in general (3.29) cannot be deduced from a $\Delta \hat{I}$ that is polynomial in the current and its derivatives. The above argument highlights the necessity of the restriction of (3.28) to functionals I_k constructed from (3.14). Still the flow equation for W_k can be extracted as follows. Let us study the flow of $(I_k)_{\mathbf{a}} = W_{k, \mathbf{a}} = \phi_{\mathbf{a}}$ which also is of interest as ϕ is an input in the flow (3.28). $I_k = \phi$ falls into the allowed class of I_k as

$$\hat{I}_{\mathbf{a}} = (\hat{I}_k)_{\mathbf{a}} = \frac{\delta}{\delta J^{\mathbf{a}}} \rightarrow (I_k)_{\mathbf{a}} = W_{k, \mathbf{a}} = \phi_{\mathbf{a}}. \quad (3.30)$$

Moreover, $\Delta I_k = 0$. Consequently, the flow of the functional I_k introduced in (3.30) reads

$$\dot{W}_{k, \mathbf{a}} + \Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}] \phi_{\mathbf{a}} - (\Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]) \phi_{\mathbf{a}} = 0. \quad (3.31)$$

With $\frac{\delta}{\delta J} 1 = 0$ the second term on the left hand side can be rewritten as follows

$$\begin{aligned} \Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}] \phi_{\mathbf{a}} &= \Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}] (\frac{\delta}{\delta J} + \phi)_{\mathbf{a}} \\ &= (\frac{\delta}{\delta J} + \phi)_{\mathbf{a}} \Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]. \end{aligned} \quad (3.32)$$

We emphasise that the first line in (3.32) is not an operator identity. For the second line in (3.32) we have used the bosonic nature of the regulator term and the representation $\frac{\delta}{\delta J} + \phi = e^{-W} \frac{\delta}{\delta J} e^W$. This also entails that $\phi_{\mathbf{a}} (\Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]) = (\Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]) \phi_{\mathbf{a}}$. We have already mentioned that $\partial_t \phi_{\mathbf{a}}[J] \neq 0$ as the t -derivative is taken at fixed J . For the same reason we can commute t -derivatives with J -derivatives: $\partial_t W_{k, \mathbf{a}}[J] = (\partial_t W_k[J])_{, \mathbf{a}}$.

We conclude that the flow of $W_{k,\mathbf{a}}$ can be written as a total derivative

$$\left[\partial_t W_k + (\Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]) \right]_{,\mathbf{a}} = 0, \quad (3.33)$$

which upon integration yields

$$\partial_t W_k + (\Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]) = 0. \quad (3.34)$$

Eq. (3.34) agrees with (3.24)².

3. Standard flow

For its importance within applications we also discuss the standard quadratic flow. In this case the flow (3.28) reduces to

$$\left(\partial_t + \dot{R}^{\mathbf{ab}} \frac{\delta}{\delta J^{\mathbf{a}}} \frac{\delta}{\delta J^{\mathbf{b}}} + 2\dot{R}^{\mathbf{ab}} \phi_{\mathbf{a}} \frac{\delta}{\delta J^{\mathbf{b}}} \right) I_k[J] = 0, \quad (3.35)$$

and (3.29) turns into $\Delta I_k = \dot{R}^{\mathbf{ab}} \phi_{\mathbf{b}} \phi_{\mathbf{a}}$ which does not vanish for $\phi \neq 0$. That proves that there is no \hat{I} leading to $I_k = W_k$. The flow of $(I_k)_{\mathbf{a}} = W_{k,\mathbf{a}}$ follows as

$$\begin{aligned} (\partial_t W_k[J])_{,\mathbf{a}} &= - \left(\dot{R}^{\mathbf{bc}} \frac{\delta}{\delta J^{\mathbf{b}}} \frac{\delta}{\delta J^{\mathbf{c}}} + 2\dot{R}^{\mathbf{bc}} \phi_{\mathbf{b}} \frac{\delta}{\delta J^{\mathbf{c}}} \right) W_{k,\mathbf{a}} \\ &= - \left[\dot{R}^{\mathbf{bc}} (W_{k,\mathbf{bc}} + \phi_{\mathbf{b}} \phi_{\mathbf{c}}) \right]_{,\mathbf{a}}. \end{aligned} \quad (3.36)$$

Both sides in (3.36) are total derivatives w.r.t. $J^{\mathbf{a}}$. Integration leads to

$$\dot{W}_k[J] = -\dot{R}^{\mathbf{ab}} (W_{k,\mathbf{ab}} + \phi_{\mathbf{a}} \phi_{\mathbf{b}}), \quad (3.37)$$

where we have put the integration constant to zero. For the reordering in (3.37) we have used that the regulator $R^{\mathbf{ab}}$ is bosonic. Eq. (3.37) agrees with (3.25). It also follows straightforwardly from (3.34) for quadratic regulators.

4. Flow of amputated correlation functions

The results of the previous sections translate directly into similar ones for amputated correlation functions $I_k[J(\phi)]$ with the following k -dependent choice of the current

$$J^{\mathbf{a}}(\Phi) = [S + \Delta S]_{\Phi=0}^{\mathbf{ba}} \Phi_{\mathbf{b}}, \quad \Phi_{\mathbf{a}} = (P_k)_{\mathbf{ba}} J^{\mathbf{b}}, \quad (3.38)$$

introducing the classical propagator P_k . With (3.38) the flow for general correlation functions $O_k[J(\Phi)]$ is computed as

$$\partial_t O_k[J(\Phi)] = \left[\partial_t O_k[J] + \Phi_{\mathbf{a}} (\partial_t \Delta S)_{\Phi=0}^{\mathbf{ab}} O_{k,\mathbf{b}} \right]_{J=J(\Phi)}, \quad (3.39)$$

in particular valid for $O_k = I_k$. The t -derivative on the lhs of (3.39) is taken at fixed Φ : the first term on the rhs of (3.39) is the flow (3.28) at fixed J , and the second term stems from the k -dependence of $J(\Phi)$. For example, in the presence of a regulator the effective Lagrangian $S_{\text{eff}}[\Phi]$ (2.17) turns into

$$S_{\text{eff}_k}[\Phi] := -W_k[J(\Phi)], \quad (3.40)$$

and hence has the flow (3.39) with (3.24). This flow further simplifies for quadratic regulators $R^{\mathbf{ab}} \hat{\phi}_{\mathbf{a}} \hat{\phi}_{\mathbf{b}}$. For this choice we arrive at

$$\partial_t S_{\text{eff}_k}[\Phi] = \frac{1}{2} (\dot{P}_k)_{\mathbf{ab}} \left(S_{\text{eff}_k}^{\mathbf{ab}} - S_{\text{eff}_k}^{\mathbf{a}} S_{\text{eff}_k}^{\mathbf{b}} - 2J^{\mathbf{a}} S_{\text{eff}_k}^{\mathbf{b}} \right). \quad (3.41)$$

Often (3.41) is rewritten in terms of the interaction part of the effective Lagrangian defined as $S_{\text{int}_k} = S_{\text{eff}_k} + \frac{1}{2} [S + \Delta S]_{\Phi=0}^{\mathbf{ba}} \Phi_{\mathbf{a}} \Phi_{\mathbf{b}}$. The flow of S_{int_k} follows as

$$\partial_t S_{\text{int}_k}[\Phi] = \frac{1}{2} (\dot{P}_k)_{\mathbf{ab}} \left(S_{\text{int}_k}^{\mathbf{ab}} - S_{\text{int}_k}^{\mathbf{a}} S_{\text{int}_k}^{\mathbf{b}} \right), \quad (3.42)$$

where we dropped the Φ -independent term $-(\partial_t \ln P_k)_{\mathbf{a}}^{\mathbf{a}}$. Flows for S_{eff_k} and its N -point insertions can be found e.g. in [6, 9, 12, 13, 42]. They are closely related to Callan-Symanzik equations for N -point insertions for $R \propto k^2$ with a possible mass renormalisation, see also [217]. The flows (3.41), (3.42) can be extended to Φ -dependent P_k by using the general DS equations (2.12) in the presence of a regulator, see e.g. [46, 152]. Then it also nicely encodes reparameterisation invariance.

We close this section with a remark on the structure of the flows (3.28),(3.39). They equate the scale derivative of a correlation function to powers of field derivatives of the same correlation function. The latter are unbounded, and the boundedness of the flow must come from a cancellation between the different terms. Hence, within truncations the question of numerical stability of these flows arises, see [77].

C. Flows in terms of mean fields

1. Derivation

In most situations it is advantageous to work with the flow of 1PI quantities like the effective action, formulated

² We have fixed the integration constant to precisely match (3.24).

as functionals of the mean field $\phi_{\mathbf{a}} = W_{,\mathbf{a}}$. In other words, we would like to trade the dependence on the current J and its derivative $\frac{\delta}{\delta J}$ in (3.28) for one on the expectation value ϕ and its derivative $\frac{\delta}{\delta\phi}$. Similarly to (2.18) we define the effective action $\Gamma = \Gamma[\phi, R]$ as

$$\Gamma[\phi, R] = \sup_J (J^{\mathbf{a}}\phi_{\mathbf{a}} - W[J, R]) - \Delta S'[\phi, R]. \quad (3.43)$$

where

$$\Delta S'[\phi, R] = \sum_{n \geq 2} R^{\mathbf{a}_1 \dots \mathbf{a}_n} \phi_{\mathbf{a}_1} \dots \phi_{\mathbf{a}_n}. \quad (3.44)$$

The exclusion of the linear regulator terms in $\Delta S'_k$ is necessary as they simply would remove the dependence on the linear regulator. $\Gamma[\phi, R]$ is the Legendre transform of $W[J, R]$, where the cut-off term has been subtracted for convenience. For $R \rightarrow 0$ (3.43) reduces to (2.18). The definition (3.43) constrains the possible choices of the operators coupled to J to those which at least locally admit a Legendre transform of $W[J, R]$. Eq. (3.43) implies

$$\gamma^{\mathbf{a}}_{\mathbf{b}} J^{\mathbf{b}} = (\Gamma + \Delta S')^{,\mathbf{a}}, \quad \phi_{\mathbf{a}} = W_{,\mathbf{a}}, \quad (3.45)$$

as well as

$$G_{\mathbf{ac}}(\Gamma + \Delta S)^{,\mathbf{cb}} = \gamma^{\mathbf{b}}_{\mathbf{a}}, \quad (3.46)$$

with

$$G_{\mathbf{ac}} = W_{,\mathbf{ac}}. \quad (3.47)$$

Here $\gamma^{\mathbf{b}}_{\mathbf{a}}$ leads to the minus sign in fermionic loops, see appendix A. For quadratic regulators (3.3) the above relations read

$$\gamma^{\mathbf{a}}_{\mathbf{b}} J^{\mathbf{b}} = \Gamma^{,\mathbf{a}} + 2R^{\mathbf{ab}}\phi_{\mathbf{b}}, \quad (3.48)$$

and

$$G_{\mathbf{ac}}(\Gamma^{,\mathbf{cb}} + 2R^{\mathbf{bc}}) = \gamma^{\mathbf{b}}_{\mathbf{a}}. \quad (3.49)$$

For (3.48),(3.49) we have used (3.5) and the bosonic nature of $R^{\mathbf{bc}}$. The operator $G[\phi]$ in (3.46) is the full field dependent propagator. With (3.46) we are able to relate derivatives w.r.t. J to those w.r.t. ϕ via

$$\frac{\delta}{\delta J^{\mathbf{a}}} = G_{\mathbf{ab}} \frac{\delta}{\delta\phi_{\mathbf{b}}}, \quad (3.50)$$

where we have used that $\phi_{\mathbf{b},\mathbf{a}} = W_{,\mathbf{ab}} = G_{\mathbf{ab}}$. As in the case of the Schwinger functional we are not only interested in the flow of Γ but in that of general correlation functions \tilde{I} as functions of ϕ . This is achieved by defining $I[J, R]$ as a functional of $J[\phi]$:

$$\tilde{I}[\phi, R] = I[J(\phi), R]. \quad (3.51)$$

We emphasise that \tilde{I} is not necessarily 1PI, it only is formulated in terms of such quantities. Still, all 1PI quantities can be constructed from the class of \tilde{I} .

One-parameter flows for \tilde{I} are derived by using trajectories $R(k)$. We extend the notation introduced in the last section for flows of \tilde{I} with

$$\tilde{I}_k[\phi] = \tilde{I}[\phi, R(k)], \quad (3.52)$$

and

$$\Gamma_k[\phi] = \Gamma[\phi, R(k)]. \quad (3.53)$$

For reformulating (3.28) in terms of \tilde{I}_k we need the relation between $\partial_t \tilde{I}_k = \partial_t|_{\phi} \tilde{I}_k$ and $\partial_t I_k = \partial_t|_J I_k$, see also appendix B. With (3.50) we rewrite $I_{k,\mathbf{a}} = G_{\mathbf{ab}} \tilde{I}_k^{,\mathbf{b}}$, and it follows from (3.51) that

$$\partial_t \tilde{I}_k[\phi] = \partial_t I_k[J] + (\partial_t J^{\mathbf{a}}[\phi]) G_{\mathbf{ab}} \tilde{I}_k^{,\mathbf{b}}, \quad (3.54)$$

with $\partial_t J^{\mathbf{a}}[\phi] = \partial_t|_{\phi} J^{\mathbf{a}}[\phi]$. Now we insert the flow for I_k , (3.28), in (3.54). With (3.50) the operator $\Delta S_1[J, \dot{R}] = \Delta S^{\mathbf{a}}[J, \dot{R}] \frac{\delta}{\delta J^{\mathbf{a}}}$ is rewritten in terms of $G_{\mathbf{ab}} \frac{\delta}{\delta\phi_{\mathbf{b}}}$. As it is more convenient to use an expansion in plain derivatives $\frac{\delta}{\delta\phi_{\mathbf{b}}}$ we also employ the identity $\Delta S_1[J, \dot{R}] = \Delta S_1[\phi, \dot{R}]$, the terms that contain at least one derivative w.r.t. J are equivalent to those containing at least one derivative w.r.t. ϕ . Note that this fails to be true for higher derivative terms, ΔS_n with $n > 1$. Together with (3.50), (3.54) the above considerations lead to the flow (3.28) as an equation for \tilde{I}_k

$$\left(\partial_t - \left((\partial_t J^{\mathbf{a}}) G_{\mathbf{ab}} + \Delta S_{\mathbf{b}}[\phi, \dot{R}] \right) \frac{\delta}{\delta\phi_{\mathbf{b}}} \right) \tilde{I}_k[\phi] = \Delta \tilde{I}_k[\phi], \quad (3.55)$$

where $\Delta S_{\mathbf{b}}$ is defined in (C.1b). It can be easily computed for general regulators. However, the higher the order of derivatives is in the regulator term, the more loop terms are contained in ΔS_1 . For further illustration we have detailed the simplest case of the standard flow in appendix D. We proceed by evaluating (3.55) for a specific simple \tilde{I}_k : we use $\tilde{I}_k[\phi] = \phi$ already introduced via $\hat{I}_{\mathbf{a}} = \frac{\delta}{\delta J^{\mathbf{a}}}$ in (3.30). For this choice we have $\Delta \tilde{I}_k = 0$ and $\partial_t \tilde{I}_k = 0$, and the flow (3.55) reads

$$\partial_t J^{\mathbf{a}} G_{\mathbf{ab}} + (\Delta S_{\mathbf{b}}[\phi, \dot{R}]) = 0. \quad (3.56)$$

Here $(\Delta S_{\mathbf{b}}[\phi, R])$ is the linear expansion coefficient of ΔS_1 in a power expansion in derivatives w.r.t. ϕ , see also (C.1). Note that (3.56) already comprises the flow equation for Γ_k : it follows from the definition of the current in (3.45) that $J^{\mathbf{b}} \gamma^{\mathbf{a}}_{\mathbf{b}} = (\Gamma + \Delta S')^{,\mathbf{a}}$. Moreover $\partial_t(\Gamma_k^{,\mathbf{a}}) = (\partial_t \Gamma_k)^{,\mathbf{a}}$ as the partial t -derivative is taken at

fixed ϕ . Then (3.56) contracted with $(\Gamma_k + \Delta S')^{\mathbf{b}\mathbf{a}}$ comprises $\partial_t(\Gamma_{k,\mathbf{b}})$ and is a total derivative w.r.t. ϕ which can be trivially integrated. This can be best seen for quadratic regulators (3.3) for which (3.56) boils down to

$$\partial_t J^{\mathbf{a}} + (G \dot{R} G)_{\mathbf{b}\mathbf{c}} \Gamma_k^{\mathbf{c}\mathbf{b}\mathbf{d}} \gamma^{\mathbf{a}}_{\mathbf{d}} - \phi_{\mathbf{b}} \dot{R}^{\mathbf{b}\mathbf{a}} = 0, \quad (3.57)$$

see also (D.1) in appendix D. We also remark that an alternative derivation of the identity (3.56) solely makes use of structural considerations which prove useful for general flows: for 1PI \tilde{I}_k the related term in (3.55) is not 1PI, whereas the other terms are. Accordingly these terms have to vanish separately³, which implies that the expression in the parenthesis has to vanish leading to (3.56). With $\partial_t J^{\mathbf{a}} G_{\mathbf{a}\mathbf{b}} = -(\Delta S_{\mathbf{b}}[\phi, \dot{R}])$ the coefficient of $\frac{\delta}{\delta\phi_{\mathbf{b}}}$ in (3.55) takes the form

$$\Delta S_{\mathbf{b}}[\phi, \dot{R}] - (\Delta S_{\mathbf{b}}[\phi, \dot{R}]) = \Delta S_{\mathbf{a}\mathbf{b}}[\phi, \dot{R}] \frac{\delta}{\delta\phi_{\mathbf{a}}}, \quad (3.58)$$

where $\Delta S_{\mathbf{a}\mathbf{b}} \frac{\delta}{\delta\phi_{\mathbf{a}}}$ is the part of the operator $\Delta S_{\mathbf{b}}$ containing at least one ϕ -derivative. $\Delta S_{\mathbf{a}\mathbf{b}}$ follows from (3.58), see also (C.1b). With (3.58) the operator in the flow (3.55) is

$$\Delta S_2[\phi, \dot{R}] = \Delta S_{\mathbf{a}\mathbf{b}}[\phi, \dot{R}] \frac{\delta^2}{\delta\phi_{\mathbf{a}} \delta\phi_{\mathbf{b}}}, \quad (3.59)$$

that part of $\Delta S[G \frac{\delta}{\delta\phi} + \phi, \dot{R}]$ containing at least two ϕ -derivatives, and we arrive at

$$\left(\partial_t + \Delta S_2[\phi, \dot{R}] \right) \tilde{I}_k[\phi] = \Delta \tilde{I}_k, \quad (3.60)$$

for general functionals \tilde{I}_k as defined with (3.14) and (3.51). The functional $\Delta \tilde{I}_k$ originates in the explicit t -scaling of \hat{I} . The partial t -derivative on the left hand side of (3.60) is taken at fixed ϕ , and the operator ΔS_2 , (3.59), accounts for inserting the regulator \dot{R} into the Green functions contained in correlation functions \tilde{I}_k . We also provide a representation of $\Delta S_2[\phi, \dot{R}]$ that only makes direct use of $\Delta S[G \frac{\delta}{\delta\phi} + \phi, \dot{R}]$,

$$\Delta S_2[\phi, \dot{R}] = \Delta S[G \frac{\delta}{\delta\phi} + \phi, \dot{R}] \quad (3.61)$$

$$- \left(\Delta S[G \frac{\delta}{\delta\phi} + \phi, \dot{R}], \phi_{\mathbf{b}} \right) \frac{\delta}{\delta\phi_{\mathbf{b}}} - (\Delta S[G \frac{\delta}{\delta\phi} + \phi, \dot{R}]),$$

where $(G \frac{\delta}{\delta\phi})_{\mathbf{b}} = G_{\mathbf{b}\mathbf{c}} \frac{\delta}{\delta\phi_{\mathbf{c}}}$. The relatively simple insertion operator ΔS_2 in terms of derivatives w.r.t. ϕ is related to the structural dependence of \tilde{I}_k on ϕ and R that is fixed

by the definitions (3.14),(3.51). In turn, changing the definition of I_k, \tilde{I}_k leads to different flows. The construction of I_k, \tilde{I}_k is a natural one as it includes general Green functions $\langle \phi^n \rangle$ as building blocks. Still, it might be worth exploring the flows of different correlation functions for specific problems, whose setting admit more natural variables than the \tilde{I}_k .

Let us now come back to the remark on numerical stability at the end of section III B. In contradistinction to the flows (3.28),(3.39) the flow (3.60) relates the scale derivative of a correlation function to a polynomial of the full propagator, field derivatives of the effective action and the correlation function itself. In most cases both sides of the flow (3.60) are bounded, ensuring numerical stability and hence better convergence towards physics [77]. A notable exception is the case where the Legendre transform from W_k to $\Gamma_k + \Delta S'_k$ is singular. This either hints at a badly chosen truncation, or it relates to physical singularities that show up in the propagator G , see also [56]. In the scale-regime where such a singularity occurs one might switch back to the flow of W_k or $S_{\text{eff}k}$ [92]. In the vicinity of $S_{\text{eff},\mathbf{a}\mathbf{b}} \approx 0$ the flows (3.41),(3.42) are bounded.

2. Flow of the effective action

As in the case of the flow equation for I_k we describe the content and the restrictions of (3.60) within basic examples. From its definition (3.43) it follows that its flow is closely related to that of W_k ,

$$\partial_t \Gamma_k[\phi] - \partial_t W_k[J] - \Delta S'[\phi, \dot{R}] = 0, \quad (3.62)$$

where we have used (3.45) for $j^{\mathbf{a}}(\phi_{\mathbf{a}} - W_{k,\mathbf{a}}[J]) = 0$. Inserting the flow (3.34) for the Schwinger functional we are led to

$$\partial_t \Gamma_k[\phi] - (\Delta S[G \frac{\delta}{\delta\phi} + \phi, \dot{R}]) + \Delta S'[\phi, \dot{R}] = 0. \quad (3.63)$$

More explicitly it reads

$$\begin{aligned} & \partial_t \Gamma_k[\phi] - \dot{R}^{\mathbf{a}} \phi_{\mathbf{a}} - \sum_{n \geq 2} \dot{R}^{\mathbf{a}_1 \dots \mathbf{a}_n} \\ & \times \left[(G \frac{\delta}{\delta\phi} + \phi)_{\mathbf{a}_1} \dots (G \frac{\delta}{\delta\phi} + \phi)_{\mathbf{a}_{n-1}} - \phi_{\mathbf{a}_1} \dots \phi_{\mathbf{a}_{n-1}} \right] \phi_{\mathbf{a}_n} \\ & = 0. \end{aligned} \quad (3.64)$$

The explicit form of the flow (3.64) allows us to read off the one particle irreducibility of $\Gamma_k[\phi]$ as a consequence of that of the classical action $S[\phi]$: the flow preserves

³ Strictly speaking, one also has to use that the span of 1PI \tilde{I} generates all 1PI quantities.

irreducibility and hence it follows recursively from that of $S[\phi]$.

As for the Schwinger functional there is no \hat{I} with $\Delta\tilde{I} = 0$ leading to $\tilde{I}_k = \Gamma_k$. The related consistency equation reads

$$\begin{aligned} \Delta\tilde{I}_k[\phi] &= \Delta S_2[\phi, \dot{R}]\Gamma_k \\ &+ (\Delta S[G_{\frac{\delta}{\delta\phi}} + \phi, \dot{R}]) - \Delta S'[\phi, \dot{R}]. \end{aligned} \quad (3.65)$$

The right hand side of (3.65) does not vanish for all ϕ implying $\Delta\tilde{I}_k \neq 0$. Moreover, in general (3.65) cannot be deduced from a $\Delta\hat{I}$ polynomial in the current J and its derivatives. Again this highlights the necessity of restricting \tilde{I}_k to those constructed from (3.14) and (3.51).

Similarly to the derivation of the flow of W_k we can derive the flow of Γ_k , (3.63), from that of its derivative, $\Gamma_k^{\cdot\mathbf{a}}$. We use $\hat{I}^{\mathbf{a}} = \gamma^{\mathbf{a}\mathbf{b}}J^{\mathbf{b}}$. The corresponding \hat{I}_k derived from (3.14b) as $\hat{I}_k^{\mathbf{a}} = \gamma^{\mathbf{a}\mathbf{b}}J^{\mathbf{b}} - \Delta S^{\cdot\mathbf{a}}[J, R]$. The second operator $\Delta S^{\cdot\mathbf{a}}$ originates from the commutator term $\gamma^{\mathbf{a}\mathbf{b}}[\Delta S, J^{\mathbf{b}}]$. The commutator gives the right ϕ -derivative of $\Delta S[G_{\frac{\delta}{\delta J}} + \phi, R]$ at fixed J , see appendix C. Contracted with $\gamma^{\mathbf{a}\mathbf{b}}$ we arrive at the left derivative, where we have also used the bosonic nature of ΔS . The corresponding \tilde{I}_k reads with (3.45)

$$\tilde{I}_k = \Gamma_k^{\cdot\mathbf{a}} + \Delta S^{\cdot\mathbf{a}}[\phi, R] - (\Delta S^{\cdot\mathbf{a}}[G_{\frac{\delta}{\delta\phi}} + \phi, R]), \quad (3.66)$$

Moreover, $\Delta\tilde{I}_k = 0$. The choice (3.66) boils down to $\tilde{I}_k = \Gamma_k^{\cdot\mathbf{a}}$ in the standard case. For general flows the last term on the right hand side of (3.66) is non-trivial by itself. Indeed, its flow can be separately studied and follows from $\hat{I} = \Delta S^{\cdot\mathbf{a}}[G_{\frac{\delta}{\delta J}}, R]$ and $\hat{F} = \partial_t \Delta S^{\cdot\mathbf{a}}[G_{\frac{\delta}{\delta J}}, R]$. This leads to $\Delta\hat{I} = \Delta S^{\cdot\mathbf{a}}[G_{\frac{\delta}{\delta J}}, \dot{R}]$ and $\Delta\tilde{I}_k = (\Delta S^{\cdot\mathbf{a}}[G_{\frac{\delta}{\delta\phi}} + \phi, \dot{R}])$. Inserting this into the flow (3.60) we are led to

$$\begin{aligned} &(\partial_t + \Delta S_2[\phi, \dot{R}]) (\Delta S^{\cdot\mathbf{a}}[G_{\frac{\delta}{\delta\phi}} + \phi, R]) \\ &= (\Delta S^{\cdot\mathbf{a}}[G_{\frac{\delta}{\delta\phi}} + \phi, \dot{R}]). \end{aligned} \quad (3.67)$$

The above equation describes the flow of the functional $(\Delta S^{\cdot\mathbf{a}}[G_{\frac{\delta}{\delta\phi}} + \phi, R])$ at fixed second argument R . Using (3.67) within the flow of \tilde{I}_k of (3.66) it reads

$$\begin{aligned} \partial_t \Gamma_k^{\cdot\mathbf{a}} &= -\Delta S_2[\phi, \dot{R}] (\Gamma_k + \Delta S')^{\cdot\mathbf{a}} \\ &- \Delta S^{\cdot\mathbf{a}}[\phi, \dot{R}] + (\Delta S^{\cdot\mathbf{a}}[G_{\frac{\delta}{\delta\phi}} + \phi, \dot{R}]). \end{aligned} \quad (3.68)$$

Eq. (3.68) looks rather complicated. However, note that ΔS_2 acts on the current as $(\Gamma_k + \Delta S'_k)^{\cdot\mathbf{a}} = \gamma^{\mathbf{a}\mathbf{b}}J^{\mathbf{b}}$, see (3.45). Hence the evaluation of (3.68) is simplified if representing $\Delta S_2[\phi, \dot{R}]$ in terms of J -derivatives as all higher

J -derivatives vanish. To that end we use that the sum of all derivative terms in either ϕ or J coincide as in both cases it is given by the operator $\Delta S - (\Delta S)$. The latter can be written as the sum of all terms with two and more derivatives, ΔS_2 and the linear derivative terms, $\Delta S_{\mathbf{a}} \frac{\delta}{\delta\phi_{\mathbf{a}}}$ and $\Delta S^{\mathbf{a}}[J, \dot{R}] \frac{\delta}{\delta J^{\mathbf{a}}}$ respectively. This leads us to

$$\begin{aligned} \Delta S_2[\phi, \dot{R}] &= \Delta S_2[J, \dot{R}] \\ &- (\Delta S_{\mathbf{a}}[\phi, \dot{R}]) \frac{\delta}{\delta\phi_{\mathbf{a}}} + (\Delta S^{\mathbf{a}}[J, \dot{R}]) \frac{\delta}{\delta J^{\mathbf{a}}}. \end{aligned} \quad (3.69)$$

The validity of (3.69) follows from the above considerations, but also can be directly proven by inserting (3.50) in the first term on the right hand side. Using the representation (3.69) of $\Delta S_2[\phi, \dot{R}]$ in (3.68), only the terms in the second line of (3.69) survive as $(\Delta S_2[J, \dot{R}] J) = 0$. Furthermore $\Delta S^{\mathbf{c}}[J, \dot{R}] \frac{\delta}{\delta J^{\mathbf{c}}} \gamma^{\mathbf{a}\mathbf{b}} J^{\mathbf{b}} = (\Delta S^{\cdot\mathbf{a}}[G_{\frac{\delta}{\delta\phi}} + \phi, \dot{R}])$, and (3.68) reduces to

$$\partial_t \Gamma_k^{\cdot\mathbf{a}} = (\Delta S_{\mathbf{b}}[\phi, \dot{R}]) (\Gamma_k + \Delta S')^{\cdot\mathbf{b}\mathbf{a}} - \Delta S^{\cdot\mathbf{a}}[\phi, \dot{R}]. \quad (3.70)$$

Both terms on the right hand side of (3.70) are total derivatives w.r.t. $\phi_{\mathbf{a}}$. For the first term this follows with (3.46) and it reduces to $(\Delta S[G_{\frac{\delta}{\delta\phi}} + \phi, \dot{R}])^{\cdot\mathbf{a}}$. With this observation we arrive at

$$\partial_t \Gamma_k^{\cdot\mathbf{a}} = [(\Delta S[G_{\frac{\delta}{\delta\phi}} + \phi, \dot{R}]) - \Delta S'[\phi, \dot{R}]]^{\cdot\mathbf{a}}, \quad (3.71)$$

which upon integration yields (3.63).

3. Standard flow

The standard flow relates to regularisations $\Delta S_k[\phi]$ quadratic in the fields $\phi_{\mathbf{a}}$. We also restrict ourselves to bosonic R 's, that is no mixing of fermionic and bosonic fields in the regulator. Then, the flow of \tilde{I}_k can be directly read off from (3.60)

$$\partial_t \tilde{I}_k[\phi] + (G \dot{R} G)_{\mathbf{bc}} \tilde{I}_k^{\cdot\mathbf{cb}}[\phi] = 0. \quad (3.72)$$

The flow equation for Γ_k is extracted from $\tilde{I}_k^{\cdot\mathbf{a}} = \Gamma_k^{\cdot\mathbf{a}}$. This \tilde{I}_k can be constructed from $\hat{I}^{\mathbf{a}} = \gamma^{\mathbf{ba}}J_{\mathbf{b}}$: we get $\hat{I}_k^{\cdot\mathbf{a}} = \gamma^{\mathbf{ba}}J_{\mathbf{b}} - R^{\mathbf{ab}} \frac{\delta}{\delta J^{\mathbf{b}}}$. Inserting this operator into (3.14) we arrive at $\tilde{I}_k^{\cdot\mathbf{a}} = \gamma^{\mathbf{ba}}J_{\mathbf{b}} - R^{\mathbf{ab}}\phi_{\mathbf{b}} = \Gamma_k^{\cdot\mathbf{a}}$. Its flow is read off from (3.60) as

$$\partial_t \Gamma_k^{\cdot\mathbf{a}} = -(G \dot{R} G)_{\mathbf{bc}} \Gamma_k^{\cdot\mathbf{cba}} = [G_{\mathbf{bc}} \dot{R}^{\mathbf{bc}}]^{\cdot\mathbf{a}}, \quad (3.73)$$

where we again have used (3.46). The flow (3.73) matches (3.56) and can be trivially integrated in ϕ ,

$$\dot{\Gamma}_k = G_{\mathbf{bc}} \dot{R}^{\mathbf{bc}}, \quad (3.74)$$

where we have put the integration constant to zero. Eq. (3.74) is the standard flow equation of Γ_k as derived in [10] (up to the normalisation $\frac{1}{2}$ absorbed in R). It matches the flow of W_k , (3.34), when using (3.46) and the definition of Γ_k in (3.53)

$$\partial_t \Gamma_k[\phi] = -\partial_t W_k[J] - \dot{R}^{\mathbf{ab}} \phi_{\mathbf{a}} \phi_{\mathbf{b}} = \dot{R}^{\mathbf{bc}} G_{\mathbf{bc}}. \quad (3.75)$$

In (3.75) we have used that $(\partial_t J^{\mathbf{a}})(\phi_{\mathbf{a}} - W_{k,\mathbf{a}}) = 0$, see (3.23). Note that we could have used (3.75) instead of evaluating $\tilde{I}_k = \phi$ for deriving (3.57) with help of

$$\begin{aligned} \partial_t \gamma^{\mathbf{ba}} J_{\mathbf{b}}[\phi] &= \partial_t (\Gamma_k)^{\cdot \mathbf{a}} + \dot{R}^{\mathbf{ab}} \phi_{\mathbf{b}} \\ &= (\partial_t \Gamma_k)^{\cdot \mathbf{a}} + \dot{R}^{\mathbf{ab}} \phi_{\mathbf{b}}. \end{aligned} \quad (3.76)$$

The derivatives in (3.76) commute as the partial t -derivative is taken at fixed ϕ . Indeed, it is the flow of the Schwinger functional W_k which is at the root of both derivations. The flow of W_k equals that of the effective Lagrangian $S_{\text{eff},k}$ generating amputated connected Green functions. The relation between the flows $\partial_t \Gamma_k$ and $\partial_t S_{\text{eff},k}$, in particular the (in-)equivalence within truncations, has been explored in [12, 13, 45, 77, 96, 97], see also the reviews [17–19, 21–24]. The numerical stability of the flows has been compared in [77].

Finally let us study the consistency condition (3.65) in the present case. It reads $\partial_t \Gamma_k + (G \dot{R} G)_{\mathbf{bc}} \Gamma_k^{\cdot \mathbf{bc}} = 0$, which does not match (3.74). Hence there is no \hat{I}_k leading to $\tilde{I}_k = \Gamma_k$ and $\Delta \tilde{I}_k = 0$. Again this underlines the importance of (3.14) for devising flows: first one constructs an I_k or \tilde{I}_k from (3.14). Their flow is given by (3.28) and (3.60) respectively.

4. Initial condition for general flows

For 1PI correlation functions \tilde{I}_k the lhs of (3.60) consists of 1PI graphs in the full propagator G . Furthermore, (3.60) is only one loop exact if $\partial_x^2 \Delta S_k[x]$ does not depend on x , that is for $n \leq 2$, see also [54]. For $n = 2$ the flow (3.60) boils down to the flow (3.72), whereas $\partial_t \tilde{I}_k = 0$ for $n = 1$. For $n > 2$ we have higher loop terms in (3.60). Appropriately chosen $R^{\mathbf{a}_1 \dots \mathbf{a}_n}$ render all loops finite. In the class of R that provide momentum cut-offs, these loops can be localised about the cut-off scale. Then the flows (3.60) are finite and numerically tractable, sharing most of the advantages with the standard flow (3.72) with $n = 2$. Indeed, for specific physical problems, in particular theories with non-linear symmetries, the general choice in (3.60) can pay-off. However, we emphasise

that for general flows the limit $R \rightarrow \infty$ has to be studied carefully. Here, it is understood that $R \rightarrow \infty$ entails a specific limit procedure characterised by some parameter, i.e. the standard momentum regularisation $R_k \rightarrow \infty$ for $k \rightarrow \Lambda$. For practical purposes an accessible limit of the effective action Γ_k is required as it usually serves as the initial condition for the flow. In particular regulator terms $\Delta S_k[\phi]$ that, after appropriate field rescaling, tend towards finite expressions which are more than quadratic in the fields require some care. The general case can be classified as follows. For a regularisation ΔS of a theory with classical action $S[\hat{\phi}]$ and a given limit procedure $R \rightarrow \infty$ we can find field transformations $\hat{\phi} \rightarrow f(R)\hat{\phi}$ with $f(R \rightarrow \infty) = 0$ that render $S + \Delta S$ finite:

$$\lim_{R \rightarrow \infty} (S + \Delta S)[f(R)\hat{\phi}] = \hat{S}[\hat{\phi}]. \quad (3.77)$$

For R that diverge for all $\hat{\phi}$ -modes \hat{S} only depends on ΔS . In the standard case with $\hat{\phi} = \hat{\varphi}$ and $\hat{S}[\hat{\phi}] = \hat{S}^{\mathbf{ab}} \hat{\phi}_{\mathbf{a}} \hat{\phi}_{\mathbf{b}}$ with field-independent $\hat{S}^{\mathbf{ab}}$, the effective action Γ_k tends towards the classical action S of the theory⁴. In general, the corresponding effective action Γ_k tends towards

$$\Gamma[\phi, R] \rightarrow S[\varphi(\phi)] + \left| \det \frac{\partial \phi}{\partial \varphi}(\phi) \right| + \hat{\Gamma}[\phi], \quad (3.78a)$$

where $\hat{\Gamma}$ is given by

$$\hat{\Gamma}[\phi] = -\ln \left(\int [d\hat{\phi}] e^{-\hat{S}[\hat{\phi} + \phi] + \hat{S}[\phi] + \hat{\phi}_{\mathbf{a}} \hat{S}^{\cdot \mathbf{a}}[\phi]} \right), \quad (3.78b)$$

and $[d\hat{\phi}]$ is the flat $\hat{\phi}$ -measure including renormalisation effects. The term in the exponent comprises the Taylor expansion of $\hat{S}[\phi + \hat{\phi}]$ about ϕ leaving out the first two terms,

$$-\sum_{n \geq 2} \frac{1}{n!} \hat{\phi}_{\mathbf{a}_n} \dots \hat{\phi}_{\mathbf{a}_1} \hat{S}^{\mathbf{a}_1 \dots \mathbf{a}_n}. \quad (3.79)$$

The representation (3.78b) relates $\hat{\Gamma}$ to a Wilsonian effective action. We emphasise that $\langle \hat{\phi} \rangle \neq \phi$, the mean field computed from the path integral (3.78b) is not the original mean field. Indeed we compute

$$\hat{\Gamma}^{\cdot \mathbf{a}}[\phi] = (\phi_{\mathbf{a}} - \langle \hat{\phi}_{\mathbf{b}} \rangle) \hat{S}^{\cdot \mathbf{ba}}. \quad (3.80)$$

Eq. (3.80) also entails that $\hat{\Gamma}$ has no classical part due to the classical action $\hat{S}[\phi]$ in the exponent in (3.78b). Only

⁴ More precisely all power-counting irrelevant couplings tend to zero.

those limits $(\Delta S_k, \hat{S})$ admitting the computation of the effective action $\hat{\Gamma}$ in (3.78b) provide suitable initial conditions for the flow (3.60). They lead to consistent flows as defined in [54]. The standard limit $(\Delta S_k, S^{\mathbf{ab}}\phi_{\mathbf{a}}\phi_{\mathbf{b}})$ leads to a ϕ -independent $\hat{\Gamma}$: the explicit integration of (3.78b) gives $\frac{1}{2} \ln \det S^{\mathbf{ab}}$ up to renormalisation terms stemming from $[d\hat{\phi}]$. Such a flow was coined complete flow in [54] as it connects the classical action with the full effective action. The flow provides for the complete integration of quantum effects, and the theory is determined by the parameters in the classical action S . The requirement of convexity of the effective action constrains the set of parameters in S , which can be evaluated with help of the regulator dependence, see [56]. The general case (3.78) with non-trivial, but accessible $\hat{\Gamma}$ was coined consistent flow. Eq. (3.78) also covers the interesting class of proper-time flows [57–61], where $\hat{\Gamma}$ comprises a full non-trivial quantum theory [52–55]. A detailed discussion of the general situation will be given elsewhere.

D. General variations

In the previous sections we have studied one-parameter flows (3.11). These flows can be used to compute observables in the full theory starting from simple initial conditions like the classical or perturbation theory. For the question of stability of the flow or its dependence on background fields present in the regulator we are also interested in general variations (3.9) of the regulator. In particular functional optimisation as introduced in section V is based on studying general variations w.r.t. R . These variations are also useful for the investigation of physical instabilities [56]. They can be straightforwardly derived with the generalisation of (3.16):

$$\hat{F}[J, \frac{\delta}{\delta J}] = \delta R^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n}} \hat{I}[J, \frac{\delta}{\delta J}], \quad (3.81)$$

with

$$\Delta \hat{I}[J, \frac{\delta}{\delta J}] = \left[\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n}}, \hat{I}[J, \frac{\delta}{\delta J}] \right]. \quad (3.82)$$

The corresponding $\hat{F}[J, \frac{\delta}{\delta J}, R]$ follows with the commutator

$$\begin{aligned} & \left[\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n}}, R^{\mathbf{b}_1 \dots \mathbf{b}_n} \frac{\delta}{\delta J^{\mathbf{b}_1}} \dots \frac{\delta}{\delta J^{\mathbf{b}_n}} \right] \\ &= \delta R^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta J^{\mathbf{a}_1}} \dots \frac{\delta}{\delta J^{\mathbf{a}_n}} \end{aligned} \quad (3.83)$$

as

$$\begin{aligned} & \hat{F}[J, \frac{\delta}{\delta J}, R] \\ &= \left(\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n}} + \Delta S[\frac{\delta}{\delta J}, \delta R] \right) \hat{I}[J, \frac{\delta}{\delta J}, R]. \end{aligned} \quad (3.84)$$

With (3.81) and (3.84) the derivation of one-parameter flows in the previous sections directly carries over to the present case. Therefore we read off the response of I_k and \tilde{I}_k to general variations from (3.28) and (3.60) respectively:

$$\begin{aligned} & \left(\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n}} + \Delta S_1[J, \delta R] \right) I[J, R] \\ &= \Delta I[J, \delta R], \end{aligned} \quad (3.85)$$

and

$$\begin{aligned} & \left(\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n}} + \Delta S_2[\phi, \delta R] \right) \tilde{I}[\phi, R] \\ &= \Delta \tilde{I}[\phi, \delta R], \end{aligned} \quad (3.86)$$

with $\Delta \tilde{I}[\phi, \delta R] = \Delta I[J(\phi), \delta R]$. For the choice $R = R(k)$ and $\delta R = dt \dot{R}$ the flows (3.85), (3.86) reduce to the one-parameter flows (3.28), (3.60).

IV. RENORMALISATION GROUP FLOWS

A. RG flows of general correlation functions

The flows (3.28) and (3.60) comprise the successive integrating-out of degrees of freedom in a general quantum theory. The standard example is the integration of momentum modes, but the formalism allows for more general definition of modes. The current J and the regulator R couple to $\hat{\phi}(\varphi)$, which is not necessarily the fundamental field $\hat{\phi} = \varphi$. In any case, with R we have introduced a further scale k , thus modifying the RG properties of the theory. Moreover, at any infinitesimal flow step $k \rightarrow k - \Delta k$ there is a natural k -dependent reparameterisation of the degrees of freedom. Taking this reparameterisation into account should improve numerical stability. Hence the appropriate choice of \tilde{I}_Λ at the initial scale Λ is affected by the proper book keeping of the anomalous scaling, which becomes crucial in the presence of fine-tuning problems. It also is relevant for studying fixed point solutions of the flow. Hence the representation of RG rescalings in the presence of a regulator is a much-studied subject, e.g. [40–50, 124–127].

From the formal point of view canonical transformations on the functional space govern both RG rescalings and general flows presented here. This point of view shall be developed elsewhere. In most practical applications an appropriate k -dependent RG rescaling is simply incorporated by hand, see reviews [16–27, 29]. We emphasise that contrary to claims in the literature the incorporation of RG rescalings is not a matter of consistency but rather one of numerical stability and optimisation. We will come back to this issue later in chapter V.

The formalism introduced in the previous chapter allows us to derive RG equations in the presence of the regulator. In general we deal with theories that depend on a number of fundamental couplings \vec{g} , which also includes mass parameters. We are interested in the response of the theory to an infinitesimal total scale change of some scale s , e.g. $s = k$, the flow parameter, or $s = \mu$, where μ is the renormalisation group scale of the full theory. The couplings and the currents may depend on this scale, $\vec{g} = \vec{g}(s)$, $J^{\mathbf{a}} = J^{\mathbf{a}}(s)$. An infinitesimal variation is introduced by the operator $s \frac{d}{ds}$. Here we consider a general linear operator D_s with

$$D_s = s \partial_s + \gamma_g^i g_i \partial_{g_j} + \gamma_J^{\mathbf{a}} J^{\mathbf{b}} \frac{\delta}{\delta J^{\mathbf{a}}}, \quad (4.1)$$

with

$$D_s W = 0,$$

where the partial s -derivative is taken at fixed J (and \vec{g}), see appendix B, and the anomalous dimensions γ_J do not mix fermionic and bosonic currents. With J -independent matrices γ we only consider linear dependences of the currents. More general relations are easily introduced but should be studied separately in the specific situation that requires such a setting. Still we remark that non-linear relations can be reduced to linear ones by coupling additional composite operators to the currents. A relevant non-trivial example for (4.1) is $s \partial_s = \mu \partial_\mu$ with renormalisation scale (or cut-off scale) μ of $W[J, 0]$ and γ_g, γ_J the corresponding anomalous dimensions of couplings and fields respectively. We also could use $s \partial_s = \mu \partial_\mu + \partial_t$. We emphasise that the operator D_s accounts for more than multiplicative renormalisation. The matrices γ_g, γ_J are not necessarily diagonal and the multi-index \mathbf{a} possibly includes composite operators. Hence (4.1) naturally includes the renormalisation of composite operators (e.g. in NPI flows) or effects due to additive renormalisation. The operator D_s does not commute with derivatives w.r.t. J . Still $D_s W = 0$ can

be easily lifted to identities for general N -point functions with

$$D_s W_{\mathbf{a}_1 \dots \mathbf{a}_N} = (D_s W)_{\mathbf{a}_1 \dots \mathbf{a}_N} - \sum_{i=1}^N \gamma_J^{\mathbf{b}} \mathbf{a}_i W_{\mathbf{a}_1 \dots \mathbf{a}_{i-1} \mathbf{b} \mathbf{a}_{i+1} \dots \mathbf{a}_N}, \quad (4.2)$$

where we have used the commutator

$$[D_s, \frac{\delta}{\delta J^{\mathbf{a}}}] = -\gamma_J^{\mathbf{b}} \mathbf{a} \frac{\delta}{\delta J^{\mathbf{b}}}. \quad (4.3)$$

The derivation of D_s -flows for functionals I_k is done along the same lines as that of the t -flow in section III. First we define an operator \hat{F} similarly to (3.16) with

$$\hat{F} = D_s \hat{I} \quad \text{and} \quad \Delta \hat{I} = [D_s, \hat{I}]. \quad (4.4)$$

With $D_s W_k = 0$ it follows that $F_k = \Delta I_k$ which does not vanish in general. We shall use that still $\Delta I_k = 0$ for $\hat{I} = 1$. The only further input needed is the commutator of the regulator term ΔS with the differential operator D_s defined in (4.1). For its determination we compute

$$[\gamma_J^{\mathbf{a}} J^{\mathbf{b}} \frac{\delta}{\delta J^{\mathbf{a}}}, R^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta J^{\mathbf{a}_1}} \dots \frac{\delta}{\delta J^{\mathbf{a}_n}}] = -n \gamma_J^{\mathbf{a}_1} \mathbf{b} R^{\mathbf{b} \mathbf{a}_2 \dots \mathbf{a}_n} \frac{\delta}{\delta J^{\mathbf{a}_1}} \dots \frac{\delta}{\delta J^{\mathbf{a}_n}}. \quad (4.5)$$

where we have used the symmetry properties (3.4) of R . Eq. (4.5) enables us to compute the commutator $[D_s, \Delta S]$. For the sake of brevity we introduce a short hand notation for the symmetrised contraction of γ with R ,

$$(\gamma_J T)^{\mathbf{a}_1 \dots \mathbf{a}_n} = \sum_{i=1}^n \gamma_J^{\mathbf{a}_i} \mathbf{b} T^{\mathbf{a}_1 \dots \mathbf{a}_{i-1} \mathbf{b} \mathbf{a}_{i+1} \dots \mathbf{a}_n}, \quad (4.6)$$

for a given n . The commutator of ΔS with the differential operator D_s takes the simple form

$$[D_s, \Delta S[\frac{\delta}{\delta J}, R]] = \Delta S[\frac{\delta}{\delta J}, (D_s - \gamma_J)R]. \quad (4.7)$$

With the above preparations the derivation of the RG flow boils down to simply replacing \hat{R} in the commutator (3.19) with $(D_s - \gamma_J)R$ and allowing for a non-zero $F_k = \Delta I_k$. We finally arrive at

$$(D_s + \Delta S_1[\frac{\delta}{\delta J}, (D_s - \gamma_J)R]) I_k = \Delta I_k, \quad (4.8)$$

where $\Delta \hat{I} = [D_s, \hat{I}]$. The term ΔI_k contains the s -scaling inflicted by the operator \hat{I} , and $\Delta S_1 I_k$ contains the additional scaling inflicted by the operator ΔS . In summary (4.8) comprises general scalings in the presence of the regulator, and reduces to the flow (3.28) for $s = k$, up

to an additional k -dependent RG rescaling. We also emphasise that for the derivation of (4.8) only the linearity of the operator D_s has been used.

An explicit example for the content of (4.8) is provided by the RG equation of N -point functions $I_k^{(N)} = \langle \phi_{\mathbf{a}_1} \cdots \phi_{\mathbf{a}_N} \rangle$ as defined in (2.11). Then $D_s = D_\mu$, implementing RG rescalings in the full theory. Furthermore we assume that the operator ΔS does not spoil the RG invariance of the theory, i.e. the commutator (4.7) vanishes. The requirements on the regulator R leading to a vanishing commutator are further evaluated in the next section IV B. The RG equation for I_k is read off from (4.8) as $(D_\mu + N\gamma_J^{\mathbf{b}\mathbf{a}_1})(I_k^{(N)})_{\mathbf{b}\mathbf{a}_2 \cdots \mathbf{a}_N} = 0$, where ΔI_k produces the explicit scaling $N\gamma$ of the N -point function. This is the usual RG equation for N -point functions as expected. For connected N -point functions it is put down in (4.2).

The general equation (4.8) simplifies in the case of quadratic regulators,

$$\left(D_s + [(D_s - \gamma_J)R]^{\mathbf{ab}} \frac{\delta}{\delta J^{\mathbf{a}}} \frac{\delta}{\delta J^{\mathbf{b}}} + 2\phi^{\mathbf{a}} [(D_s - \gamma_J)R]^{\mathbf{ab}} \frac{\delta}{\delta J^{\mathbf{b}}} \right) I_k = \Delta I_k, \quad (4.9)$$

where

$$[(D_s - \gamma_J)R]^{\mathbf{ab}} = D_s R^{\mathbf{ab}} - 2\gamma_J^{\mathbf{a}\mathbf{c}} R^{\mathbf{cb}}. \quad (4.10)$$

In the last equality in (4.10) we have used $R^{\mathbf{ab}} = (-1)^{\mathbf{ab}} R^{\mathbf{ba}}$ and the fact that γ_J does not mix fermionic and bosonic currents. The general s -scaling of the Schwinger functional for quadratic regulator is derived similarly to the flow (3.37): we use $\hat{I} = \frac{\delta}{\delta J^{\mathbf{a}}}$ which leads to $\Delta I_k = -\gamma_J^{\mathbf{b}\mathbf{a}} W_{k,\mathbf{b}}$. Moreover we have $D_s W_{k,\mathbf{a}} + \gamma_J^{\mathbf{b}\mathbf{a}} W_{k,\mathbf{b}} = (D_s W_k)_{,\mathbf{a}}$. Inserting this into (4.8) we arrive at

$$[D_s W_k + (G_{\mathbf{bc}} + \phi_{\mathbf{b}}\phi_{\mathbf{c}}) [(D_s - \gamma_J)R]^{\mathbf{bc}}]_{,\mathbf{a}} = 0. \quad (4.11)$$

Upon integration we are led to

$$D_s W_k = -(G_{\mathbf{ab}} + \phi_{\mathbf{a}}\phi_{\mathbf{b}}) [(D_s - \gamma_J)R]^{\mathbf{ab}}. \quad (4.12)$$

Eq. (4.12) entails the response of the theory to a general scaling including the flow (3.28) as well as RG rescalings. For $s = \mu$ (4.12) expresses the modification of the RG equation $D_\mu W[J, 0] = 0$ in the presence of the regulator.

B. RG flows in terms of mean fields

We proceed by turning (4.8) into an equation formulated in terms of 1PI quantities and fields. This is done

by repeating the steps in the derivation of (3.60), and hence we shorten the details. First we lift (3.54) to operators D_s . This requires the definition of the action of D_s on functionals $F[\phi]$ as provided in appendix B:

$$D_s = (s\partial_s + \gamma_g^i g_i \partial_{g_j} + \gamma_\phi^{\mathbf{b}\mathbf{a}} \phi_{\mathbf{b}} \frac{\delta}{\delta \phi_{\mathbf{a}}}). \quad (4.13)$$

With $\tilde{I}_k = I_k[J(\phi)]$ we rewrite $D_s \tilde{I}_k$ in terms of \tilde{I}_k as

$$D_s \tilde{I}_k[\phi] = D_s I_k[J] + ((D_s - \gamma_J)J[\phi])^{\mathbf{a}} G_{\mathbf{ab}} \tilde{I}_k^{\mathbf{b}}[\phi], \quad (4.14)$$

where $(\gamma_J J)^{\mathbf{a}} = \gamma_J^{\mathbf{a}\mathbf{b}} J^{\mathbf{b}}$. In (4.14) we have used that $D_s|_J I_k = D_s I_k - (\gamma_J J[\phi])^{\mathbf{a}} I_{k,\mathbf{a}}$. In case D_s stands for a total derivative w.r.t. s , the second term on the right hand side of (4.14) has to vanish, $(D_s - \gamma_J)J = 0$. Then, keeping track of dependences on ϕ or J is irrelevant. With (4.14) we get

$$(D_s + \Delta S_2[\phi, (D_s - \gamma_J)R]) \tilde{I}_k[\phi] = \Delta \tilde{I}_k - \Delta_{\mathbf{b}} \tilde{I}_k^{\mathbf{b}}, \quad (4.15a)$$

where

$$\Delta_{\mathbf{b}} = (D_s J)^{\mathbf{a}} G_{\mathbf{ab}} + (\Delta S_{\mathbf{b}}[\phi, (D_s - \gamma_J)R]). \quad (4.15b)$$

We emphasise that γ_ϕ in $D_s \tilde{I}_k$ is at our disposal. Now, as in the case of the t -flow for \tilde{I}_k , we simplify the above equation by solving it for $\tilde{I}_k = \phi$ following from $\hat{I}_k = \frac{\delta}{\delta J}$. Then, $D_s \tilde{I}_k = \gamma_\phi \phi$ and $\Delta \tilde{I}_k = -\gamma_J \phi$. This leads to

$$\Delta_{\mathbf{b}} = -(\gamma_\phi + \gamma_J)^{\mathbf{a}\mathbf{b}} \phi_{\mathbf{a}}. \quad (4.16)$$

Inserting this into (4.15) the D_s -flow equation for \tilde{I}_k reads

$$(D_s + \Delta_{\mathbf{a}} \frac{\delta}{\delta \phi_{\mathbf{a}}}) \tilde{I}_k + \Delta S_2[\frac{\delta}{\delta \phi}, (D_s - \gamma_J)R] \tilde{I}_k = \Delta \tilde{I}_k, \quad (4.17)$$

where

$$D_s + \Delta_{\mathbf{a}} \frac{\delta}{\delta \phi_{\mathbf{a}}} = s\partial_s + \gamma_g^i g_i \partial_{g_j} - \gamma_J^{\mathbf{b}\mathbf{a}} \phi_{\mathbf{b}} \frac{\delta}{\delta \phi_{\mathbf{a}}}. \quad (4.18)$$

The dependence on γ_ϕ has completely dropped out. Its rôle has been taken over by $-\gamma_J$. In other words, however we choose the fields ϕ to scale under D_s , the RG flow (4.15) shows its natural RG scaling induced by $D_s W = 0$ and $D_s J = \gamma_J J$. For the t -flows studied in section III this translates into $\partial_t \phi = 0$, corresponding to the natural choice $\gamma_\phi = 0$. As γ_ϕ is at our disposal we take the natural choice

$$\gamma_\phi = -\gamma_J, \quad (4.19)$$

for which $\Delta_{\mathbf{b}} \equiv 0$. With the choice (4.19) we arrive at

$$(D_s + \Delta S_2[\phi, (D_s + \gamma_\phi)R]) \tilde{I}_k[\phi] = \Delta \tilde{I}_k[\phi], \quad (4.20)$$

where $\Delta \tilde{I}$ derived from (3.14) with $\Delta \hat{I} = [D_s, \hat{I}]$. Eq. (4.20) is the ϕ -based representation of (4.8), and hence comprises general explicit and implicit scalings in the presence of the regulator. A special case are those s -scalings with $(D_s + \gamma_\phi)R = 0$ leading to $\Delta S_2 \tilde{I}_k = 0$. For these choices of the pairs (R, D_s) the s -scaling of the regularised theory remains unchanged in the presence of the regulator. If D_s stands for a scale-symmetry of the full theory such as the RG invariance with $s = \mu$, regulators with $(D_s + \gamma_\phi)R = 0$ preserve the RG properties of the full theory, see [48, 49]. We shall discuss this interesting point later in section VIII B.

The above equations (4.8),(4.20) can be straightforwardly lifted to include general variations (3.85),(3.86) by

$$D_s \rightarrow D_R = \delta R^{\mathbf{a}_1 \dots \mathbf{a}_n} \frac{\delta}{\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n}} \Big|_s + \delta s D_s \quad (4.21)$$

with variations $\delta R(k)$ about $R(k)$ and $\delta s(R, \delta R), s(R)$. The operator D_R stands for the total derivative w.r.t. R , hence using D_R in (3.86) simply amounts to rewriting a total derivative w.r.t R in terms of partial derivatives. These general variations are important if it comes to stability considerations of the flow as well as discussing fixed point properties.

We close this section by illustrating the content of the RG flow (4.20) within some examples. First we note that by following the lines of the derivation for the t -flow of Γ_k , (3.63), we can derive the RG flow of the effective action. It is given with the substitutions $\partial_t \rightarrow D_s$ and $\hat{R} \rightarrow (D_s + \gamma_\phi)R$ in (3.63). For quadratic regulators (4.20) reduces to

$$D_s \tilde{I}_k + (G [(D_s + \gamma_\phi)R] G)_{\mathbf{ab}} \tilde{I}_k^{\mathbf{ab}} = \Delta \tilde{I}_k, \quad (4.22)$$

where

$$[(D_s + \gamma_\phi)R]^{\mathbf{ab}} = D_s R^{\mathbf{ab}} + 2\gamma_\phi^{\mathbf{a}}{}_{\mathbf{c}} R^{\mathbf{cb}}. \quad (4.23)$$

The D_s -flow of the effective action Γ_k is derived with the choice $\hat{I}^{\mathbf{a}} = \gamma^{\mathbf{a}}{}_{\mathbf{b}} J^{\mathbf{b}}$. This leads to $\tilde{I}_k^{\mathbf{a}} = \Gamma_k^{\mathbf{a}}$ and $\Delta \tilde{I}_k^{\mathbf{a}} = \gamma_{\mathbf{J}}^{\mathbf{a}}{}_{\mathbf{b}} \Gamma_k^{\mathbf{b}}$. By also using the commutator $[D_s, \frac{\delta}{\delta \phi_{\mathbf{a}}}] = \gamma_{\mathbf{J}}^{\mathbf{a}}{}_{\mathbf{b}} \frac{\delta}{\delta \phi_{\mathbf{b}}}$ we are led to

$$[D_s \Gamma_k - G_{\mathbf{bc}} [(D_s + \gamma_\phi)R]^{\mathbf{bc}}]_{,\mathbf{a}} = 0. \quad (4.24)$$

This is trivially integrated and we arrive at

$$D_s \Gamma_k = G_{\mathbf{bc}} [(D_s + \gamma_\phi)R]^{\mathbf{bc}}, \quad (4.25)$$

where we have set the integration constant to zero. The lhs of (4.25) can be projected onto the anomalous dimensions γ with appropriate derivatives w.r.t. fields and momenta. Then the rhs is some linear combination of γ 's. These relations can be solved for the γ 's, see e.g. [42, 48, 49]. With the choice $s = \mu$ and $(D_\mu + \gamma_\phi)R = 0$ we are led to the equation $D_\mu \Gamma_k = 0$, the regularised effective action satisfies the RG equation of the full theory. This interesting case is further discussed in section VIII B.

V. OPTIMISATION

An important aspect concerns the optimisation of truncated flows. Optimised flows should lead to results as close as possible to the full theory within each order of a given systematic truncation scheme. This is intimately linked to numerical stability and the convergence of results towards physics as already mentioned in the context of RG rescalings in the last section. By now a large number of conceptual advances have been accumulated [68–79], and are detailed in sections VB, VC. In particular [72] offers a structural approach towards optimisation which allows for a construction of optimised regulators within general truncation schemes. Still a fully satisfactory set-up requires further work. In the present section we take a functional approach, which allows us to introduce a general setting in which optimisation can be accessed. This is used to derive a functional optimisation criterion, which admits the construction of optimised regulators as well as providing a basis for further advances.

A. Setting

The present derivation of flows is based on the existence of a *finite* Schwinger functional W and finite correlation functions $\mathcal{O}[\phi]$ for the full theory. These quantities are modified by the action of an R -dependent operator, $\mathcal{O}[\phi] \rightarrow \mathcal{O}[\phi, R]$ with $\mathcal{O}[\phi] = \mathcal{O}[\phi, 0]$, see section III A. One-parameter flows (3.86) connect initial conditions, that are well under control, with the full theory. For most theories these flows can only be solved within approximations. Typically truncated results for correlation functions $\mathcal{O}[\phi, 0]$ show some dependence on the chosen flow trajectory $R(k)$ not present for full flows by definition. Naturally the question arises whether we can single

out regulators $R(k)$ that minimise this non-physical regulator dependence.

Consider a general systematic truncation scheme: at each order of this systematic expansion we include additional independent operators to our theory, thus successively increasing the number of independent correlation functions. At each expansion step these correlation functions take a range of regulator-dependent values. This regulator dependence should be rather small if the truncation scheme is well adapted to the physics under investigation. In extremal cases the truncation scheme may only work for a sub-set of well-adapted regulators but fail for others. An optimisation of the truncation scheme is achieved if at each successive expansion step and for the set of correlation functions included in this step we arrive at values that are as close as possible to the physical ones of the full theory. In all cases such an optimisation of the truncation scheme is wished for as it increases the reliability and accuracy of the results, in the extremal case discussed above it even is mandatory.

General correlation functions $\mathcal{O}[\phi]$ are either given directly by $\tilde{I}[\phi]$ or can be constructed from them as the \tilde{I} include all moments of the Schwinger functional, $\tilde{I}^{(N)}$, see (2.11). From now on we restrict ourselves to $\tilde{I}[\phi]$. Most relations directly generalise to correlation functions $\mathcal{O}[\phi]$, in particular to physical observables, except those whose derivation exploits the flows of \tilde{I} . The constraint of quickest convergence can be cast into the form of an equation on the single iteration steps within a given truncation scheme. We expand a correlation function $\tilde{I}[\phi, R]$ in orders of the truncation

$$\tilde{I}_k^{(i)}[\phi, R] = \tilde{I}_k^{(i-1)}[\phi, R] + \Delta^{(i)}\tilde{I}_k[\phi, R], \quad (5.1)$$

where $\Delta^{(i)}\tilde{I}$ adds the contribution of the i th order. With adding the subscript k and keeping the variable R we wish to make explicit the two qualitatively different aspects of the R -dependence of $\tilde{I}^{(i)}[\phi, R]$. Firstly, the $\tilde{I}^{(i)}[\phi, R]$ depend on the functional form of $R(k)$ that singles out a path in theory space. Secondly, k is specifying that point on the path belonging to the value k of the cut-off scale ranging from $k/\Lambda \in [0, 1]$. If we could endow the space of theories with a metric, optimisation could be discussed locally as a stationary constraint at each k . The resulting flows are geodesic flows, and k turns into a geodesic parameter. For now we put aside the problem of defining a natural metric or norm on the space of theories, but we shall come back to this important point later.

The full correlation function in the physical theory is given by $\tilde{I}[\phi] = \tilde{I}_0^{(\infty)}[\phi, R]$ and shows no R -dependence

except for a possible R -dependent renormalisation group reparameterisation, not present for RG invariant quantities. Therefore, optimisation of a correlation function \tilde{I} at a given order i of an expansion scheme is simply minimising the difference

$$\min_{R(k)} \|\tilde{I}[\phi] - \tilde{I}_0^{(i)}[\phi, R]\| = \min_{R(k)} \left\| \sum_{n=i+1}^{\infty} \Delta^{(n)}\tilde{I}_0 \right\|, \quad (5.2)$$

on the space of one-parameter flows $R(k)$. An optimal trajectory $R_{\text{opt}}(k)$ is one where the minimum (5.2) is achieved. As already mentioned in the last paragraph, for the general discussion we leave aside the subtlety of specifying the norm $\|\cdot\|$. The constraint (5.2) also fixes the freedom of RG rescalings for a given \tilde{I} with fixed RG scheme in the full theory.

How can such an optimisation (5.2) be achieved? A priori we cannot estimate how close to physics the results are, that were obtained with a specific regulator and truncation step. If we could, we knew the physical results in the first place and there would be no need for any computation. Hence an optimisation of the i th order within a general truncation scheme has to be based either on structural aspects of the flow or on an evaluation of successive truncation steps; both procedures allow to evaluate (5.2) within the given i th order. For correlation functions \tilde{I} with

$$\left\| \sum_{n=i+1}^{\infty} \Delta^{(n)}\tilde{I} \right\| = \sum_{n=i+1}^{\infty} \|\Delta^{(n)}\tilde{I}\|, \quad (5.3)$$

we can reduce (5.2) to a constraint on $\tilde{I}^{(i)}$ at a given order i . The minimum in (5.2) is approached for regulators minimising each term $\|\Delta^{(n)}\tilde{I}\|$ separately. In this case optimised regulators $R_{\text{opt}}(k)$ are those with

$$\|\Delta^{(i)}\tilde{I}_0[\phi, R_{\text{opt}}(k)]\| = \min_{R(k)} \|\Delta^{(i)}\tilde{I}_0[\phi, R(k)]\|, \quad (5.4)$$

for almost all i, ϕ . Eq. (5.4) is the wished for relation applicable at each order of the truncation. Note that (5.4) also eliminates the freedom of a k -dependent RG scaling of general correlation functions. It picks out that implicit RG scaling which minimises the norm of $\Delta^{(i)}\tilde{I}_0$. One could argue that an optimisation with (5.4) possibly gives close to optimal convergence even if (5.3) is not strictly valid: in the vicinity of optimal regulators sub-leading orders $\tilde{I} - \tilde{I}^{(i+1)}$ are small in comparison to the leading rest term $\Delta^{(i)}\tilde{I}$ and a partial cancellation between them should not have a big impact on the optimisation. Still it is dangerous to rely on such a scenario. For its

importance we discuss the general situation more explicitly: assume that we deal with m_{\max} observables λ_m^{phys} , $m = 1, \dots, m_{\max}$, built off some set of $\tilde{I}[\phi, R]$'s. Examples are critical exponents, physical masses, particle widths etc.. Within the i th order of a given truncation scheme and a flow trajectory $R(k)$ we get $\lambda_m^{(i)}[R]$ taking values in an interval $[\lambda_m^{\min(i)}, \lambda_m^{\max(i)}]$. By construction the extremisation picks out either $\lambda_m^{\min(i)}$ or $\lambda_m^{\max(i)}$. This procedure entails an optimisation if $\lambda_m^{\text{phys}} \notin [\lambda_m^{\min(i)}, \lambda_m^{\max(i)}]$ (subject to the correct choice of the closest extremum). In turn, if $\lambda_m^{\text{phys}} \in [\lambda_m^{\min(i)}, \lambda_m^{\max(i)}]$ a procedure picking out the boundary points decouples from optimisation, only by chance it provides close to optimal results. Indeed this scenario is likely to be the standard situation at higher order of the truncation scheme. An indication for this case is the failure of finding coinciding extrema for all observables, in particular if these extrema are far apart. The resolution of this problem calls for an observable-independent optimisation based on (5.2).

The evaluation of the optimisation (5.2) is more convenient in a differential form. This equation can be directly derived from (5.2). However, there exists an alternative point of view which might also be fruitful: truncated flows may be amended with functional relations valid in the full theory. The hope is to carry over some additional information from the full theory that is not present in the truncation of the flow. This is the idea behind the use of symmetry relations such as STIs together with flows. In the context of optimisation the key relation is the regulator independence of the full theory. By construction the end-points of one-parameter flows $\tilde{I}[\phi] = \tilde{I}_0[\phi, R]$ are correlation functions in the full theory, being trivially independent of the path $R(k)$ in regulator space: k is a further variable of R and any local variation of such a path about a regulator $R^{\mathbf{a}_1 \dots \mathbf{a}_n k}$ does not change $\tilde{I}_0[\phi, R]$. Moreover, in section IV we have seen that there is the freedom of k -dependent RG scalings of the full theory, and the apparent independence of $\tilde{I}[\phi] = \tilde{I}_0[\phi, R]$ on the path $R(k)$ for full flows is expressed in the relation

$$\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n k} \frac{\delta \tilde{I}_0[\phi, R]}{\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n k}} = \delta(\ln \mu) D_\mu \tilde{I}_0[\phi, R], \quad (5.5)$$

for all $\tilde{I}[\phi]$. The variation on the lhs of (5.5) stands for the total derivative w.r.t. $R^{\mathbf{a}_1 \dots \mathbf{a}_n k}$ also including possible R -dependent RG scalings as in D_R , (4.21). The rhs of (5.5) accounts for a possible integrated R -dependence of the renormalisation scheme at $k = 0$: $\delta\mu(R, \delta R), \mu(R)$. For RG invariant $\tilde{I}[\phi]$ the rhs of (5.5) vanishes. For RG variant $\tilde{I}[\phi]$ the rhs can always be absorbed in an ap-

propriate redefinition of the variation w.r.t. R , though technically this might be difficult. The relation of (5.5) to the optimisation (5.2) is provided by enforcing (5.5) already for the i th order of the truncation scheme and absorbing the RG scaling on the rhs in an appropriate redefinition of the R -variation. Also assuming (5.3) we are led to

$$\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n k} \frac{\delta \|\tilde{I}_0[\phi, R] - \tilde{I}_0^{(i)}[\phi, R]\|}{\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n k}} = 0, \quad (5.6)$$

which is the differential form of (5.2). Eq. (5.5) is an integrability condition for the flow. Its relation to reparameterisations of the flow and the initial condition $\tilde{I}[\phi, R(\Lambda)]$ become more evident by using

$$\tilde{I}_0[\phi, R] = \tilde{I}_\Lambda[\phi, R] + \int_\Lambda^0 \frac{dk}{k} \partial_t \tilde{I}_k[\phi, R]. \quad (5.7)$$

Inserting (5.7) in (5.5) leads to

$$\begin{aligned} D_R \tilde{I}[\phi, R] \Big|_{R(\Lambda)} + \int_\Lambda^0 \frac{dk}{k} \partial_t \left[D_R \tilde{I}[\phi, R] \right]_{R(k)} \\ = \delta(\ln \mu) D_\mu \tilde{I}_0[\phi, R], \end{aligned} \quad (5.8)$$

with D_R defined in (4.21). The integrand in (5.8) is a total derivative, and with using that $\delta R|_{R=0} = \delta\mu$ the lhs in (5.8) equals the rhs. A variation of the initial regulator $R(\Lambda)$ in general entails that $\tilde{I}[\phi, R(\Lambda)]$ cannot be kept fixed by adjusting an appropriate RG scaling. For example, a different momentum dependence of $R(\Lambda)$ leads to different composite operators coupled to the theory via ΔS , and hence physically different theories. For sufficiently large regulators these differences are usually sub-leading. Neglecting this subtlety we conclude that in general a change of regulator with a vanishing rhs and fixed initial conditions $\tilde{I}[\phi, R(\Lambda)]$ entails a k -dependent RG scaling of the flow.

B. Principle of Minimum Sensitivity

For the sake of simplicity we only discuss couplings λ 's and not general functionals \tilde{I} or \mathcal{O} . Eq. (5.6), evaluated for one or several observables λ_m , $m = 1, \dots, m_{\max}$, at some order i of a given truncation scheme can be viewed as a constraint for truncated flows. This implies the search for local extrema of observables λ_m in regulator space. However, not knowing λ_{phys} we have to resort to (5.5), most conveniently written as

$$\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n k} \frac{\delta \lambda_m}{\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n k}} = 0. \quad (5.9)$$

Eq. (5.9) can be seen as a symmetry constraint as suggested in the last section or as an optimisation with the assumption (5.3). As a constraint, (5.9) can have several solutions or none (the extremum could be a point on the boundary in regulator space). Eq. (5.9) in its integral form, only allowing global changes along the full flow trajectory, is related to the principle of minimum sensitivity (PMS) [67], which has been introduced to the functional RG in [68], for further applications see [69–71]. Its limitations have been discussed in [74]. Practically such a PMS extremum has been evaluated by computing observables $\lambda_1, \dots, \lambda_{m_{\max}}$ for a class of regulators $R(\alpha_1, \dots, \alpha_j)$ labelled with $\alpha_1, \dots, \alpha_j$. Strictly speaking, m_{\max} should increase with the order i of the truncation, as the number of observables increase with the order i of the truncation scheme. The functional derivatives w.r.t. R turn into ordinary ones and we are left with the problem of finding a coinciding extremum for these λ . As already mentioned before, even if they exist at all, these extrema need not coincide. There are several options of how to proceed in such a situation. We can constrain the set of regulators by fixing the value of some $\lambda_1, \dots, \lambda_r$ to their physical value to all orders of the truncation, thereby sacrificing a part of the predictive power. Such a procedure resolves (if r is big enough) the above mentioned problem and the optimisation is done for the other observables $\lambda_{r+1}, \dots, \lambda_{m_{\max}}$ in this smaller set of regulators, see [68]. One also could argue that optimised values for each of these variables are obtained at their extrema. A regulator that optimises the flow of λ_1 is not necessarily optimising that for other λ_m . This idea has been used in [71] and in general requires the use of supplementary constraints. Both procedures have to be used with care as already discussed in general in the last section V A. Within the present explicit procedure this analysis hints at several short-comings: firstly, fixing the values of r observables does not necessarily lead to small flow operators ΔS_2 , and possibly constrains the values for $\lambda_{r+1}, \dots, \lambda_m$ to regions that are far from their physical values. Secondly, non-coinciding optimal regulators also could hint at a badly working truncation scheme, or badly chosen λ_m . We emphasise again that searching for a solution of (5.9) for some variable $\tilde{I}^{(i)}$ equals an optimisation (5.2) only as long as the physical value $\tilde{I}^{(\infty)}$ is not included in the range of possible values of $\tilde{I}^{(i)}$. It is mainly for this reason that an observable-independent optimisation is wished for.

C. Stability criterion

The above mentioned problems are also directly related to the fact that the preceding use of (5.5), (5.9) is not a constructive one; it does not allow us to devise an optimal regulator that limits the contribution of higher orders of the truncation by construction. Moreover, an optimisation as in section V B always involves considerable numerical effort. A constructive optimisation criterion, directly based on the fundamental optimisation condition (5.2) and on the structure of the functional RG, has first been suggested in [72]. The construction there also emphasises the link between optimisation, optimal convergence and global stability of the flows. We shall show later in section V D that the criterion developed in [72–76] relates to the local use of (5.5).

The key point in [72] is the observation that optimisation of any systematic expansion implies quickest convergence of the expansion towards physics. Consequently we can turn the question of optimisation into that of quickest convergence. The latter allows to devise constructive optimisation conditions. In [72] it was pointed out that for the standard flow (3.73) any such expansion includes an expansion in powers of the propagator $G = 1/(\Gamma_k^{(2)}[\phi] + R)$. Hence minimising the norm of the propagator G relates to stability and fastest convergence. Consider regulators introducing an IR cut-off with $R = R(p^2)$ as discussed at the end of section III A. The norm implicitly used in [72] is the operator norm on L_2 : $\|G[\phi_0, R]\| = \sup_{\|\psi\|_{L_2}=1} \{\|G[\phi_0, R]\psi\|_{L_2}\}$, where $\|\psi\|_{L_2} = (\int |\psi|^2)^{1/2}$ is the L_2 -norm. The norm $\|G[\phi_0, R]\|_{L_2}$ is directly related to the biggest spectral value of G at ϕ_0 , and hence is sensitive on the growth of the maximum of G^n for $n \rightarrow \infty$. A canonical choice for ϕ_0 is a field maximising $\|G[\phi, R]\|$ on the space of fields ϕ . Within a truncation scheme that uses an expansion in powers of the field a natural choice for ϕ_0 is the expansion point. Reformulating the optimisation criterion of [72] in the present setting leads to

$$\{R_{\text{stab}}\} = \left\{ R \quad \text{with} \quad \|G[\phi_0, R]\|_{L_2} \leq \|G[\phi_0, R']\|_{L_2} \right. \\ \left. \forall R' \quad \text{and} \quad R'(k_{\text{eff}}^2) = R(k_{\text{eff}}^2) = c k_{\text{eff}}^2 \right\}. \quad (5.10)$$

The normalisation constant c is at our disposal. The condition $R'(k_{\text{eff}}^2) = c k_{\text{eff}}^2$ is required for identifying a parameter $k'(k_{\text{eff}})$ at which the norm of the propagator is taken. Eq. (5.10) allows to construct optimised regulators for general truncations schemes, even though the

key demand of stability might necessitate supplementary constraints, see e.g. section VIII E. At a given order it singles out a set of stability inducing regulators as (5.10) does not restrict the shape of R_{stab} . An optimisation with (5.10) entails in the limit of large truncation order the PMS condition (5.5), if the latter admits a solution [74]. If the PMS condition has several solutions, by construction (5.10) is likely to pick out that closer to the physical value.

The criterion (5.10) has very successfully been applied to the derivative expansion [75, 76], where also the above statements have been checked. In its leading order, the local potential approximation (LPA), a particularly simple optimised regulator is provided by

$$R_{\text{opt}}(p^2) = (k^2 - p^2)\theta(k^2 - p^2), \quad (5.11)$$

where θ is the Heaviside step function. By now (5.11) is the standard choice in the field. It is a solution of (5.10) with $k_{\text{eff}}^2 = \frac{1}{2}k^2$ and $c = 1$. As a solution of (5.10) in LPA it only is optimised for the LPA but not beyond, as has been already remarked in [73]. Beyond LPA a solution to (5.10) has to meet the necessary condition of differentiability to the given order. The related supplementary constraint is provided in (8.42). Solutions to (5.11) with (8.42) exist, being simple enhancements of (5.11) [79]. We add that (5.11) works within truncation schemes where the full momentum dependence of correlation functions is included from the onset.

D. Functional optimisation

In summary much has been achieved for our understanding as well as the applicability of optimisation procedures within the functional RG. Still, the situation is not fully satisfactory, in particular given its key importance for the reliability of functional RG methods. In the present section we exploit the functional equation (5.5) to devise an optimisation criterion based on stability as well as discussing in more detail the link between stability-related criteria and the PMS condition. We also aim at the presentation of fundamental relations and concepts that are possibly helpful for making further progress in this area.

1. Local optimisation

So far we have only discussed the implications of (5.5) in its integrated form as done within the PMS optimi-

sation in section V B. Such a procedure always requires the integration of the flow and hence involves considerable numerical effort. On the practical side, the classes of regulators usually used for the PMS are not sufficiently dense for resolving the local structure: for the standard choice of a momentum regulator we parameterise quite generally $R(p^2) = p^2 r(x)$ with $x = p^2/k^2$. Then, a variation of R is a variation of r and as such an integral condition as it implies a variation at all scales k . Consequently a resolution of the local (in k and \mathbf{a}_1) information of (5.5) is only obtained for regulator classes $\{R\}$ which include as differences $R_1 - R_2$ smeared out versions δ_ϵ of the delta function in k : $(R_1 - R_2)^{\mathbf{a}_1 \dots \mathbf{a}_n k} \propto \delta_\epsilon(k - k_{\text{eff}})\Delta R^{\mathbf{a}_1 \dots \mathbf{a}_n}$. It is convenient to include these variations functionally: evaluating (5.5) for variations local in k we turn (5.5) into a local condition on $\tilde{I}[\phi, R]$. As such it is the local form of the integrability condition (5.5) and can be read off from (5.7) and (5.8),

$$\oint \delta \tilde{I}[\phi, R] = 0, \quad (5.12)$$

the integral in (5.12) describing a small closed curve in the space of regulators. Within truncations, (5.12) is a non-trivial, physically relevant constraint. For example, gradient flows cease to be gradient flows within truncations that violate (5.12). In turn, this property is kept intact if (5.12) is satisfied. A consequence of (5.5) and its local form (5.12) is

$$\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n k'} \frac{\delta \tilde{I}[\phi, R(k)]}{\delta R^{\mathbf{a}_1 \dots \mathbf{a}_n k'}} = D_R \tilde{I}[\phi, R(k)], \quad (5.13)$$

for all $\tilde{I}[\phi, R(k)]$ and variations δR that vanish at Λ . The right hand side in (5.13) accounts for a total scale variation of the end-point $R(k)$ with D_s as defined in (4.21). We emphasise again that (5.12) and (5.13) are non-trivial constraints within truncations. Moreover, at finite $k \neq 0, \infty$ the rhs in general does not agree with $\delta(\ln \mu) D_\mu \tilde{I}[\phi, R(k)]$ even for full flows, as already mentioned in section III A: firstly, a general variation w.r.t. R leads to the flow (4.20) with (4.21), a special case being the one parameter flow (3.60) with $\delta R = dk \partial_k R$ and $D_s = \partial_t$. Secondly, in the presence of two different regulator functions R, R' at some fixed scales k, k' the two regularised theories cannot completely agree as they differ by their coupling to different composite operators $\Delta S[\phi, R]$ and $\Delta S[\phi, R']$. Still it might be possible to identify hyper-surfaces of regularised theories at the same physical cut-off scale k_{eff} . So far k was just a parameter labelling one-parameter flows, only its end-point

$k = 0$ (and to some extent $R = \infty$) defining a specific theory. For $k \neq 0$ this is a priori not clear, the trivial example being two momentum regularisations $R(p^2)$ and $R'(p^2) = R(c^2 p^2)/c^2$. Obviously k cannot be the physical cut-off scale in both cases. In this trivial case it is simple to identify the relative effective cut-off scale for R, R' with $k = k_{\text{eff}}$ and $k'(k_{\text{eff}}) = ck_{\text{eff}}$. In general the natural relation $k'(k)$ is less obvious, apart from not being unique anyway. Nevertheless let us assume for the moment that we have overcome this subtlety. Then we can define a variation of R on hyper-surfaces $\{R_{\perp}\}_{k_{\text{eff}}} = \{R(k_{\text{eff}})\}$ regularising the theory under investigation at the same physical cut-off scale k_{eff} . Stability of the flow is achieved by minimising its action on the set $\{R_{\perp}\}$ and (5.13) translates into

$$\delta R_{\perp}^{\mathbf{a}_1 \cdots \mathbf{a}_n k'} \frac{\delta \tilde{I}[\phi, R]}{\delta R^{\mathbf{a}_1 \cdots \mathbf{a}_n k'}} \Bigg|_{R=R_{\text{stab}}} = \delta \ln \mu D_{\mu} \tilde{I}[\phi, R_{\text{stab}}], \quad (5.14)$$

lifting (5.5) to non-vanishing regulators. Eq. (5.14) is a non-trivial constraint already for full flows. Subject to a given foliation of the space of theories with $\{R_{\perp}\}$ for all cut-off scales k_{eff} , (5.14) entails maximal (in-)stability of the flow at its solutions R_{stab} . With (5.13) we rewrite (5.14) as

$$D_{R_{\perp}} \tilde{I}[\phi, R] \Big|_{R=R_{\text{stab}}} = 0, \quad (5.15)$$

where we have absorbed the RG rescaling on the rhs of (5.14) in $D_{R_{\perp}} = D_R(\delta R = \delta R_{\perp})$. A solution $R_{\text{stab}}(k)$ of (5.15) is achieved by varying the flows of variables \tilde{I} in regulator space. In its form (5.15) it cannot be used to construct regulators R_{stab} . To that end we have to rewrite (5.15) as a criterion on the flow operator ΔS_2 . This is done as follows: if a one-parameter flow $\tilde{I}_k[\phi] = \tilde{I}[\phi, R(k)]$ obeys the constraint (5.15) for all k , so must $\partial_t \tilde{I}_k$. Varying $\partial_t \tilde{I}_k$ with δR_{\perp} it follows with (3.60) and (5.15) that

$$\left(D_{R_{\perp}} \Delta S_2[\phi, \dot{R}] \right) \tilde{I}[\phi, R] \Big|_{R=R_{\text{stab}}} = 0, \quad (5.16)$$

where we have used that ∂_t and D_R commute up to RG scalings. For most practical purposes the RG scaling will be neglected and (5.16) boils down to

$$\delta R_{\perp}^{\mathbf{a}_1 \cdots \mathbf{a}_n} \frac{\delta \Delta S_2[\frac{\delta}{\delta \phi}, \dot{R}]}{\delta R^{\mathbf{a}_1 \cdots \mathbf{a}_n}} \tilde{I}[\phi, R] \Big|_{R=R_{\text{stab}}} = 0. \quad (5.17)$$

Finding a globally stable one-parameter flow $R_{\text{stab}}(k)$ amounts to demanding the validity of (5.16) for all \tilde{I}

and k . This implies that the variation of ΔS_2 in the directions δR_{\perp} has to vanish at all scales k and all index values $\mathbf{a}_1 \cdots \mathbf{a}_n$, that is pointwise zero. Clearly there is the danger of overconstraining the regulator. In practical applications we limit ourselves to a restricted set of \tilde{I} for which we solve (5.16). As any truncation scheme is based on the assumption of dominance of certain degrees of freedom the related $\{\tilde{I}_{\text{rel}}\}$ should be taken. Then the choices $R_{\text{stab}}(k)$ lead to extrema of the action of $\Delta S_2[\frac{\delta}{\delta \phi}, \dot{R}_{\text{stab}}]$ on $\{\tilde{I}_{\text{rel}}\}$ for all scales k . Such a flow, if it exists, is either most stable (minimal ΔS_2) or most unstable (maximal ΔS_2). Eq. (5.16) implements the PMS condition (5.5) on $\{\tilde{I}_{\text{rel}}\}$, as the k -flow vanishes identically at $k = 0$ and integrating (5.16) over all scales still is zero. We also emphasise that (5.16) defines local (in-)stability. We could have global extrema at the boundary of the hyper-surface $\{R_{\perp}\}$ defined with k_{eff} .

2. Optimisation and effective cut-off scale

So far we have not fixed the hyper-surfaces $\{R_{\perp}\}$ which amounts to the definition of a metric on the space of regularised theories. Before embarking on a discussion of natural definitions of such metrics we would like to elucidate the subtleties within a simple example: assume we restrict ourselves to the set of regulators given by a specific flow $R_{\text{base}}(k)$ and possibly momentum dependent RG rescaling of $R_{\text{base}}(k)$. Then the definition of a natural (relative) physical cut-off scale is uniquely possible; the set of regulators $\{R_{\perp}\}_k$ is defined by those regulators with correlation functions $\tilde{I}[\phi, R]$ that only differ by RG rescalings (fixed physics) from $\tilde{I}[\phi, R_{\text{base}}]$. Note in this context that the RG scalings also change the field ϕ . The $\{R_{\perp}\}_k$ cover the restricted space of regulators we started with, and by definition (5.16) is satisfied for all $R \in \{R_{\perp}\}_k$. This should be the case as their physical content is indistinguishable. In turn, if we had chosen another foliation the result would have been different. Then, necessarily $R(k), R(ck) \in \{R_{\perp}\}_k$ for at least one regulator R and (5.16) differentiates between them even though the one-parameter flows $R(ck)$ and $R(k)$ are the same. Suitable foliations are those where the hyper-surfaces $\{R_{\perp}\}$ do not contain such pathologies.

So far k is only a parameter that provides a scale ordering without identifying physical scales (except for $k = 0$). Consequently we have to answer the question of how to define the distance d of two points R and R' in theory space given by their set of correlation functions $I[\phi, R]$,

$I[\phi, R']$, or more generally $\mathcal{O}[\phi, R]$, $\mathcal{O}[\phi, R']$. To that end we define

$$d_{\mathcal{O}}[R, R'] = \sup_{\phi \in \mathcal{S}} \{ \|\mathcal{O}[\phi, R] - \mathcal{O}[\phi, R']\| \}, \quad (5.18)$$

where the supremum is taken in an appropriate space of fields \mathcal{S} , and we have to specify an appropriate norm $\|\cdot\|$. A natural choice for \mathcal{S} is the configuration space of the theory under investigation. However, the definition (5.18) only is useful if $d_{\mathcal{O}}$ is finite for almost all R, R' . This can be achieved by turning $\mathcal{O} \rightarrow f(\mathcal{O})$ in an operator or functional that has a spectrum that is bounded from below and above, e.g. $\mathcal{O} \rightarrow 1/(C + |\mathcal{O}|^2)$ with positive constant C . Alternatively, one can restrict the space of fields ϕ , e.g. with $\phi \in \mathcal{S}_C = \{\phi \mid \|\mathcal{O}[\phi, R]\|, \|\mathcal{O}[\phi, R']\| < C\}$. Here, the constant $C < \infty$ is introduced to get rid of singular fields with $\mathcal{O}[\phi, R] = \infty$ that possibly would render the distance $d = \infty$ for all R, R' . Obviously allowing for these fields would spoil the construction. We could also evaluate the norm in (5.18) for a specific configuration $\phi = \phi_0$ with $\mathcal{S} = \{\phi_0\}$. This is an appropriate choice if ϕ_0 could be singled out by the truncation scheme, e.g. as the expansion point in an expansion in powers of the field.

As general flows (4.20) for \tilde{I}_k depend on $\Gamma_k^{(n)}$ via ΔS_2 which is the crucial input for the optimisation, a natural choice for \mathcal{O} is the effective action $\mathcal{O}[\phi, R] = \Gamma[\phi, R] - \Gamma[0, R]$ ⁵, or its second derivative $\Gamma^{(2)}$. Of course, any correlation function \tilde{I}_k (or set of correlation functions) that entails the full information about the theory and has no explicit regulator dependence is as good as the above suggestion. From now on we drop the subscript \mathcal{O} , keeping it only if discussing a specific choice for \mathcal{O} . The distance d between two regularisation paths $R(k), R'(k')$ of a theory at the effective cut-off scale $k = k_{\text{eff}}$ is given by

$$d[R, R'](k) = \min_{k'} d[R(k), R'(k')], \quad (5.19)$$

which implicitly defines the relative effective cut-off scale $k'(k)$ as that k' for which the minimum (5.19) is obtained

$$d[R(k), R'(k'(k))] = d[R, R'](k). \quad (5.20)$$

In general $d[R, R'](k) = d[R', R](k'(k)) \neq d[R', R](k)$. A priori, $k'(k)$ is not necessarily continuous. Indeed one

can even construct pathological regulators that lead to discontinuities in $k'(k)$. In most theories such subtleties are avoided by using regularity restrictions on the regulators $R(k)$ such as monotonicity in k : $R(k) \leq R(k')$ for $k < k'$.

The basic building block of the flow operator ΔS_2 is the full propagator $G = 1/(\Gamma^{(2)} + R)$, and it would seem natural to use d_G . However, $d_G[R, R']$ does not qualify directly for measuring the distance: for physically close regularisations R, R' the distance $d_{\Gamma^{(2)}}[R, R']$ is necessarily small⁶. Then, $d_G[R, R']$ is determined by the difference $(R - R')$ evaluated in the regularised regime which has no physical implication. Still, d_G can be turned into a simple relation for the effective cut-off scale k_{eff} with

$$d_{G, \text{sup}}[R, \infty] = \|G[R]\|_{\text{sup}} = \frac{1}{Z_{\phi}} k_{\text{eff}}^{\dim_G}, \quad (5.21)$$

with

$$\|G[R]\|_{\text{sup}} = \sup_{\phi} \{ \|G[\phi, R]\|_{L_2} \}, \quad (5.22)$$

where the supremum is taken in configuration space. The norm $\|\cdot\|_{L_2}$ is the operator norm on L_2 already used for the criterion (5.10). In (5.21) \dim_G is the momentum dimension of G , e.g. $\dim_G = -2$ for bosons and $\dim_G = -1$ for fermions. Z_{ϕ} is the wave function renormalisation of the field ϕ , and makes the definition of k_{eff} invariant under RG rescalings. In most cases the norm (5.22) will be evaluated in momentum space where it reads explicitly

$$\|G[R]\|_{\text{sup}} = \sup_{\phi, \|\psi\|_2=1} \left\{ \left(\int_p |G[\phi, R]\psi|^2(p) \right)^{1/2} \right\}. \quad (5.23)$$

Note that the use of Z_{ϕ} is not necessary as long as one uniquely fixes the endpoint of the flows, the theory at vanishing regulator. If one allows for simultaneous RG rescalings of the flow trajectories the prefactor in (5.21) arranges for an RG invariant k_{eff} . For including relative RG rescalings of trajectories the supremum in (5.21) also has to be taken over RG transformations. For most practical purposes these more general scenarios are not of interest.

The expression $k_{\text{eff}}^{\dim_G}$ relates to the biggest spectral value the propagator $G[\phi, R]$ can achieve for all fields ϕ . Therefore k_{eff} is the smallest relevant scale and hence is

⁵ In (3.74) we have put an integration constant to zero, here we choose it to be $-\Gamma[0, R]$. At finite temperature the effective action Γ cannot be renormalised that way as $\Gamma[0, R]$ is related to the thermal pressure.

⁶ More precisely this applies to the distance $d_{f(\Gamma^{(2)})}[R, R']$ where the function $|f(x)|$ is bounded from above.

the effective cut-off. In the limit $k \rightarrow 0$ the effective cut-off scale k_{eff} tends towards the smallest mass scale in the theory ⁷.

As an example we study a scalar theory with R in the leading order derivative expansion: $\Gamma_k[\phi] = \int (\frac{1}{2}\phi p^2 \phi + V_k[\phi])$. For regulators providing a momentum cut-off we can adjust k as a physical cut-off scale by taking as a reference regulator the sharp cut-off

$$R_{\text{sharp}}(p^2) = p^2(1/\theta(p^2 - k^2) - 1). \quad (5.24)$$

For R_{sharp} it is guaranteed that k^2 is the momentum scale below which ϕ modes do not propagate. Inserting (5.24) in (5.21) with $Z_\phi = 1$, the effective cut-off scale is

$$k_{\text{eff}}(k) = \sqrt{k^2 + V_{k,\text{min}}^{(2)}}, \quad (5.25)$$

where $V_{k,\text{min}}^{(2)}$ is the minimal value for $V_k^{(2)}$. Hence, in theories with a mass gap the effective cut-off scale k_{eff} does not tend to zero but settles at the physical mass scale of the theory. In the present example $k_{\text{eff}}^2(k=0) = V_{0,\text{min}}^{(2)}$, the minimum of the second derivative of the full effective potential. Note that the full effective potential is convex and hence $V_{0,\text{min}}^{(2)} \geq 0$.

3. Optimisation criterion

The analysis of the previous two sections allows to put forward a general optimisation criterion in a closed form:

$$D_{R_\perp} \tilde{I}[\phi, R] \Big|_{R=R_{\text{stab}}} = 0, \quad (5.26a)$$

with

$$\{R_\perp\} = \left\{ R \text{ with } \|G[R]\|_{\text{sup}} = \frac{1}{Z_\phi} k_{\text{eff}}^{\text{dim}_G} \right\}, \quad (5.26b)$$

where $\tilde{I}[\phi, R]$ are correlation functions in the given order of the truncation. The norm $\|\cdot\|_{\text{sup}}$ and the effective cut-off k_{eff} have been introduced in (5.21). For the sake of completeness of the definition (5.26) we recall its properties here: dim_G is the momentum dimension of G , and the effective cut-off k_{eff} is related to the biggest spectral

value of the propagator $k_{\text{eff}}^{\text{dim}_G}/Z_\phi$. The norm in (5.26) is the supremum of the L_2 operator norm,

$$\|G[R]\|_{\text{sup}} = \sup_\phi \{\|G[\phi, R]\|_{L_2}\}, \quad (5.27)$$

see also (5.23). If the theory or the truncation scheme admits a natural expansion point ϕ_0 , the supremum in (5.27) might be substituted by evaluating the propagator at ϕ_0 , e.g. a configuration ϕ_0 for which the minimum of the effective potential is achieved.

As shown in section VD 1, the constraint in (5.26) can be rewritten as the constraint of minimal action of ΔS_2 , (5.16):

$$\left(D_{R_\perp} \Delta S_2[\phi, \dot{R}] \right) \tilde{I}[\phi, R] \Big|_{R=R_{\text{stab}}} = 0. \quad (5.28)$$

The criterion (5.26) is not bound to specific truncation schemes. The trivial starting point at $R \equiv \infty$ is evaluated for $k_{\text{eff}}(R \equiv \infty) = \infty$ (assuming $d_g < 0$), the end-point at $R \equiv 0$ represents the mass gap of the theory, $k_{\text{eff}}(R \equiv 0) = (\|1/\Gamma^{(2)}\|_{\text{sup}})^{1/\text{dim}_G}$. The monotonic parameter k_{eff} defines the effective cut-off scale and interpolates between the classical theory at $k_{\text{eff}} = \infty$ and the full theory at $k_{\text{eff}}(0)$. If the theory undergoes a phase transition, in particular if it is first order, the monotonicity of $k_{\text{eff}}(k)$ within truncations is at stake. If this happens it hints at a truncation scheme that is not well-adapted. Nonetheless it can be dealt with in (5.26), it simply demands a more careful comparison of regulators at an effective cut-off scale defined by (5.27). Indeed, such pathologies can be avoided if restricting the space of regulators to those with monotonicity in k , $R(k) \leq R(k')$ for $k < k'$ which entails that regulators implement a true mode (scale) ordering. There are further secondary regularity constraints, but we do not want to overburden the criterion (5.26) with technicalities.

The general form of the optimisation criterion (5.26) is achieved by substituting $\|G\|_{\text{sup}}$ by a general norm $d_{\mathcal{O}}$ as defined in (5.19). For example, an interesting option can be found in [83]. In most cases the norm (5.27) applied to G supposedly is the natural choice: the propagator G is the key input in ΔS_2 , any iterative truncation scheme involves powers of G and hence the importance of its supremum is enhanced within each iteration step ⁸.

Even in its form (5.28) the optimisation criterion (5.26) shows some dependence on the correlation function un-

⁷ In a regime with anomalous momentum scaling $G \propto p^{\text{dim}_G - 2\kappa_\phi}$ one should rather define $\|G[R]\|_{\text{sup}} = k_{\text{eff}}^{\text{dim}_G - 2\kappa_\phi}/Z_\phi$ with dimensionful Z_ϕ .

⁸ First investigations within LPA reveal the suggested equivalence of different choices for $d_{\mathcal{O}}$, see also [78].

der investigation. Bearing in mind the discussion about observable-independent optimisation we apply this idea to (5.28). First of all, ΔS_2 depends on $\Gamma^{(2)}$ (and possibly higher derivatives of Γ). The functional optimisation implies (5.26) for these correlation functions which maximises the physics content of ΔS_2 . Consequently the derivative in (5.28) is taken at

$$D_{R_\perp} \Gamma^{(2)} \Big|_{R_{\text{stab}}} = 0. \quad (5.29a)$$

Eq. (5.29a) facilitates the evaluation of (5.28) as it only requires the evaluation of derivatives w.r.t. the explicit R -dependence. An optimisation for almost all relevant correlation functions \tilde{I} within a given truncation order implies the vanishing of the operator $D_{R_\perp} \Delta S_2[\phi, \dot{R}]$ on the span of these \tilde{I} . Assuming that we can embed this span in a normed vector space $\mathcal{V}_{\tilde{I}}$ we arrive at a correlator-independent optimisation

$$\left\| D_{R_\perp} \Delta S_2[\phi, \dot{R}] \right\|_{R=R_{\text{stab}}} = 0, \quad (5.29b)$$

with (5.29a) with the operator norm $\|\cdot\|$ on $\mathcal{V}_{\tilde{I}}$. The optimisation (5.29) minimises the action of ΔS_2 on correlation functions \tilde{I} within a given truncation order. The representation (5.29) allows for a clear understanding of the result of the optimisation with the example of the two-point function. Eq. (5.29a) entails that for optimal regulators R_{stab} the spectrum of $\Gamma^{(2)}$ at the effective cut-off scale k_{eff} is as close as possible (for the set of regulators $R_\perp(k_{\text{eff}})$) to that of the full two-point function at $k=0$: the physics content of $\Gamma^{(2)}$ is optimised. It also implies a monotonic evolution of the spectral values of $\Gamma^{(2)}$ for optimal regulators. In case $\Gamma^{(2)}$ has negative spectral values at the initial scale, e.g. a non-convex potential, the above investigations lead to one k -independent spectral value, up to RG rescalings.

The criterion (5.26), (5.29) can be rewritten as a simple criterion on the full propagator and the full vertices. For its importance and for the sake of simplicity we concentrate on the standard flow (3.72) with

$$\begin{aligned} \Delta S_2 &= (G \dot{R} G)_{\text{bc}} \frac{\delta}{\delta \phi_c} \frac{\delta}{\delta \phi_b} \\ &= \partial_t |_{\Gamma^{(2)}} (G - G_0)_{\text{bc}} \frac{\delta}{\delta \phi_c} \frac{\delta}{\delta \phi_b}, \end{aligned} \quad (5.30)$$

where G_0 is an appropriate R -independent normalisation, that leads to well-defined insertions for correlation functions \tilde{I} if applying $(G - G_0) \frac{\delta^2}{\delta \phi^2}$. In the presence of a mass gap a possible choice is e.g. $G_0 = G[\phi, R=0]$. The partial t -derivative at fixed $\Gamma^{(2)}$ commutes with D_{R_\perp} at

R_{stab} . There, $D_{R_\perp} = D_{R_\perp} |_{\Gamma^{(2)}}$. Now we use that the second functional derivative $\delta^2 / \delta \phi^a \delta \phi^b$ does not vanish on almost all \tilde{I} . Therefore a vanishing norm (5.29b) implies

$$\|\partial_t |_{\Gamma^{(2)}} D_{R_\perp} |_{\Gamma^{(2)}} (G - G_0)\|_{R=R_{\text{stab}}} = 0. \quad (5.31)$$

The norm in (5.31) derives from the operator norm on $\mathcal{V}_{\tilde{I}}$, and hence is related to the truncation scheme. A solution of $\|D_{R_\perp} |_{\Gamma^{(2)}} (G - G_0)\|_{R=R_{\text{stab}}} = 0$ for all k implies a solution of (5.31). Consequently we search for extrema on the spectrum of the positive operator G . Now we use that the positive operator G vanishes identically for $R = \infty$ and tends towards the full propagator $G[\phi, R=0]$ with positive spectrum at vanishing regulator. Then with (5.29b) and (5.31) we conclude that optimal flows maximise G at a given k_{eff} for all spectral values, with the constraint that $-\partial_t G \geq 0$ is a positive operator. The latter constraint guarantees that the maximisation is globally valid for all k . We conclude that optimal flows are those where $G[\phi, R]$ is already as close as possible to the full propagator for a given cut-off scale k_{eff} . This criterion can be cast into the form

$$d_{\theta_\lambda(G)}[R_{\text{stab}}, 0] = \min_{R_\perp} d_{\theta_\lambda(G)}[R_\perp, 0], \quad (5.32a)$$

for all $\lambda \in \mathbb{R}^+$ with $\{R_\perp\}$ as defined in (5.26), and θ_λ is defined via its action on eigenvectors $|\psi_{\lambda_G}\rangle$ of G

$$\theta_\lambda(G) |\psi_{\lambda_G}\rangle = \left[\lambda + (\lambda_G - \lambda) \theta(\lambda - \lambda_G) \right] |\psi_{\lambda_G}\rangle, \quad (5.32b)$$

with Heaviside step function $\theta(x)$ ⁹. The operator θ_λ used in (5.32a) resolves the full spectral information of G . The criterion (5.32a) entails the constraint that $G[\phi, R_{\text{stab}}]$ takes the closest spectral values (according to the norm) to the full propagator $G[\phi, 0]$ for all $R \in \{R_\perp\}$, starting from the boundary condition $G[\phi, \infty] = 0$, or alternatively at $G[\phi, 0]$. This implies a minimisation of the flow, as well as monotonicity of the spectral values of G in k : $G[\phi, 0] \geq G[\phi, R]$. These considerations enable us to reformulate (5.32a) without relying on the full propagator $G[\phi, 0]$. We are led to

$$\|\theta_\lambda(\Gamma^{(2)}[R_{\text{stab}}] + R_{\text{stab}})\| = \min_{R_\perp} \|\theta_\lambda(\Gamma^{(2)}[R_\perp] + R_\perp)\| \quad (5.32c)$$

for all $\lambda \in \mathbb{R}^+$. We remark that in deducing (5.32c) from (5.32a) we have again used $-\partial_t G \geq 0$ and $\Gamma^{(2)}[R=0] \geq$

⁹ θ_λ is required to be a bounded operator. Hence for general norms used in $d[R, R']$ (5.32b) has to be modified, see e.g. section VIII E.

0. If the distance d is defined with the L_2 -norm in the given order of the truncation, (5.32c) is also conveniently written as

$$d_{\theta_\lambda(G)}[R_{\text{stab}}, \infty] = \max_{R_\perp} d_{\theta_\lambda(G)}[R_\perp, \infty]. \quad (5.33)$$

Note that in general (5.32c) can be written as (5.33) and some supplementary constraints depending on the norm used in (5.32c), see e.g. section VIII E. For each norm these supplementary constraints are straightforwardly derived from (5.32a).

Eq. (5.32) is a simple optimisation procedure independent of the correlation functions \tilde{I} under investigation. It already works without computations of full flow trajectories. In its form the criterion (5.26) has already been successfully applied to Landau gauge QCD [142, 143], see also section VIII C. We emphasise again that the appropriate norm relates to the truncation used. The above analysis extends to general regulators. There, one also has to take into account the evolution of higher vertices $\Gamma^{(n)}$. Their properties under R_\perp -variations at R_{stab} derive from (5.29a) by taking field-derivatives. Spectral considerations are more involved but it can be shown that an optimisation for general regulators implies (5.32).

We close the section with some comments concerning the generality of (5.26), the existence of solutions, and its connection to the criterion (5.10)¹⁰:

the definition of the set R_\perp in (5.26b) guarantees the existence of R_{stab} for a general expansion scheme: within any given truncation scheme the set of $\{R_\perp\}$ is bounded by possibly smooth modifications of the sharp cut-off and the optimal cut-off (5.11) as functions on the spectrum of $\Gamma^{(2)}$ and for spectral values $\lambda(\Gamma^{(2)}) \leq Z_\phi k_{\text{eff}}^{-d_G}$. Together with positivity and monotonicity of the regulators R this proves the existence of a stable solution of (5.26), if neglecting the R_\perp -variation of $\Gamma_k^{(2)}$. Indeed such a procedure defines a further truncation scheme on top of that at hand. Note also that possibly one has to introduce a λ -ordering: we search for a solution to (5.32a), (5.32c) for a given λ on the sub-space of solutions to (5.32a), (5.32c) for $\lambda' < \lambda$.

The argument above fails for generalisations of regulator functions where the demand of positivity and monotonicity of the regulator are dropped. Still, for reasonable choices the set R_\perp sweeps out basically the area

bounded by, possibly smooth modification, of the sharp cut-off and the optimal cut-off (5.11). However, it is not guaranteed anymore that the boundary curves are themselves in R_\perp . Therefore, a strict extremisation for all momenta (spectral values) as demanded in (5.26) might fail for generalisations of (5.26). More details will be provided elsewhere.

Both criteria, (5.10) and (5.26), are based on the same key idea of global stability. In (5.10) the set of regulators $\{R_\perp\}$ is defined by normalising the regulators at some momentum. Then the inverse gap $\|G[R, \phi_0]\|_{L_2}$ of the full propagator is minimised. In (5.26) the set of regulators $\{R_\perp\}$ is defined as those with the same maximal spectral value (inverse gap) $\|G[R]\|_{\text{sup}}$ and the action of the flow operator ΔS_2 is minimised. With (5.10) one is comparing regulators with different effective cut-off scales but, roughly speaking, close physics content. Then, optimal regulators are those where this physics content is achieved for the biggest effective cut-off scale. In turn, with (5.26) we compare regulators leading to the same effective cut-off scale and single out those that lead to correlation functions as close as possible to those in the full theory.

VI. APPLICATIONS TO FUNCTIONAL METHODS

In this chapter we discuss immediate structural consequences of the setting developed so far. First of all this concerns the interrelation of functional methods like the general flows studied here, Dyson-Schwinger equations [180–190], stochastic quantisation [191–193], and the use of NPI effective actions [194–213]. All these methods have met impressive success in the last decade, in particular if it comes to physics where a perturbative treatment inherently fails. Here, we discuss structural similarities as well as functional relations between these approaches that open a path towards a combined use as well as non-trivial consistency checks of respective results. We also highlight the important aspect of practical renormalisation schemes that can be derived from general flows for either DS equations or NPI methods. However, given the scope of the present work we only outline the relevant points, leaving a more detailed analysis to future work.

¹⁰ For its connection to the PMS condition (5.9) we refer the reader to the discussion below (5.17).

A. Functional RG and DS equations

1. DSEs as integrated flows

Formally Dyson-Schwinger equations (2.14) are integrated flows. They constitute finite functional relations between renormalised Green functions as well as bare vertices. They have been successfully used for the description of the infrared sector of QCD formulated in Landau gauge, initiated in [180, 181], for a review see [182]. This approach is also tightly linked to a similar analysis in stochastic quantisation [191–193].

More recently, these investigations have been extended to finite temperature QCD, e.g. [188] and the review [187]. The formal finiteness of the DS equations is more intricate if solving them within truncations [180–190]. Here, we discuss Dyson-Schwinger equations and their flow in the presence of a standard regulator coupled to the fundamental fields. This allows us to construct a general consistent BPHZ-type renormalisation of DS equations from integrated flows being valid beyond perturbation theory. The extension of the results to the general setting is straightforward.

Recall the DS operator \hat{I} given in (2.14) with $\hat{\phi} = \hat{\varphi}$, the source J coupled to the fundamental fields: $\hat{I}_{\text{DSE}} = J - \frac{\delta S}{\delta \hat{\phi}}$. Inserting this into (3.14) leads to

$$\tilde{I}_{\text{DSE}}^a[\phi, R] = \Gamma^a[\phi, R] - \langle S^a[\hat{\phi}] \rangle \equiv 0 \quad (6.1a)$$

with

$$\hat{I}_{\text{DSE}}^a[J, \frac{\delta}{\delta J}, R] = J^a - \frac{\delta S}{\delta \hat{\varphi}_a} - 2R^{ab} \frac{\delta}{\delta J^b}. \quad (6.1b)$$

Note that $\langle S^a[\hat{\phi}] \rangle$ in (6.1a) has to be read as a function of ϕ^a . The flow of \tilde{I}_{DSE} is given by (3.60) and reads

$$\left(\partial_t + \Delta S_2[\frac{\delta}{\delta \hat{\phi}}, \dot{R}] \right) \tilde{I}_{\text{DSE}} = 0. \quad (6.2)$$

The first term in the DSE (6.1), $\Gamma_k{}^b$, already satisfies (6.2), see (3.71). This leaves us with the separate flow

$$\left(\partial_t + \Delta S_2[\frac{\delta}{\delta \hat{\phi}}, \dot{R}] \right) \langle S^a \rangle = 0. \quad (6.3)$$

Eq. (6.3) also follows directly from considering $\hat{I} = S^a[\frac{\delta}{\delta J}]$. By construction the corresponding correlation function \tilde{I} satisfies the flow equation (3.60) and is given by $\tilde{I}[\phi, R] = \langle S^a \rangle$. From the above identities we also relate t -derivatives of $\Gamma_k{}^a$ and $\langle S^a \rangle$, i.e.

$$\partial_t \Gamma_k{}^a + \Delta S_2[\frac{\delta}{\delta \hat{\phi}}, \dot{R}] \langle S^a \rangle = 0, \quad (6.4a)$$

as well as

$$\partial_t \langle S^a \rangle + \Delta S_2[\frac{\delta}{\delta \hat{\phi}}, \dot{R}] \Gamma_k{}^a = 0. \quad (6.4b)$$

Eq. (6.4) highlights the aspect of the functional RG as a differential DSE. The use of the above identities (6.1), (6.2) and (6.3) is twofold. Firstly they allow us to relate DSEs and flow equations in similar truncations, hence providing non-trivial consistency checks for both approaches. Secondly they open a path towards a combined use of functional RGs and DSEs dwelling on the advantageous features of both. For example, an infrared analysis within both functional approaches usually provides a set of possible solutions whose intersection is possibly unique. In QCD this can be directly achieved by a fixed point analysis of (6.4a) along the lines in [142, 143]. The present idea has been put forward in [51], and applies to general theories. In Landau gauge such an analysis completely fixes the infrared scaling behaviour of all Green functions, [51].

2. Renormalisation

Furthermore the flow equation in its integrated form can be used to set up an explicit renormalisation procedure within general truncation schemes. Such a renormalisation is not necessarily multiplicative but generalises the BPHZ renormalisation of perturbation theory to general expansions. As it relies on a functional equation for the effective action its consistency is guaranteed by construction. Hence it is possible to derive consistent subtraction schemes for Dyson-Schwinger equations from the integrated flow in a given truncation.

We illustrate the above statements within the standard flow (3.75) for the effective action. Assume that we have solved the theory within the i th order of a given general truncation scheme, leading to $\Gamma_k^{(i)}$. Generally the flow can be written as

$$\partial_t \Gamma = \dot{R}^{ab} G_{ab} = -\frac{1}{2} \partial_t (\ln G)_{aa} - \frac{1}{2} \dot{\Gamma}_k{}^{ab} G_{ab}. \quad (6.5)$$

In its integrated form this leads to

$$\Gamma_k = \Gamma_\Lambda - \frac{1}{2} (\ln G)_{aa} \Big|_\Lambda^k - \frac{1}{2} \int_\Lambda^k dt \dot{\Gamma}_k{}^{ab} G_{ab}. \quad (6.6)$$

The integrated flow (6.6) represents an integral equation for the effective action Γ_k with the boundary condition Γ_Λ . Note that its solution for a given k requires its solution for $k' \in [k, \Lambda]$. As such it constitutes a Dyson-Schwinger equation. It provides an explicit (re)normalisation procedure involving two different

aspects. Firstly the choice of a finite boundary condition Γ_Λ implicitly renormalises the theory: it ensures finiteness. The renormalisation conditions for the full effective action, i.e. fixing the relevant operators (of Γ_0) at some renormalisation scale μ translate to similar conditions for Γ_k for all k . In particular its choice at $k' = 0$ relates to an appropriate normalisation at $k = \Lambda$. As can be seen from the representation of the integrated flow in (6.6) the renormalisation is done in a BPHZ-type way with subtractions $-\frac{1}{2} \ln G(k' = \Lambda) + (\Gamma_\Lambda - S)$, the t -integral also comprises some sub-leading subtractions.

With (6.6) we have resolved the notorious consistency problem for explicit renormalisation procedures within Dyson-Schwinger equations. Practically it can be solved within an iteration of Γ_k about some zeroth iteration step $\Gamma_{k,0}$ for $k \in \{\Lambda, 0\}$, e.g. $\Gamma_{k,0} = S_{\text{cl}}$, the classical action. This works for paths $R(k)$, for which the initial condition Γ_Λ is sufficiently close to the classical action, an example being regulators R implementing a momentum regularisation with Λ setting a high momentum scale.

An interesting option are non-trivial $\Gamma_{k,0}$ that already incorporate some non-trivial physics content of the theory under investigation. If the zeroth iteration step is already close to the full solution the numerical effort is minimised. Accordingly such a procedure benefits from any information already collected by other means about the physics content. In comparison to the standard (numerical) solution of DS-equations involving momentum integrations one has to perform an additional t -integration. In general this is bound to increase the numerical costs. However, this additional integral comes with the benefit that now the integrand is localised in momenta and t which stabilises the numerics. Indeed, the above ideas have been used for resolving the infrared sector of QCD within the Landau gauge thus furthering the evidence for the Kugo-Ojima/Gribov-Zwanziger confinement scenario in this gauge [142, 143], and providing a general consistent renormalisation procedure for related DS-studies [182, 183, 189, 190]. This aspect will be further discussed in section VIII. We also remark that the present analysis can be extended to the stochastic quantisation [191–193]. There it helps that we do not rely on an explicit path integral representation. This shall be detailed elsewhere.

Still the question arises whether (6.6) can be used more directly for setting up a renormalisation procedure for functional equations in the full theory at $k = 0$, solved iteratively within a given general truncation scheme

$$\Gamma_k^{(i)}[\phi, R] = \Gamma_k^{(i-1)}[\phi, R] + \Delta^{(i)}\Gamma_k[\phi, R], \quad (6.7)$$

as introduced in (5.1) for general \tilde{I}_k . Assume we have managed to construct regulators R that lead to a suppression of modes in the path integral related to orders $i > i_k$ of our truncation scheme. As an example we take the derivative expansion. Here we can use regulators that suppress at $k = \Lambda$ all momentum-dependent fields, $i_\Lambda = 0$. By decreasing k we add more and more derivatives, $i_k \rightarrow \infty$ with $k \rightarrow 0$, either continuously switching on their effects or adding more and more derivatives in discrete steps.

If R implements the truncation in discrete steps the flow only is non-zero at the discrete set of k_i . Integrating the flow from $k_i < k_1 < k_{i+1}$ and $k_{i+1} < k_2 < k_{i+2}$ we arrive at

$$\begin{aligned} \Gamma^{(i+1)} &= \Gamma^{(i)} - \frac{1}{2} \left((\ln G)_{aa}^{(i+1)} - (\ln G)_{aa}^{(i)} \right) \\ &\quad - \frac{1}{2} \int_{k_1}^{k_2} dt \dot{\Gamma}^{,ab} G_{ab}. \end{aligned} \quad (6.8)$$

Eq. (6.8) recursively implements the renormalisation at a given order $i + 1$ of the truncations by subtraction of appropriate terms of the order i . Naively the integral in (6.8) can be performed as $\dot{\Gamma}^{,ab}$ only is non-zero at k_{i+1} . However, this has to be done carefully for similar reasons to those that do not allow for a naive integration of sharp-cut-off flows: at k_{i+1} , the flow $\dot{\Gamma}^{,ab}$ is singular and G jumps. Nonetheless, as in the case of the sharp cut-off (6.8) can be easily integrated within explicit iteration schemes. For example, perturbation theory within BPHZ-renormalisation can be reproduced with (6.8) but it extends to general schemes as well as general functional relations and correlation functions \tilde{I} of the theory that require explicit renormalisation if it comes to truncations.

B. Composite operators and NPI flows

The analysis of the last section extends naturally to flows in the presence of composite operators, in particular to flows of NPI effective actions [194–196]. Flows with the coupling to composite operators have been considered in e.g. [23, 47, 85, 87–90]. Flows for the 2PI effective action have been studied in [85, 87, 90].

In the presence of sources for composite operators the renormalisation of these operators has to be taken into account. In particular, the construction of practical consistent renormalisation schemes within truncations poses a challenge, see e.g. [197–207]. Such a renormalisation

has to respect the symmetry and symmetry breaking pattern of the theory under investigation. We discuss the use of general flows for the construction of consistent subtraction schemes in general truncations by extending the renormalisation ideas of the last section. We also discuss the direct relation between flows in the presence of composite operators and NPI effective actions, relying on the interpretation of the regulator R as a source for a composite operator.

1. Linear flows

The structure of the flows (3.28),(3.60) always allows us to reduce the order of derivatives in ΔS_k at the expense of introducing further tensorial currents. In general we have

$$\begin{aligned} & \left(\frac{\delta}{\delta J^{\mathbf{a}_1 \cdots \mathbf{a}_n}} \frac{\delta}{\delta J^{\mathbf{a}'_1 \cdots \mathbf{a}'_m}} \right)^i \left[e^{J^{\mathbf{b}_1 \cdots \mathbf{b}_n} \hat{\phi}_{\mathbf{b}_1 \cdots \mathbf{b}_n} + J^{\mathbf{b}_1 \cdots \mathbf{b}_m} \hat{\phi}_{\mathbf{b}_1 \cdots \mathbf{b}_m}} \right. \\ & \quad \left. \times e^{J^{\mathbf{b}_1 \cdots \mathbf{b}_{n+m}} \hat{\phi}_{\mathbf{b}_1 \cdots \mathbf{b}_n} \hat{\phi}_{\mathbf{b}_{n+1} \cdots \mathbf{b}_{m+n}}} \right] \\ &= \left(\frac{\delta}{\delta J^{\mathbf{a}_1 \cdots \mathbf{a}_{n+m}}} \right)^i \left[e^{J^{\mathbf{b}_1 \cdots \mathbf{b}_n} \hat{\phi}_{\mathbf{b}_1 \cdots \mathbf{b}_n} + J^{\mathbf{b}_1 \cdots \mathbf{b}_m} \hat{\phi}_{\mathbf{b}_1 \cdots \mathbf{b}_m}} \right. \\ & \quad \left. \times e^{J^{\mathbf{b}_1 \cdots \mathbf{b}_{n+m}} \hat{\phi}_{\mathbf{b}_1 \cdots \mathbf{b}_n} \hat{\phi}_{\mathbf{b}_{n+1} \cdots \mathbf{b}_{m+n}}} \right], \quad (6.9) \end{aligned}$$

with $\mathbf{a}_{n+j} = \mathbf{a}'_j$. Eq. (6.9) is valid for all $i \in \mathbb{N}$. We also could have substituted only a part of the derivatives, obviously the relation is not unique. In case the source term $J^{\mathbf{b}_1 \cdots \mathbf{b}_{n+m}} \hat{\phi}_{\mathbf{b}_1 \cdots \mathbf{b}_n} \hat{\phi}_{\mathbf{b}_{n+1} \cdots \mathbf{b}_{m+n}}$ was not present in the Schwinger functional $W[J]$ it has to be added. Note that the derivatives w.r.t. t are taken at fixed arguments J and $\hat{\phi}$ respectively. Hence the reduction to lower powers of derivatives is accompanied by holding the corresponding Green functions fixed. With (6.9) a part of the regulator term (3.2) with n th order derivatives, is reduced to order $n-m+1$ by adding a further source term to $W[J]$

$$J^{\mathbf{a}} \hat{\phi}_{\mathbf{a}} \rightarrow J^{\mathbf{a}} \hat{\phi}_{\mathbf{a}} + J^{\mathbf{a}_1 \cdots \mathbf{a}_m} \hat{\phi}_{\mathbf{a}_1} \cdots \hat{\phi}_{\mathbf{a}_m} = J^{\mathbf{a}'} \hat{\phi}_{\mathbf{a}'}, \quad (6.10)$$

where

$$\gamma^{\mathbf{a}'\mathbf{b}'} = (\gamma \oplus (\otimes \gamma)^m)^{\mathbf{a}'\mathbf{b}'}, \quad (6.11)$$

with enlarged multi-indices $\mathbf{a}' = \mathbf{a}, \mathbf{a}_1 \cdots \mathbf{a}_m$ and $\gamma = (\gamma^{\mathbf{a}\mathbf{b}})$. Eq. (6.10) implies $\hat{\phi}_{\mathbf{a}_1 \cdots \mathbf{a}_m} = \hat{\phi}_{\mathbf{a}_1} \cdots \hat{\phi}_{\mathbf{a}_m}$. With

(6.10) we are led to

$$\begin{aligned} & \left(R^{\mathbf{a}_1 \cdots \mathbf{a}_n} \frac{\delta}{\delta J^{\mathbf{a}_1}} \cdots \frac{\delta}{\delta J^{\mathbf{a}_n}} \right)^i e^{J^{\mathbf{a}'} \hat{\phi}_{\mathbf{a}'}} \\ &= \left(R^{\mathbf{a}'_1 \cdots \mathbf{a}'_{n-m+1}} \frac{\delta}{\delta J^{\mathbf{a}'_1}} \cdots \frac{\delta}{\delta J^{\mathbf{a}'_{n-m+1}}} \right)^i e^{J^{\mathbf{a}'} \hat{\phi}_{\mathbf{a}'}} \quad (6.12) \end{aligned}$$

with $R^{\mathbf{a}'_1 \cdots \mathbf{a}'_{n-m+1}} \hat{\phi}_{\mathbf{a}'_1} \cdots \hat{\phi}_{\mathbf{a}'_{n-m+1}} = R^{\mathbf{a}_1 \cdots \mathbf{a}_n} \hat{\phi}_{\mathbf{a}_1} \cdots \hat{\phi}_{\mathbf{a}_n}$. The above relation is not unique, and we could have further reduced the order of derivatives by identifying additional products $\hat{\phi}_{\mathbf{a}_1 \cdots \mathbf{a}_m} = \hat{\phi}_{\mathbf{a}_1} \cdots \hat{\phi}_{\mathbf{a}_m}$ for $n-m \geq m$. By recursively using (6.10),(6.12) with general m we can substitute ΔS by an expression with only quadratic derivative terms, and the flow reduces to the standard form of the flow equation (3.72). Reducing ΔS one step further we arrive at first order derivatives w.r.t. J and (3.60) boils down to

$$\partial_t \tilde{I}_k[\phi] = 0. \quad (6.13)$$

It seems that (6.13) is rather trivial but it should be read as a fixed point equation for the flow. When evaluating $\tilde{I}_k^{\mathbf{a}} = \gamma^{\mathbf{a}\mathbf{b}}(J^{\mathbf{b}} - R^{\mathbf{b}}) = \Gamma_k^{\mathbf{a}}$ resulting from $\hat{I}_k^{\mathbf{a}} = \gamma^{\mathbf{a}\mathbf{b}} J^{\mathbf{b}}$ the flow (6.13) reads

$$\partial_t \Gamma_k^{\mathbf{a}}[\phi] = \dot{R}^{\mathbf{a}}, \quad (6.14)$$

where the partial t -derivatives is taken at fixed fields $\phi_{\mathbf{a}}$. Eq. (6.14) yields upon integration

$$\partial_t \Gamma_k[\phi] = \dot{R}^{\mathbf{a}} \phi_{\mathbf{a}}, \quad (6.15)$$

which also can be read off from (3.64). If $R^{\mathbf{a}_1 \cdots \mathbf{a}_n} = 0$ for $n \neq 2$, (6.15) resembles the standard flow equation with $G \rightarrow \phi_{a_1 a_2}$, in particular for $\hat{\phi} = \hat{\phi}$. However, even for general n its integration is trivial: we exploit that for $k=0$ the regulator vanishes, $R=0$ and get

$$\Gamma_k[\phi] = \Gamma_0[\phi] + R^{\mathbf{a}} \phi_{\mathbf{a}} = \Gamma[\phi] + \Delta S_k[\phi]. \quad (6.16)$$

Eq. (6.16) can directly be obtained by evaluating the Legendre transformation (3.53) for the present scenario. For regulator terms linear in ϕ , $\Delta S_k[\phi] = R^{\mathbf{a}} \phi_{\mathbf{a}}$, there is a simple relation between the Schwinger functional of the full theory and that of the regularised theory: $W_k[J] = W_0[J - R]$. Moreover $\Delta S'_k[\phi] = 0$. With these observations we can rewrite (3.53) for linear ΔS_k as

$$\begin{aligned} \Gamma_k &= \sup_J (J^{\mathbf{a}} \phi_{\mathbf{a}} - W[J - R]) \\ &= \sup_J ((J - R)^{\mathbf{a}} \phi_{\mathbf{a}} - W[J - R]) + \Delta S_k \\ &= \Gamma + \Delta S_k. \end{aligned} \quad (6.17)$$

In (6.17) we have used that the supremum over the space of functions J is the same as that over the space of functions $J - R$. Strictly speaking, the last equality in (6.17) is only valid for the subset of regulators R that can be absorbed in currents J .

From the above definitions and the flow (6.15) we can step by step resolve the composite operators $\phi^{\mathbf{a}}$ by using the related equations of motion. Here we show how such a procedure can be used to finally recover the regularised effective action $\Gamma_k[\phi_a]$ in (3.53) and the general flows (3.60). The equations of motion for $\phi_{a_1 \dots a_{n_i}}$ for $n_i \geq 2$ read

$$\frac{\delta \Gamma_k[\phi]}{\delta \phi_{a_1 \dots a_{n_i}}} = 0, \quad \forall n_i \geq 2. \quad (6.18)$$

Using the solution $\bar{\phi}(\phi_a) = (\phi_a, \bar{\phi}_{a_1 a_2}, \dots, \bar{\phi}_{a_1 \dots a_{n_N}})$ of (6.18) in (6.16), we end up with the effective action (3.53). As $\Delta S'_k = 0$ for linear regulators we have

$$\Gamma_k[\phi_a] = \Gamma_k[\bar{\phi}] - \Delta S_k[\hat{\phi}(\phi_a)], \quad (6.19)$$

where $\Delta S_k[\hat{\phi}(\phi_a)] = \sum_i R^{a_1 \dots a_{n_i}} \hat{\phi}_{a_1 \dots a_{n_i}}[\phi_a]$. Due to the linearity of the t -derivative the flow (6.15) holds true also for the effective action $\Gamma_k[\phi_a]$. This statement reads more explicitly

$$\begin{aligned} \partial_t \Gamma_k[\phi_a] &= \partial_t|_{\bar{\phi}} \Gamma_k[\bar{\phi}] + \Gamma_k^{\mathbf{a}}[\bar{\phi}] \partial_t \bar{\phi}_{\mathbf{a}}[\phi_a] \\ &= \partial_t|_{\bar{\phi}} \Gamma_k[\bar{\phi}]. \end{aligned} \quad (6.20)$$

The second term on the rhs of the first line in (6.20) vanishes due to the equations of motion (6.18) for $n_i \geq 2$ and due to $\partial_t \bar{\phi}_a[\phi_a] = 0$ for the fundamental field $\bar{\phi}_a := \phi_a$, that is not a solution to the related equations of motion but a general field. Hence the flow equation for the 1PI effective action reads

$$\partial_t \Gamma_k[\phi_a] = \dot{R}^{\mathbf{a}} \bar{\phi}_{\mathbf{a}}[\phi_a]. \quad (6.21)$$

The equations of motion (6.18) relate the fields $\bar{\phi}_{\mathbf{a}}[\phi_a]$ to a combination of Green functions

$$\bar{\phi}_{\mathbf{a}}[\phi_a] = \langle \hat{\phi}_{\mathbf{a}}[\hat{\phi}_a] \rangle_{J^{\mathbf{a}}=(J^a, 0)}. \quad (6.22)$$

The relations (6.22) can be written in terms of functional ϕ -derivatives as

$$\bar{\phi}_{\mathbf{a}} = \left(\hat{\phi}_{\mathbf{a}} \left[G_{ab} \frac{\delta}{\delta \phi_b} + \phi_a \right] \right). \quad (6.23)$$

As an example we use (6.23) for the two-point function $\hat{\phi}_{a_1 a_2} = \hat{\phi}_{a_1} \hat{\phi}_{a_2}$ and $(\phi_{\mathbf{a}}) = (\phi_a, \phi_{a_1 a_2})$. It follows

$$\begin{aligned} \bar{\phi} &= \left(\phi_a, \left((G \frac{\delta}{\delta \phi} + \phi)_{a_1} (G \frac{\delta}{\delta \phi} + \phi)_{a_2} \right) \right) \\ &= (\phi_a, G_{a_1 a_2} + \phi_{a_1} \phi_{a_2}). \end{aligned} \quad (6.24)$$

Inserting (6.23) into the flow (6.21) we recover the flow (3.63). The relation (6.23) also leads to the general flows (3.60) starting at the trivial flow in (6.13), $\partial_t \tilde{I}_k = 0$. The flow for $\tilde{I}_k[\phi_a] = \tilde{I}_k[\bar{\phi}(\phi_a)]$ reads

$$\partial_t \tilde{I}_k[\phi_a] - \tilde{I}_k^{\mathbf{a}}[\bar{\phi}] \partial_t \bar{\phi}_{\mathbf{a}} = 0, \quad (6.25)$$

similarly to (6.20). In (6.25) we have used (6.13), there is no explicit t -dependence. In contradistinction to (6.20) the remaining term on the rhs of (6.25) does not vanish as general correlation functions do not satisfy the equations of motion (6.18). Note also that the fields $\bar{\phi}$ trivially satisfy the flows (6.25). The fields $\bar{\phi}(\phi_a)$ belong to the correlation functions \tilde{I}_k and hence they obey the flow equation

$$\partial_t \bar{\phi}_{\mathbf{a}}[\phi_a] + \Delta S_2[\phi_a, \dot{R}] \bar{\phi}_{\mathbf{a}}[\phi_a] = 0. \quad (6.26)$$

Inserting (6.26) into (6.25) we arrive at the flow

$$\partial_t \tilde{I}_k[\phi_a] + (\Delta S_2 \bar{\phi}_{\mathbf{a}}) \tilde{I}_k^{\mathbf{a}}[\bar{\phi}] = 0, \quad (6.27)$$

which implies (3.60). The latter statement follows only after some algebra from (6.27). For its proof one has to consider that ΔS_2 acts linearly on \tilde{I}_k which it does not on general correlation functions \mathcal{O}_k ¹¹. However, it is more convenient to work with the flow (3.28) for $I_k[J^{\mathbf{a}}]$ and with the definition $I_k[J^a] = I_k[J^{\mathbf{a}} = (J^a, 0)]$. By using the equivalence of J -derivatives (6.9) valid for the I_k , the flow for $I_k[J^a]$ derives from that of $I_k[J^{\mathbf{a}}]$ as $(\partial_t + \Delta S_1[J^a, \dot{R}]) I_k[J^a] = 0$, implying the flow (3.60) for $\tilde{I}[\phi_a]$. It is worth noting that truncated flows derived from either the representation (3.60) or (6.27) differ. This fact can be used for consistency checks of truncations as well as an improvement in case one of the representations is better suited within a given truncation.

Accordingly there is a close link between NPI formulations of the effective action and general flows. Moreover, it is possible to switch back and forth between these formulations, thereby combining their specific advantages.

2. 2PI flows

As an explicit example we study the standard flow related to the quadratic regulator term

$$\Delta S_k[\frac{\delta}{\delta J}] = R^{ab} \frac{\delta}{\delta J^a} \frac{\delta}{\delta J^b}, \quad (6.28)$$

¹¹ The proof can be worked out for N -point functions (6.23) from where it extends straightforwardly.

which can be linearised in terms of 2PI quantities

$$\hat{\phi}_{a_1 a_2} = \hat{\phi}_{a_1} \hat{\phi}_{a_2}, \quad (6.29)$$

where $\hat{\phi}_a$ is not necessarily a fundamental field. For $\hat{\phi}_{a_1 a_2}$ as defined in (6.29) the relation (6.9) reads

$$\begin{aligned} & \frac{\delta}{\delta J^{a_1}} \frac{\delta}{\delta J^{a_2}} e^{J^b \hat{\phi}_b + J^{b_1 b_2} \hat{\phi}_{b_1} \hat{\phi}_{b_2}} \\ &= \frac{\delta}{\delta J^{a_1 a_2}} e^{J^b \hat{\phi}_b + J^{b_1 b_2} \hat{\phi}_{b_1} \hat{\phi}_{b_2}}, \end{aligned} \quad (6.30)$$

Using (6.30) we reduce (6.28) to a linear regulator at the expense of also keeping the corresponding 2-point functions fixed,

$$\partial_t \phi_{a_1 a_2} = \partial_t (G + \phi_{a_1} \phi_{a_2}) = \partial_t G = 0. \quad (6.31)$$

We substitute ΔS_k in (3.13),(3.14) with

$$\Delta S_k \left[\frac{\delta}{\delta J} \right] = R^{a_1 a_2} \frac{\delta}{\delta J^{a_1}} \frac{\delta}{\delta J^{a_2}} \rightarrow R^{a_1 a_2} \frac{\delta}{\delta J^{a_1 a_2}}, \quad (6.32)$$

and are lead to (6.13), $\partial_t \tilde{I}_k[\phi] = 0$. The effective action and its flow are functions of the field ϕ_a and the two-point function ϕ_{ab} :

$$\Gamma_k[\phi_{\mathbf{a}}] = \Gamma[\phi_{\mathbf{a}}] + R^{ab} \phi_{ab}, \quad (6.33)$$

with $(\phi_{\mathbf{a}}) = (\phi_{a_1}, \phi_{a_1 a_2})$ and

$$\partial_t \Gamma_k[\phi_{\mathbf{a}}] = \dot{R}^{ab} \phi_{ab}. \quad (6.34)$$

The flow (6.34) resembles the standard flow equation (3.60) and follows directly from the definition of Γ_k in (6.33). It also follows by integration w.r.t. ϕ from (6.14) with $\frac{\delta \dot{\Gamma}_k}{\delta \phi_{ab}} = \gamma^{ab a' b'} \dot{R}_{a' b'}$ and $\frac{\delta \dot{\Gamma}_k}{\delta \phi_a} = 0$. The equation of motion in ϕ_{ab} according to (6.18) is given by

$$\left. \frac{\delta \Gamma_k[\phi_{\mathbf{a}}]}{\delta \phi_{ab}} \right|_{\phi=\bar{\phi}} = 0. \quad (6.35)$$

Its solution (6.23) reads $\bar{\phi}_{\mathbf{a}} = (\phi_a, \bar{\phi}_{a_1 a_2})$ with

$$\bar{\phi}_{ab} = G_{ab} + \phi_a \phi_b. \quad (6.36)$$

The above relations lead to the standard flow equation for the 1PI effective action $\Gamma_k[\phi_a] = \Gamma_k[\phi_a, \bar{\phi}_{ab}] - R^{bc} \phi_b \phi_c$ defined in (6.19). With (6.35) it follows that [85, 87, 90]

$$\begin{aligned} \partial_t \Gamma_k[\bar{\phi}(\phi)] &= \partial_t|_{\bar{\phi}} \Gamma_k[\bar{\phi}] + \Gamma_k^{\mathbf{a}}[\bar{\phi}] \partial_t \bar{\phi}_{\mathbf{a}}[\phi] \\ &= \partial_t|_{\bar{\phi}} \Gamma_k[\phi, \bar{\phi}]. \end{aligned} \quad (6.37)$$

Using the flow (6.34) in (6.37) we arrive at

$$\partial_t \Gamma_k[\phi_a] = \dot{R}^{bc} G_{bc}, \quad (6.38)$$

the standard flow (3.74). Hence linear flows of 2PI quantities and its fixed point equations reflect the standard flow equation and offer the possibility of using 2PI expansions as well as results in standard flows.

3. Renormalisation

The setting in the present work hinges on the bootstrap idea that the path integral, more precisely the Schwinger functional $W[J, R]$, is finite and uniquely defined. Resorting to Weinberg's idea of non-perturbative renormalisability [36] this simply implies the existence of a finite number of relevant operators in the theory. If not only the fundamental fields $\hat{\phi} = \hat{\varphi}$ are coupled to the path integral but also general composite operators $\hat{\phi}^{\mathbf{a}}$ some care is needed. As an example let us consider $\hat{\phi}^4$ -theory in $d = 4$ dimensions in the presence of a source for $\hat{\phi}^6(x)$. More generally we deal with a Schwinger functional $W[J, R]$ with $J^{\mathbf{a}} \hat{\phi}_{\mathbf{a}} = J^a \hat{\varphi}_a + J^{a_1 \dots a_6} \hat{\varphi}_{a_1} \dots \hat{\varphi}_{a_6}$. The composite $\hat{\varphi}^6(x)$ operator is coupled with the choice $J^{a_1 \dots a_6} \hat{\varphi}_{a_1} \dots \hat{\varphi}_{a_6} = \lambda_6 \int_x \hat{\varphi}^6(x)$. However, at face value we have changed the theory to a $\hat{\varphi}^6(x)$ -theory with coupling λ_6 that is not perturbatively renormalisable in $d = 4$. Still, within functional RG methods one can address the question whether such the theory is consistent. In particular if the theory admits a non-trivial ultraviolet fixed point the problem of perturbative non-renormalisability is cured. Leaving aside the problem of its UV-completion the flow equation can be used to generate the IR-effective action from some finite initial condition. Then, the flow equation introduces a consistent BPHZ-type renormalisation.

In turn, as long as the composite operator $\phi^{\mathbf{a}}$ is renormalisable we deal with the standard renormalisation of composite operators [218]. Moreover, functional RG flows can be used to actually define finite generating functionals in the presence of composite operators as well as practical iterative renormalisation procedures [47, 85]. The general case is covered by the RG equations (4.8),(4.20) and the full flows (4.20). In particular we deal with a matrix $\gamma_{\phi}^{\mathbf{a} \mathbf{c}}$ of anomalous dimensions, and the corresponding renormalisation conditions, for the general perturbative setting see e.g. [218]. We resort again to the above example of $\hat{\varphi}^4$ -theory in $d = 4$ but coupled to the 2-point function: $J^{\mathbf{a}} \phi_{\mathbf{a}} = J^a \hat{\varphi}_a + J^{a_1 a_2} \hat{\varphi}_{a_1} \hat{\varphi}_{a_2}$. We have extended the number of (independent) relevant operators $\langle \hat{\varphi}^2(x) \rangle$, $\langle (\partial \hat{\varphi})^2(x) \rangle$ and $\langle \hat{\varphi}^4(x) \rangle$ with $\langle \hat{\phi}(x, x) \rangle$ and $\langle \hat{\phi}(x, x) \hat{\varphi}^2(x) \rangle$ and $\langle \hat{\phi}^2(x, x) \rangle$, where $\hat{\phi}(x, y) = \hat{\varphi}(x) \hat{\varphi}(y)$. The anomalous dimensions of these operators are related by the matrix γ_{ϕ} and coincide naturally on the equations of motions.

Apart from these more formal questions there is the important issue of practical renormalisation, i.e. consis-

tently renormalising the theory order by order within a given truncation scheme. The general flows (3.60) together with the considerations of this section allow to construct such a renormalisation. Again we outline the setting within the 2PI effective action with $\mathbf{a} = a, a_1 a_2$ and $\hat{\phi}_{\mathbf{a}} = (\hat{\phi}_a, \hat{\phi}_{a_1} \hat{\phi}_{a_2})$. As distinguished from the last section VIB 2 we couple a quadratic regulator to the fields,

$$\Delta S[\hat{\phi}, R] = R^{\mathbf{ab}} \hat{\phi}_{\mathbf{a}} \hat{\phi}_{\mathbf{b}}, \quad (6.39)$$

where we also allow for insertions of the operators $\hat{\phi}_a \hat{\phi}_{b_1 b_2}$ and $\hat{\phi}_{a_1 a_2} \hat{\phi}_{b_1 b_2}$. The regulator (6.39) leads to the standard flow (3.72) for general correlation functions, for the effective action it is given by (3.74). In the present case it reads

$$\begin{aligned} \partial_t \Gamma_k[\phi] &= \dot{R}^{ab} G_{ab} + \dot{R}^{ab_1 b_2} G_{ab_1 b_2} + \dot{R}^{a_1 a_2 b} G_{a_1 a_2 b} \\ &+ \dot{R}^{a_1 a_2 b_1 b_2} G_{a_1 a_2 b_1 b_2} G_{a_1 a_2 b_1 b_2}. \end{aligned} \quad (6.40)$$

In the first term on the rhs of (6.40) we could also identify $G_{ab} = \phi_{ab} - \phi_a \phi_b$, see (6.24). 2PI expansions relate to loop (coupling) expansions in the field $\phi_{\mathbf{a}}$ and hence, via the equations of motion, to resummations of classes of diagrams. For general expansion schemes we refer to the results of section VIA 2 that straightforwardly translate to the present multi-index situation.

We proceed by discussing an iterative loop-wise resolution of the flow (6.40) that leads to a BPHZ-type renormalisation of diagrams as in the standard case. This analysis is not bound to the 2PI example considered above as the index \mathbf{a} could comprise higher N -point functions. From now on we consider the general case. Still we keep the simple quadratic regulator (6.39). Assume that we have resolved the theory at i th loop order leading to a finite i -loop contribution $\Gamma_k^{(i)}$, the full effective action being $\Gamma_k = \sum_i \Gamma_k^{(i)}$. Then, the $i+1$ st order reads in differential form

$$\partial_t \Gamma_k^{(i+1)} = \dot{R}^{\mathbf{ab}} G_{\mathbf{ab}}^{(i)}, \quad (6.41)$$

and is finite. At one loop, $i = 1$, its integration results in

$$\Gamma_k^{(1)}[\phi] = -\frac{1}{2} (\ln G)_{\mathbf{a}}^{\mathbf{a}} \Big|_{\Lambda}^k + \Gamma_{\Lambda}^{(1)}, \quad (6.42)$$

where the Λ -dependent terms arrange for a BPHZ-type renormalisation procedure and, in a slight abuse of notation, G stands for the classical propagators of the fields $\phi_{\mathbf{a}}$. The superscript (1) indicates the one loop order, not the one point function. The subtraction at Λ makes the

rhs finite. Γ_{Λ} ensures the Λ -independence as well as introducing a finite (re-)normalisation. For $i = 2$ we have to feed $\Gamma^{(1)}[\phi]$ and its derivatives into the rhs of the flow (6.41). Again the t -integration can be performed as the rhs is a total derivative w.r.t. t . It is the same recursive structure which reproduces renormalised perturbation theory from a loop-wise integration of the 1PI flow. At two loop the flow (6.41) reads

$$\dot{R}^{\mathbf{ab}} G_{\mathbf{ab}}^{(2)} = -\dot{R}^{\mathbf{ab}} G_{\mathbf{ac}} \Gamma^{(1), \mathbf{cd}} G_{\mathbf{db}}, \quad (6.43)$$

assuming no coupling dependence of R . The two-point function at one loop, $\Gamma^{(1), \mathbf{cd}}$, is the second derivative of (6.42) w.r.t. the field $\phi_{\mathbf{a}}$, and (6.43) turns into a total t -derivative. Finally we arrive at the two-loop contribution

$$\begin{aligned} \Gamma_k^{(2)} &= \frac{1}{8} \Gamma^{\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 \mathbf{a}_4} (G - G|_{\Lambda})_{\mathbf{a}_1 \mathbf{a}_2} (G - G|_{\Lambda})_{\mathbf{a}_3 \mathbf{a}_4} \\ &- \frac{1}{12} \Gamma^{\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3} \Gamma^{\mathbf{a}_1 \mathbf{a}_5 \mathbf{a}_2} (G - G|_{\Lambda})_{\mathbf{a}_1 \mathbf{a}_2} (G - G|_{\Lambda})_{\mathbf{a}_3 \mathbf{a}_4} \\ &\quad \times ((G - G|_{\Lambda}) + 3G|_{\Lambda})_{\mathbf{a}_5 \mathbf{a}_6} \\ &+ \frac{1}{2} \Gamma_{\Lambda}^{(1), \mathbf{a}_1 \mathbf{a}_2} (G - G|_{\Lambda})_{\mathbf{a}_1 \mathbf{a}_2} + \Gamma_{\Lambda}^{(2)}. \end{aligned} \quad (6.44)$$

Higher orders follow similarly. Such a procedure allows for a constructive renormalisation of the theory under investigation, and also facilitates formal considerations concerning the renormalisation of general truncations schemes. The first two terms in (6.44) are already finite due to the subtractions. The terms proportional to $3G$ in the third line of (6.44) and in the 4th line constitute finite (re-) normalisations. Eq. (6.44) stays finite if the vertices and propagators are taken to be full vertices and propagators in the sense of an RG improvement. Within the 2PI example considered in (6.40) the integrated flow (6.44) is the consistently renormalised result for the 2PI effective action at two loop. It translates into a resummed renormalised 1PI effective action by using the equation of motion (6.18) for the composite field ϕ_{ab} . However, the above result also applies to N PI effective actions or more general composite operators coupled to the theory: the integrated flow (6.44) constitutes a finite BPHZ-type renormalised perturbative expansion. Moreover, the above method straightforwardly extends to general expansion schemes: in general the integrated flow constitutes a finite BPHZ-type renormalised expansion. The consistency of the renormalisation procedure is guaranteed by construction.

The renormalisation conditions for the full theory are set implicitly with the choice of the effective action at

the initial cut-off scale Λ . We emphasise that any RG scheme that derives from a functional truncation to the flow (3.60), and in the particular the loop expansion (6.41), is consistent with the truncation. Moreover, the iterative structure displayed in (6.41), (6.42) and (6.44) allows us to discuss general renormalisation conditions in the present setting. By adding the operator $\hat{\phi}_{ab}$ we have extended the number of relevant vertices in the effective action and hence the number of renormalisation conditions. In case $\phi_{\mathbf{a}}$ includes only marginal and irrelevant operators the renormalisation proof can be mapped to that of the 1PI case.

The basic example is provided by $(\phi_{\mathbf{a}}) = (\phi_a, \phi_{a_1 a_2})$, where the field $\phi_{a_1 a_2}$ with $\hat{\phi}_{a_1 a_2} = \hat{\phi}_{a_1} \hat{\phi}_{a_2}$ counts like $\phi_{a_1} \phi_{a_2}$. RG conditions for e.g. the 2-point function and the 4-point function

$$\frac{\delta\Gamma}{\delta\phi_a \delta\phi_b}, \quad \frac{\delta\Gamma}{\delta\phi_{a_1} \cdots \delta\phi_{a_4}}, \quad (6.45)$$

trigger additional RG conditions for

$$\frac{\delta\Gamma}{\delta\phi_{ab}}, \quad \frac{\delta\Gamma}{\delta\phi_{a_1 a_2} \delta\phi_{a_3 a_4}}, \quad \frac{\delta\Gamma}{\delta\phi_{a_1 a_2} \delta\phi_{a_3} \delta\phi_{a_4}}. \quad (6.46)$$

Using the relation (6.9) between derivatives w.r.t. ϕ_a and ϕ_{ab} we are left with the same number of independent RG conditions as in the 1PI case. In other words, the matrix $\gamma^{\mathbf{a}\mathbf{b}}$ is highly symmetric. This symmetry can be imposed on the level of Γ_Λ and evolves with the flow as its rhs only depends on (derivatives of) Γ_k . We observe that formally any choice of Γ_Λ independently fixes these RG conditions at all scales (via the flow) but violates the relation (6.9). A priori there is nothing wrong with such a procedure that simply relates to an additional additive renormalisation (at 1PI level) and can be absorbed in a possibly k -dependent rescaling of the 2PI fields. The above discussion extends to the general case with fields $\phi_{\mathbf{a}}$. We shall detail these observations and structures elsewhere and close with the remark that for general truncation schemes that do not admit a direct resolution of the flow as in perturbation theory, the costs relate to an additional t -integration as already discussed in the 1PI case of section VIA.

VII. APPLICATIONS TO GAUGE THEORIES

The generality of the present approach fully pays off in gauge theories, and the present work was mainly triggered by related investigations. In flow studies for gauge

theories [108–151] and gravity [161–170] with the standard quadratic regulator one has to deal with modified Slavnov-Taylor identities [108–127]. These identities tend towards the Slavnov-Taylor identities of the full theory in the limit of vanishing regulator. It is crucial to guarantee this limit towards physical gauge invariance.

The subtlety of modified Slavnov-Taylor identities can be avoided for thermal flows. This is achieved by either modifying the thermal distribution [133, 134], or by constructing the thermal flow as a difference of Callan-Symanzik flows at zero and finite temperature in an axial-type gauge [19, 135]. The resulting thermal flows are gauge invariant. We remark that Callan-Symanzik flows in axial gauges at zero temperature [128–131] are formally gauge invariant, but the approach towards the full theory at vanishing regulator has severe consistency problems. This problem is related to the missing locality in momentum space combined with the incomplete gauge fixing [122]. One expects a better convergence for Callan-Symanzik flows within covariant or Abelian gauges [132]

Alternatively one can resort to gauge-invariant degrees of freedom [159, 160], gauge-covariant degrees of freedom [152–158], or higher order regulator terms with regulators $R^{\mathbf{a}_1 \cdots \mathbf{a}_n}$ with $n > 2$. Then, N -point functions directly relate to observables and allow for the construction of gauge-invariant flows. In general such a parameterisation is paid for with non-localities, in particular in theories with a non-Abelian gauge symmetry.

In this chapter we discuss the structural aspects of the above formulations. In particular we deal with the question of convenient representations of symmetry identities that facilitates their implementation during the flow. Moreover we discuss the related question of adjusted parameterisations of gauge theories, and evaluate the fate of symmetry constraints in gauge-invariant formulations.

A. Parameterisation

In gauge fixed formulations of gauge theories, and in particular in strongly interacting regimes, the propagators and general Green functions are only indirectly related to physical observables. Firstly, only combinations of them are gauge invariant and secondly, the relevant degrees of freedom in the strongly interacting regime are

not the perturbative ones¹². Good choices are observables that serve as order parameters; e.g. the Polyakov loop¹³

$$P(\vec{x}) = \text{Tr } \mathcal{P} \exp \int_0^\beta A_0(x) d\tau, \quad (7.1)$$

and its two-point function $\langle P(\vec{x}) P^\dagger(\vec{y}) \rangle$ in the case of the confinement-deconfinement phase transition. These observables fall into the class of I_k defined in (3.14). For the Polyakov loop variable (7.1) the corresponding operator is $\hat{I} = P(\vec{x})[A_0 = \frac{\delta}{\delta J^0}]$ which implies $\hat{I}_k = \hat{I}$, see (3.14b). Hence their flow can still be described in terms of field propagators and vertices of the fundamental fields via (3.14),(3.51). It amounts to the following procedure: compute the flow of propagators and vertices, even though partially decoupling in the phase transition. Then, the flow of relevant observables \tilde{I} is computed from this input with the flow (3.60), i.e. the heavy quark potential from the flow of the Wilson loop or Polyakov loop. Such a procedure allows for a direct computation of physical quantities from the propagators and vertices of the theory in a given parameterisation, and it applies to gauge fixed as well as gauge invariant formulations. It also emphasises the key rôle played by the propagators of the theory, and matches their key importance within the functional optimisation developed in section V.

One also can use appropriate fields $\hat{\phi}$ coupled to the theory. In the above example of the confinement-deconfinement phase transition a natural choice is provided by the gauge invariant field $\hat{\phi}(x) = P(\vec{x})$ with (7.1). Such a choice has to be completed by additional $\hat{\phi}^a$ that cover the remaining field degrees of freedom. Alternatively one can integrate out the remaining degrees of freedom and only keep that of interest. Another interesting option are gauge covariant degrees of freedom, e.g. $\hat{\phi}_{\mu\nu}(x) = F_{\mu\nu}$ or $\hat{\phi}_{\mu\nu}(x) = \tilde{F}_{\mu\nu}$, that is the dual field strength [137, 138]. Both choices can be used to derive (partially) gauge invariant effective actions, and aim at a description of gauge theories in terms of physical variables.

We emphasise that the above suggestions usually generate non-local and non-polynomial effective actions even at the initial scale. We have to keep in mind that gauge

theories are formulated as path integrals over the gauge field supplemented with a polynomial and local classical action. Gauge fixing is nothing but the necessity to deal with a non-trivial Jacobian that arises from the decoupling of redundant degrees of freedom, and Slavnov-Taylor identities (STIs) carry the information of this reparameterisation. If coupling gauge invariant or gauge covariant degrees of freedom to the theory the necessity of decoupling the redundant degrees of freedom remains, and hence the symmetry constraints are still present. In a gauge invariant setting the corresponding STIs turn into a subset of DSEs. Their relevance might be hidden by the fact of manifest gauge invariance, but still they carry the information about locality. In other words, approximations to gauge invariant effective actions or general correlation functions still can be in conflict with the Slavnov-Taylor identities and hence violate physical gauge invariance. Indeed it is helpful to explicitly gauge fix the theory within a choice that simplifies the relation $\phi = \phi(A)$ for gauge-fixed fields A as it makes locality more evident in the variables ϕ . For example, in case of the confinement-deconfinement phase transition we choose $\hat{\phi}(\vec{x}) = P(\vec{x})$ defined in (7.1), and use the Polyakov gauge or variations thereof, e.g. [214–216].

In summary we conclude that it is vital to study the fate of symmetry constraints such as the Slavnov-Taylor identities for general flows, be they gauge invariant or gauge variant. This is done in the next three sections VII B, VII C, VII D.

B. Modified Slavnov-Taylor identities

The propagators and vertices of a gauge theory are constrained by gauge invariance of the theory. A non-trivial symmetry $I_k \equiv 0$ is maintained during general flows (4.8), (4.20): if $I_k \equiv 0$ is satisfied at the starting scale, its flow vanishes as it is proportional to I_k . In particular this is valid for $D_s = \partial_t$. The corresponding flows include that of modified Ward-Takahashi or Slavnov-Taylor identities for the effective action [114, 120, 122], and that of Nielsen identities [160] for gauge invariant flows [159, 160].

The above statements imply that the generator of the flow, D_s , commutes with the generator of the modified symmetry \hat{I}_k . Within truncations this property does not hold, and it is not sufficient to guarantee the symmetry at the starting scale. Consequently a symmetry relation $I_k \equiv 0$ should be read as a fine-tuning condition which has to be solved at each scale. This is technically rather

¹² Basically by definition; the relevant degrees of freedom should only weakly interact.

¹³ The definition (7.1) only applies in the case of periodic boundary conditions for the gauge field.

involved, and any simplification is helpful. Here we aim at a discussion of different representations of symmetry constraints and their flows.

1. STI

First we concentrate on a pure non-Abelian gauge theory with general gauge fixing $\mathcal{F}[A]$. For its chief importance we shall explain the structure with sources coupled to the fundamental fields φ , and a standard quadratic regulator term $R^{ab}\varphi_a\varphi_b$. We keep the condensed notation and refer the reader to [217] for some more details. The Schwinger functional is given by

$$e^{W[J,Q]} = \int d\hat{\varphi} d\lambda e^{-S[\hat{\varphi}] + J^{\mathbf{a}}\hat{\varphi}_{\mathbf{a}} + Q^{\mathbf{a}}\mathfrak{s}\hat{\varphi}_{\mathbf{a}}}. \quad (7.2)$$

In (7.2) we have also included source terms $Q^{\mathbf{a}}\mathfrak{s}\hat{\varphi}_{\mathbf{a}}$ for the symmetry variations of the fields as introduced in section II. Here \mathfrak{s} generates BRST transformations defined below in (7.8). The fields $\hat{\varphi}_{\mathbf{a}}[\hat{\varphi}]$ depend on the fundamental fields $\hat{\varphi}_a$ given by

$$(\hat{\varphi}_a) = (A_i, C_\alpha, \bar{C}_\alpha), \quad (7.3)$$

where we have dropped the hats on the component fields. The component fields in (7.3) read more explicitly $A_i = A_\mu^i(x)$, the gauge field, and $C_\alpha = C_a(x)$, $\bar{C}_\alpha = \bar{C}_a(x)$, the ghost fields. A more explicit form of the source term in the case of $\hat{\varphi} = \hat{\varphi}$ reads

$$\begin{aligned} J^a\hat{\varphi}_a &= J_i A_i + \bar{J}_\alpha C_\alpha - J_\alpha \bar{C}_\alpha \\ &= \int_x (J_\mu^a(x) A_\mu^a(x) + \bar{J}^a(x) C^a(x) + \bar{C}^a(x) J^a(x)). \end{aligned} \quad (7.4)$$

The action S in the path integral (7.2) is given by

$$\begin{aligned} S[\hat{\varphi}, \lambda] &= S_{\text{YM}}[\hat{\varphi}] \\ &\quad -\omega(\lambda) + \lambda_\alpha \mathcal{F}_\alpha(A) - \bar{C}_\alpha M^{\alpha\beta} C_\beta, \end{aligned} \quad (7.5)$$

with

$$M^{\alpha\beta} = \mathcal{F}^\alpha_{,i} D_i^\beta(A), \quad \omega(\lambda) = \frac{\xi}{2} \lambda_\alpha \lambda_\alpha, \quad (7.6)$$

the latter equation for ω leading to the standard gauge fixing term $\frac{1}{2\xi} \mathcal{F}^\alpha \mathcal{F}_\alpha$ upon integration over λ . Then, in a less condensed notation, (7.5) turns into

$$\begin{aligned} S[\hat{\varphi}] &= \frac{1}{4} \int_x F_{\mu\nu}^a F_{\mu\nu}^a \\ &\quad - \frac{1}{2\xi} \int_x \mathcal{F}^a \mathcal{F}^a - \int_x \bar{C}^a \frac{\partial \mathcal{F}^a}{\partial A_b^\mu} D_\mu^{bc} C^c. \end{aligned} \quad (7.7)$$

Matter fields and a Higgs sector can be straightforwardly added. The action (7.5) is invariant under the BRST transformations

$$(\mathfrak{s}\hat{\varphi}) = (D_i^\alpha C_\alpha, -\frac{1}{2} f_{\alpha\beta\gamma} C_\beta C_\gamma, \lambda_\alpha), \quad (7.8)$$

and \mathfrak{s} acts trivially on λ : $\mathfrak{s}\lambda_\alpha = 0$. The operator \mathfrak{s} can be represented as a functional differential operator on the fields $\hat{\varphi}, \lambda$ with

$$\mathfrak{s} = (\mathfrak{s}\hat{\varphi}_a) \frac{\delta}{\delta\hat{\varphi}_a}, \quad (7.9)$$

making the anti-commuting (Grassmann) property of \mathfrak{s} explicit. The invariance of the action, $\mathfrak{s}S[\hat{\varphi}] = 0$ can be proven straightforwardly by insertion. Moreover, \mathfrak{s} is a differential with $\mathfrak{s}^2\varphi = 0$ allowing for a simple form of the symmetry constraint. The only BRST-variant term is the source term $J^a\hat{\varphi}_a$. The related Slavnov-Taylor identity (STI) is cast into an algebraic form with help of the source terms for the BRST variations (7.8) included in (7.2). For $\hat{\varphi} = \varphi$ this source terms reads

$$Q^a \mathfrak{s}\hat{\varphi}_a = Q^i D_i^\alpha C_\alpha - \frac{1}{2} \bar{Q}^\alpha f_{\alpha\beta\gamma} C_\beta C_\gamma + Q^\alpha \lambda_\alpha, \quad (7.10)$$

where $Q^\alpha \mathfrak{s}\bar{C}_\alpha = Q^\alpha \lambda_\alpha$ could also be considered as a standard source term for the auxiliary field λ . The general BRST source term reads

$$Q^{\mathbf{a}}(\mathfrak{s}\hat{\varphi})_{\mathbf{a}} = Q^{\mathbf{a}}(\mathfrak{s}\hat{\varphi})_a \hat{\varphi}_{\mathbf{a}}^a[\hat{\varphi}], \quad (7.11)$$

following with (7.9). The Slavnov-Taylor identity follows from

$$\int \mathfrak{s} \left(d\hat{\varphi} d\lambda \exp\{-S[\hat{\varphi}] + J^{\mathbf{a}}\hat{\varphi}_{\mathbf{a}} + Q^{\mathbf{a}}(\mathfrak{s}\hat{\varphi})_{\mathbf{a}}\} \right) \equiv 0. \quad (7.12)$$

Eq. (7.12) is of the form (2.12). It follows with (7.9) after a partial functional integration and $(\mathfrak{s}\hat{\varphi}_a)^{\cdot a} = D_{i,i}^\alpha C_\alpha + f_{\alpha\alpha\beta} C_\beta = 0$ (for compact Lie groups). Except for the source term $J^{\mathbf{a}}\hat{\varphi}_{\mathbf{a}}$ all terms in (7.12) are BRST-invariant: $\mathfrak{s}d\hat{\varphi} = 0$, $\mathfrak{s}S[\hat{\varphi}] = 0$, $\mathfrak{s}(Q^{\mathbf{a}}\mathfrak{s}\hat{\varphi}_{\mathbf{a}}) = 0$. The operator \mathfrak{s} commutes due to its Grassmannian nature with bosonic currents J and anti-commutes with fermionic ones. For example, for the fundamental fields and currents this entails that \mathfrak{s} commutes with J^i but anti-commutes with J^α, \bar{J}^α and $\mathfrak{s}J^a\hat{\varphi}_a = J^b\gamma^a_b(\mathfrak{s}\hat{\varphi}_a)$. Using all these properties in (7.12) leads us to the Slavnov-Taylor identity

$$\begin{aligned} &\int d\hat{\varphi} d\lambda J^{\mathbf{b}}\gamma^{\mathbf{a}}_{\mathbf{b}}(\mathfrak{s}\hat{\varphi}_{\mathbf{a}}) \exp\{-S[\hat{\varphi}] + J^{\mathbf{a}}\hat{\varphi}_{\mathbf{a}} + Q^{\mathbf{a}}\mathfrak{s}\hat{\varphi}_{\mathbf{a}}\} \\ &= J^{\mathbf{b}}\gamma^{\mathbf{a}}_{\mathbf{b}} \frac{\delta}{\delta Q^{\mathbf{a}}} e^{W[J,Q]} \equiv 0. \end{aligned} \quad (7.13)$$

Eq. (7.13) is of the form $e^W I[J, Q] \equiv 0$ leading to (3.7) with I defined in (2.10) for

$$\hat{I}_s = J^b \gamma^a_b \frac{\delta}{\delta Q^a}, \quad (7.14)$$

The operator \hat{I}_s generates BRST transformations on the Schwinger functional W . Accordingly the STI (7.13) can be written as

$$\hat{I}_s W[J, Q] \equiv 0, \quad (7.15)$$

that is the Schwinger functional is invariant under BRST transformations. The STI (7.15) can be generalised to that for correlation functions I . To that end we use that (7.15) can be multiplied by any operator \hat{I} from the left. We are led to

$$\mathcal{W}_{s,I} \equiv 0, \quad \text{with} \quad \hat{\mathcal{W}}_{s,I} = \hat{I} \hat{I}_s, \quad (7.16a)$$

where \mathcal{W}_I is derived from $\hat{\mathcal{W}}_I$ with (2.10). The symmetry relation (7.16) is a direct consequence of (7.13), which is reproduced for $\hat{I} = 1$. We can write the correlation function \mathcal{W}_I in terms of I as

$$\mathcal{W}_{s,I}[J, Q] = \hat{I}_s I[J, Q] + \delta I_{s,I}[J, Q], \quad (7.16b)$$

with

$$\widehat{\delta I} = [\hat{I}, J^b \gamma^a_b \frac{\delta}{\delta Q^a}]. \quad (7.16c)$$

For the derivation of (7.16b) we have used that $\hat{I} \hat{I}_s = \hat{I}_s \hat{I} + \widehat{\delta I}$ as well repeatedly using $[\hat{I}_s, W] = 0$, which is the STI (7.13).

For Q -independent \hat{I} the commutator $\widehat{\delta I}$ substitutes one of the J -derivatives in \hat{I} by one w.r.t. Q . Applied on e^W this generates a (quantum) BRST transformation on \hat{I} . Consequently we write

$$\widehat{\delta I}_{s,I}[J, \hat{\phi}] e^W = - \left(\mathfrak{s}[\hat{\phi}] \hat{I}[J, \hat{\phi}] \right) e^W, \quad (7.17)$$

which we evaluate at $\hat{\phi} = \frac{\delta}{\delta J}$. Accordingly, for BRST-invariant $\hat{I}[J, \hat{\phi}]$ the second term on the rhs of (7.16b) disappears. Hence, if I is the expectation value of a BRST-invariant $\hat{I}[J, \hat{\phi}]$, the second term on the rhs of (7.16b) vanishes and I is BRST-invariant, $\hat{I}_s I = 0$.

We remark that (7.16), as the flow (3.28), does not directly encode the STI for the Schwinger functional. This comes about since we have divided out the STI for W , (7.15) in its form $[\hat{I}_s, W]$ in the derivation of (7.16). In turn, it has to be trivially satisfied. Indeed, for either $\hat{I} = 1$ or $\hat{I} = W[J, Q]$, leading to $I = 1$ and $I = W$, the STI (7.16) is trivially satisfied. The situation is similar

to that of the flow (3.28) where the flow of the Schwinger functional has been divided out. Without resorting to the STI for W , (7.15), the STIs $\mathcal{W}_{s,I}$ derived with $\hat{\mathcal{W}}_{s,I}$ in (7.16a) read

$$\mathcal{W}_{s,I}[J, Q] = \left(\hat{I}_s - (\hat{I}_s W) \right) I[J, Q] + \delta I, \quad (7.18)$$

and, for $\hat{I} = 1$ or $\hat{I} = W[J, Q]$ the STI for the Schwinger functional, (7.15) follows. Hence, we shall refer to the STI (7.15) as $\mathcal{W}_{s,1} = 0$. Note also that its trivial resolution does not imply that it is not encoded in the representation (7.16b). Similarly to the derivation of its flow from the general flow (3.28), the STI for the Schwinger functional derives from $\hat{I} = \frac{\delta}{\delta J}$, inserted in (7.16). We are led to $\frac{\delta}{\delta J} \hat{I}_s W[J, Q] = 0$ which entails (7.15).

2. mSTI

So far we have adapted the analysis of the STI in its algebraic form to the present setting. Now we consider regularisations of the Schwinger functional $W[J, Q, R]$ defined in (3.1), as well as general operators $I[J, Q, R]$ defined in (3.8). The operator $\hat{I}_s[J, \frac{\delta}{\delta J}, \frac{\delta}{\delta Q}, R]$ corresponding to $I_s[J, Q, R]$ is derived from (3.8b) as

$$\hat{I}_s = (J^b - [\Delta S, J^b]) \gamma^a_b \frac{\delta}{\delta Q^a}, \quad (7.19)$$

where the second term generates BRST transformations of the regulator term ΔS , and we have used that ΔS is bosonic. As an example we compute (7.19) for the standard flow, $\hat{\phi} = \hat{\varphi}$ and a quadratic regulator term $R^{ab} \hat{\varphi}_a \hat{\varphi}_b$. This leads us to the symmetry operator

$$\hat{I}_s = (J^b - 2R^{cb} \frac{\delta}{\delta J^c}) \gamma^a_b \frac{\delta}{\delta Q^a}, \quad (7.20)$$

where we have used the symmetry properties of R in (3.5) for standard flows. The STI for the Schwinger functional (7.13) turns into [108–127]

$$\hat{I}_s W[J, Q, R] = 0, \quad (7.21)$$

with \hat{I}_s defined in (7.20). It entails that only the source terms $J^a \phi_a$ and the regulator term are BRST-variant. The relation (7.21) was coined modified Slavnov-Taylor identity (mSTI) as it encodes BRST invariance at $R = 0$, and shows its explicit breaking via the regulator term at $R \neq 0$.

The general case with \mathcal{W}_I leads to the same general STI (7.16) with all operators and correlation functions substituted by their R -dependent counterparts defined in (3.8),

$$\mathcal{W}_{s,I}[J, Q, R] \equiv 0, \quad (7.22a)$$

with

$$\mathcal{W}_{\mathfrak{s},I}[J, Q, R] = \hat{I}_{\mathfrak{s}} I[J, Q, R] + \delta I. \quad (7.22b)$$

The correlation function $\delta I[J, Q, R]$ is the R -dependent counterpart derived from (7.16c) with (3.8);

$$\widehat{\delta I}[R] = e^{-\Delta S} [\hat{I}, \hat{I}_{\mathfrak{s}}]_{R=0} e^{\Delta S}, \quad (7.23)$$

where $\Delta S = \Delta S[\frac{\delta}{\delta J}, R]$. Hence the second term on the rhs of (7.22b) still vanishes for a BRST-invariant $\hat{I}[J, \frac{\delta}{\delta J}, 0]$. The modification of BRST invariance is solely encoded in the modification of the BRST operator $\hat{I}_{\mathfrak{s}}$ in (7.20). The flow of (7.22) is governed by (3.28).

The mSTI (7.21) for the Schwinger functional follows as $\mathcal{W}_{\mathfrak{s},1} \equiv 0$ with the alternative representation (7.18). As in the case without regulator, it also can be derived from (7.22) from $\mathcal{W}_{\mathfrak{s},W_{\mathfrak{a}}}$. Inserting $\hat{I} = \frac{\delta}{\delta J}$ into (7.22b) leads to $\mathcal{W}_{W_{\mathfrak{a}}} = \frac{\delta}{\delta J^{\mathfrak{a}}} \hat{I}_{\mathfrak{s}} W[J, Q, R] \equiv 0$ and hence to (7.21).

As in the case of the flows we can turn the general mSTIs (7.22) into mSTIs for correlation functions \tilde{I} in terms of the variable ϕ . The definition of the effective action (3.43) extends to the case with external currents Q :

$$\begin{aligned} \Gamma[\phi, Q, R] &= J^{\mathfrak{a}}(\phi, Q) \phi_{\mathfrak{a}} - W[J(\phi, Q), Q, R] \\ &\quad - \Delta S'[\phi, R], \end{aligned} \quad (7.24)$$

the source J now depends on the fields ϕ and the source Q . Eq. (7.24) entails that

$$\frac{\delta W}{\delta Q^{\mathfrak{a}}} = -\frac{\delta(\Gamma + \Delta S')}{\delta Q^{\mathfrak{a}}} = -\frac{\delta \Gamma}{\delta Q^{\mathfrak{a}}}, \quad (7.25)$$

as $\Delta S'$ does not depend on Q and the Q -dependence of J cancels out. In (7.25) the Q -derivatives of W and Γ are taken at fixed arguments J and ϕ respectively. The correlation functions \tilde{I} derive from (3.51) as

$$\tilde{I}[\phi, Q, R] = I[J(\phi, Q, R), Q, R]. \quad (7.26)$$

For the mSTIs $\tilde{\mathcal{W}}_I \equiv 0$ we have to rewrite Q -derivatives at fixed J in terms of Q -derivatives at fixed ϕ . This reads

$$\frac{\delta}{\delta Q_{\mathfrak{s}}} := \frac{\delta}{\delta Q} \Big|_J = \frac{\delta}{\delta Q} \Big|_{\phi} - \gamma^{\mathfrak{a}\mathfrak{b}} \frac{\delta \Gamma^{\mathfrak{b}}}{\delta Q} G_{\mathfrak{a}\mathfrak{c}} \frac{\delta}{\delta \phi_{\mathfrak{c}}}, \quad (7.27)$$

where we have used (3.45) and (3.50). With the above relations we arrive at the modified Slavnov-Taylor identity

$$\tilde{\mathcal{W}}_{\mathfrak{s},I}[\phi, Q, R] \equiv 0, \quad (7.28a)$$

with

$$\tilde{\mathcal{W}}_{\mathfrak{s},I}[\phi, Q, R] = \hat{I}_{\mathfrak{s}} \tilde{I}[\phi, Q, R] + \widehat{\delta I}_{\mathfrak{s},I}[\phi, Q, R], \quad (7.28b)$$

where the operator $\hat{I}_{\mathfrak{s}}$ is defined in (7.20). In (7.28b) it acts on functionals of the variable ϕ . With (7.27) it can be written as

$$\hat{I}_{\mathfrak{s}} = \left(\frac{\delta \Gamma}{\delta \phi_{\mathfrak{a}}} - \Delta S^{\mathfrak{a}\mathfrak{b}} [G_{\frac{\delta}{\delta \phi}} + \phi, R] + \Delta S^{\mathfrak{a}\mathfrak{b}}[\phi, R] \right) \frac{\delta}{\delta Q_{\mathfrak{s}}}. \quad (7.28c)$$

The sum of the ΔS -terms in (7.28c) give the part of $\Delta S^{\mathfrak{a}\mathfrak{b}} [G_{\frac{\delta}{\delta \phi}} + \phi, R]$ with at least one ϕ -derivative acting to the right. The operator $\hat{I}_{\mathfrak{s}}$ defined in (7.28c) generates BRST transformations while keeping the regulator term fixed. Consequently the mSTI (7.28) entails that such a BRST transformation of \tilde{I} is given by the explicit BRST variation due to $\widehat{\delta I}$. The correlation function $\widehat{\delta I}$ is the expectation value of $\widehat{\delta I}$ defined in (7.23). Similarly to (7.17) we write

$$\widehat{\delta I}_{\mathfrak{s},I}[J, \hat{\phi}, R] = e^{-\Delta S} \left(\mathfrak{s}[\hat{\phi}] \hat{I}[J, \hat{\phi}] \right) e^{\Delta S}. \quad (7.29)$$

Eq. (7.29) entails that $\widehat{\delta I}$ vanishes for BRST-invariant correlation function \tilde{I} . In this case $\hat{I}_{\mathfrak{s}} \tilde{I} \equiv 0$. Finally we remark that the representation (7.18) of $\tilde{\mathcal{W}}$ translates into

$$\tilde{\mathcal{W}}_{\mathfrak{s},I} = \left(\hat{I}_{\mathfrak{s}} \left[\frac{\delta}{\delta Q_{\mathfrak{s}}} \right] + \left(\hat{I}_{\mathfrak{s}} \left[\frac{\delta}{\delta Q} \right] \Gamma \right) \right) \tilde{I} + \widehat{\delta I}, \quad (7.30)$$

where we have used $\hat{I} \left[\frac{\delta}{\delta Q} |_J \right] W[J, R] = -\hat{I} \left[\frac{\delta}{\delta Q} |_{\phi} \right] \Gamma[\phi, R]$, following from (7.19), (7.25) and (7.27).

We proceed with elucidating the general identity (7.28) with two examples. Firstly we discuss the standard regularisation with a quadratic regulator $R^{\mathfrak{a}\mathfrak{b}} \phi_{\mathfrak{a}} \phi_{\mathfrak{b}}$. Inserting this into (7.28) we are led to

$$\tilde{\mathcal{W}}_{\mathfrak{s},I} = \left(\frac{\delta \Gamma}{\delta \phi_{\mathfrak{a}}} - 2R^{\mathfrak{b}\mathfrak{a}} G_{\mathfrak{c}\mathfrak{b}} \frac{\delta}{\delta \phi_{\mathfrak{c}}} \right) \frac{\delta \tilde{I}}{\delta Q_{\mathfrak{s}}^{\mathfrak{a}}} + \widehat{\delta I}. \quad (7.31)$$

The second important example is provided by the mSTI for Γ . It can be read off from the alternative representation for $\tilde{\mathcal{W}}_{\mathfrak{s},I}$ in (7.30) for $I = 1$ ($\hat{I} = 1$) leading to

$$\left(\frac{\delta \Gamma}{\delta \phi_{\mathfrak{a}}} - \Delta S^{\mathfrak{b}\mathfrak{a}} [G_{\frac{\delta}{\delta \phi}} + \phi, R] G_{\mathfrak{c}\mathfrak{b}} \frac{\delta}{\delta \phi_{\mathfrak{c}}} \right) \frac{\delta \Gamma}{\delta Q^{\mathfrak{a}}} \equiv 0. \quad (7.32)$$

We emphasise that the Q -derivative in (7.32) is that at fixed ϕ and not at fixed J . It is also possible to derive it directly from (7.28) with $I = \phi$. For the standard regulator the mSTI (7.32) reads [113]

$$\frac{\delta \Gamma}{\delta \phi_{\mathfrak{a}}} \frac{\delta \Gamma}{\delta Q^{\mathfrak{a}}} - 2R^{\mathfrak{a}\mathfrak{b}} \frac{\delta \Gamma^{\mathfrak{c}}}{\delta Q^{\mathfrak{b}}} G_{\mathfrak{c}\mathfrak{a}} = 0. \quad (7.33)$$

The terms proportional to derivatives of $\Delta S'$ cancel in (7.33).

3. Flows and alternative representations

The compatibility of (7.28) with the flow is ensured by the flow (3.60) for \tilde{W}_I ,

$$(\partial_t + \Delta S_2) \tilde{W}_I = 0, \quad (7.34)$$

for the effective action and quadratic regulator see [114, 120, 122, 123]

Eq. (7.34) implies that a truncated solution to $\tilde{I}_{STI} \equiv 0$ stays a solution during the flow if the flow is consistent with the truncation. Then it suffices to solve the mSTI for the initial condition $\Gamma[\phi, Q, R_{in}]$, $\tilde{I}[\phi, Q, R_{in}]$. However, the search for consistent truncations is intricate as (7.28) involves loop terms. It is worth searching for alternative representations of the mSTI (7.28) that facilitate the construction of such truncations. For the sake of simplicity we discuss this for the mSTI (7.33) for the effective action in the presence of quadratic regulator terms. The generalisation to correlation functions \tilde{I} and general ΔS is straightforwardly done by substituting the correlation function Γ with \tilde{I} (leaving the Γ -dependence of \hat{I}_s unchanged) as well as the quadratic regulator R^{ab} with a general R . We can cast (7.33) into an algebraic form using the fact that $-R$ serves as a current for G :

$$\tilde{W}_{s,1} = \frac{\delta\Gamma}{\delta\phi_a} \frac{\delta\Gamma}{\delta Q^a} + 2R^{ab} \frac{\delta\Gamma^{,c}}{\delta Q^b} \frac{\delta\Gamma}{\delta R^{ca}}. \quad (7.35)$$

The algebraic form of the STI (7.35) can be used to ensure gauge invariance in a given non-trivial approximation to Γ by successively adding explicitly R -dependent terms. Such a procedure accounts for gauge invariance of classes of resummed diagrams. We add that in most cases it implicitly dwells on an ordering in the gauge coupling. We also remark that (7.35) seems to encode a preserved symmetry. This point of view becomes even more suggestive if introducing anti-fields [124–127]. Note that in general the related symmetry transformation is inherently non-local.

Eq. (7.35) constitutes an ordering in R . This can be made explicit by fully relying on the interpretation of R as a current. There is a simple relation between Q -derivatives and J -derivatives: BRST variations of the fundamental fields φ are at most quadratic in the fields, see (7.8). Hence, the φ -order of the BRST transformation of a composite field $s\hat{\varphi}$ is at most increased by one. Therefore, the source term $Q^a s\hat{\varphi}_a$ can be absorbed into

a redefinition of J^a ,

$$J^a \hat{\varphi}_a - R^{ab} \hat{\varphi}_a \hat{\varphi}_b + Q^c s\hat{\varphi}_c = \quad (7.36)$$

$$\left(J^a + Q^c (s\hat{\varphi}_c)^{,a}_{\hat{\varphi}=0} \right) \hat{\varphi}_a - \left(R^{ab} - \frac{1}{2} Q^c (s\hat{\varphi}_c)^{,ba} \right) \hat{\varphi}_a \hat{\varphi}_b.$$

The tensors $(s\hat{\varphi}_c)^{,ab}$ are the structure constants of the gauge group as can be seen within the example of the fundamental fields (7.3) and their BRST variation (7.8). With (7.36) we can rewrite Q -derivatives of W and Γ in terms of J, R -derivatives of W and R -derivatives and fields φ for Γ . The key relation is

$$\frac{\delta\Gamma}{\delta Q^a} = -s\phi_a + \frac{1}{2} (s\phi_a)^{,cb} \frac{\delta\Gamma}{\delta R^{bc}}, \quad (7.37)$$

where we also have to admit source terms with source $-R$ for $A_i C_\alpha$ and $C_\alpha C_\beta$. With (7.37) we can substitute the Q -derivatives in (7.35) and eliminate Q . Then the correlation function $\tilde{W}_{s,1}[\phi, R] = \tilde{W}_{s,1}[\phi, 0, R]$ reads

$$\begin{aligned} \tilde{W}_{s,1}[\phi, R] = & -\frac{\delta\Gamma}{\delta\phi_a} \left(s\phi_a + \frac{1}{2} (s\phi_b)^{,ed} \frac{\delta\Gamma}{\delta R^{de}} \right) \\ & - 2R^{ab} \left((s\phi_b)^{,c} + \frac{1}{2} (s\phi_b)^{,ed} \frac{\delta\Gamma^{,c}}{\delta R^{de}} \right) \frac{\delta\Gamma}{\delta R^{ca}}. \end{aligned} \quad (7.38)$$

At $R = 0$ the second line vanishes and we deal with the standard STI. The parameterisation (7.28) and (7.35) of the STI emphasise the gauge symmetry and are certainly convenient within a coupling expansion. The parameterisation (7.35) and (7.38) naturally relate to the 'importance-sampling' relevant in the flow equation. The latter, (7.38), requires no BRST source terms and hence reduces the number of auxiliary fields/terms.

The derivation of (7.38) highlights the fact that (7.31) also constitutes the Slavnov-Taylor identity for the 2PI effective action, e.g. [210, 211]. To that end we restrict ourselves to $\mathbf{a} = a$ and quadratic regulators R^{ab} . With the substitution $R^{ab} \rightarrow -J^{ab}$ we are led to the Slavnov-Taylor identity for $\Gamma[\phi_a, Q, -J^{ab}]$. More explicitly we have

$$J^{ab} = -R^{ab}, \quad (7.39)$$

and

$$J^a \phi_a = J^a \phi_a + J^{ab} \phi_a \phi_b, \quad (7.40)$$

with the implicit definition $\phi_a = (\phi_a, \phi_{bc} = \phi_b \phi_c)$. We perform a second Legendre transformation with

$$\begin{aligned} & \Gamma_{2PI}[\phi_a, \phi_{ab}, Q] \\ & = \sup_J (J^{ab} \phi_{ab} + \Gamma[\phi_a, Q, R^{ab} = -J^{ab}]), \end{aligned} \quad (7.41)$$

leading to $\frac{\delta\Gamma_{2\text{PI}}}{\delta\phi_{ab}} = J^{ab}$ and $\phi_{ab} = G$. Note that $\Gamma[\phi_a, Q, R^{ab}]$ already includes the standard subtraction $-J^{ab}\phi_a\phi_b$. We arrive at

$$\frac{\delta\Gamma_{2\text{PI}}}{\delta\phi_a} \frac{\delta\Gamma_{2\text{PI}}}{\delta Q^a} + 2 \frac{\delta\Gamma_{2\text{PI}}}{\delta\phi_{ab}} \frac{\delta\Gamma_{2\text{PI}}{}^c}{\delta Q^b} \phi_{ca} \equiv 0. \quad (7.42)$$

The last term on the lhs of (7.42) accounts for the BRST variation of ϕ_{ab} that derives from the BRST variations of its field content $\hat{\phi}_a\hat{\phi}_b$. The BRST variation of $\hat{\phi}_a\hat{\phi}_b$ can be added with a source term $Q^{ab}\mathfrak{s}(\hat{\phi}_a\hat{\phi}_b)$ in the path integral leading to $\Gamma_{2\text{PI}} = \Gamma_{2\text{PI}}[\phi_a, \phi_{ab}, Q_a, Q_{ab}]$. Then we have

$$\frac{\delta\Gamma_{2\text{PI}}}{\delta Q^a} = \frac{\delta\Gamma_{2\text{PI}}{}^c}{\delta Q^b} \phi_{ca} + \phi_{bc} \frac{\delta\Gamma_{2\text{PI}}{}^c}{\delta Q^a}. \quad (7.43)$$

Eq. (7.43) and the symmetry property $\phi_{ab} = \gamma^c{}_b\phi_{ca}$ lead to (7.42). Collecting the fields into a super-field $\phi_{\mathbf{a}} = (\phi_a, \phi_{bc})$, and $Q^{\mathbf{a}} = (Q^a, Q^{bc})$ with $\Gamma_{2\text{PI}} = \Gamma_{2\text{PI}}[\phi_{\mathbf{a}}, Q^{\mathbf{a}}]$, we get an appealing form of the STI (7.42)

$$\frac{\delta\Gamma_{2\text{PI}}}{\delta\phi_{\mathbf{a}}} \frac{\delta\Gamma_{2\text{PI}}}{\delta Q^{\mathbf{a}}} = 0. \quad (7.44)$$

In its spirit (7.44) is close to the mSTI written as a master equation [124–127]. As in (7.44) the master equation emphasises the algebraic structure of the mSTI but hides the symmetry-breaking nature of the identities. Nonetheless algebraic identities are useful if constructing consistent truncations as well as discussing minimal symmetry breaking due to quantisation in the sense of Ginsparg-Wilson relations [35].

As in (7.38) we can absorb Q^a -derivatives with help of (7.36), (7.37). As the source Q is a spectator of the Legendre transformation (7.41) we have $\frac{\delta\Gamma}{\delta Q} = \frac{\delta\Gamma_{2\text{PI}}}{\delta Q}$ and (7.37) reads for the 2PI effective action

$$\frac{\delta\Gamma_{2\text{PI}}}{\delta Q^a} = -(\mathfrak{s}\phi^a + \frac{1}{2}(\mathfrak{s}\phi^a)^{bc}\phi_{bc}), \quad (7.45)$$

where we have used that $R^{ab} = -J^{ab}$ and hence $\frac{\delta\Gamma}{\delta R^{ab}} = -\phi_{ab}$. Using (7.45) we arrive at

$$-\frac{\delta\Gamma_{2\text{PI}}}{\delta\phi_a} (\mathfrak{s}\phi^a + (\mathfrak{s}\phi^a)^{bc}\phi_{bc}) + \frac{\delta\Gamma_{2\text{PI}}}{\delta\phi_{ab}} \frac{\delta\Gamma_{2\text{PI}}}{\delta Q^{ab}} = 0. \quad (7.46)$$

The BRST variation of $\hat{\phi}_{ab}$ involves $\hat{\phi}_{cd}\hat{\phi}_e$ and Q^{ab} is a source for a specific tensor structure $\mathcal{T}^{abcde}\phi_{cd}\phi_e$. Within regularisations of the 2PI effective action that regularises three point functions, the source Q^{ab} can be eliminated analogously to (7.38). This is an interesting option for NPI regularisations of gauge theories, in particular in view of consistent approximations [207, 209–213].

We close this section with a short summary of the derivation of STIs without the use of BRST transformations. To that end we integrate out the auxiliary field λ and use the classical gauge-fixed action (7.7). In view of the auxiliary nature of the ghost fields we derive identities that describe the response of general correlation functions to (infinitesimal) gauge transformations \mathfrak{g}_ω of the physical fields, the gauge field and possible matter fields. Gauge-invariant correlation functions I, \tilde{I} are invariant under these transformations.

$$(\mathfrak{g}_\omega\hat{\varphi})_a = ((D\omega)_i, [\omega, C]_\alpha, [\omega, \bar{C}]_\alpha). \quad (7.47)$$

The linear operator \mathfrak{g} is bosonic as distinguished to \mathfrak{s} . It can be cast into the form (7.9) as a functional derivative operator $\mathfrak{g} = (\mathfrak{g}\hat{\varphi}_{\mathbf{a}}) \frac{\delta}{\delta\hat{\varphi}_{\mathbf{a}}}$. The related generator $\hat{I}_{\mathfrak{g}}$ reads

$$\hat{I}_{\mathfrak{g}} = \left(J^{\mathbf{a}}(\mathfrak{g}\hat{\phi})_{\mathbf{a}} - (\mathfrak{g}S[\hat{\phi}]) \right)_{\hat{\phi} = \frac{\delta}{\delta J}}, \quad (7.48)$$

leading to the STI (7.16) for $\mathcal{W}_{\mathfrak{s}, I}$. Restricting ourselves to J -independent \hat{I} 's (7.16c) reads

$$\widehat{\delta I} = (\mathfrak{g}\hat{I}[\hat{\phi}])[\hat{\phi} = \frac{\delta}{\delta J}]. \quad (7.49)$$

In the presence of a regulator term the generator of symmetry transformations turns into

$$\hat{I}_{\mathfrak{g}} = \left(J^{\mathbf{a}}(\mathfrak{g}\hat{\phi})_{\mathbf{a}} - (\mathfrak{g}(S + \Delta S))[\hat{\phi}] \right)_{\hat{\phi} = \frac{\delta}{\delta J}}, \quad (7.50)$$

leading to the mSTIs (7.22) and (7.28) for $\mathcal{W}_{\mathfrak{g}, I}$ and $\mathcal{W}_{\mathfrak{s}, I}$ respectively. We close this section with exemplifying the mSTI $\tilde{W}_{\mathfrak{g}, I}$ at $I = 1$ and the standard flow. Then, with the alternative representation (7.30) we are led to [123]

$$\mathfrak{g}\Gamma[\phi, R] = \left(\mathfrak{g} \left(\frac{1}{2\xi} \mathcal{F}^\alpha \mathcal{F}^\alpha + \Delta S \right) [G_{\frac{\delta}{\delta\phi}} + \phi] \right) - \mathfrak{g}\Delta S'[\phi, R] - \left(\mathfrak{g}(\bar{C}_\alpha \frac{\partial \mathcal{F}^\alpha}{\partial A_i} D_i^\beta C_\beta) [G_{\frac{\delta}{\delta\phi}} + \phi] \right). \quad (7.51)$$

The right hand side of (7.51) reproduces the gauge variation of the classical action $\mathfrak{g}S[\phi]$ as well as loop terms. The highest loop order (in the full propagator) is given by the highest order of the field ϕ in the gauge fixing term and the ghost term in the classical action as well as the regulator term. For linear gauges and $\hat{\phi} = \hat{\varphi}$ the modified STI (7.51) involves one loop (gauge fixing, ghost term) and two loop terms (ghost term) apart from the regulator-dependent terms. Thus a purely algebraic form of the mSTI (7.51) can be achieved for regulator terms with R involving R^{ab} and R^{abc} .

C. Gauge-invariant flows

An interesting option for flows in gauge theories is the construction of (partially) gauge-invariant flows. The gain of such formulations is twofold. Firstly they allow for a more direct computations of physical observables. Observables are gauge-invariant as opposed to Greens functions in gauge-fixed formulations. Secondly one can hope to avoid the subtleties of solving the symmetry relations in the presence of a regulator. However, gauge-invariant formulations come to a price that also has to be evaluated: if the corresponding flows are themselves far more complicated than the standard gauge-fixed flows the benefit of no additional symmetry relations is, at least partially, lost. Also, gauge invariance does not rule out the persistence of non-trivial symmetry relations, mostly formulated in the form of Nielsen identities or, alternatively, in the form of specific projections of the general Dyson-Schwinger equations valid within such a setting.

In the present work we concentrate on gauge-invariant flows formulated in mean fields and the effective action Γ_k . An alternative construction of gauge-invariant flows is based on the Wilsonian effective action $S_{\text{eff},k}$, (3.40), formulated in Wilson lines and using gauge-covariant regulators. For details we refer the reader to [152–158] and references therein.

1. Background field flows

The first and most-developed gauge-invariant flow originates in the use of the background field formalism. We couple the fundamental fields to the currents, $\phi = \varphi$ with

$$\phi = (a_i, C_\alpha, \bar{C}_\alpha), \quad (7.52)$$

where the full gauge field is defined as

$$A = \bar{A} + a. \quad (7.53)$$

The gauge field A is split into a background field configuration \bar{A} and a fluctuation field a coupled to the current. BRST transformations and gauge transformations are defined by (7.8) and (7.47) respectively at fixed background field \bar{A} , $\mathfrak{s}\bar{A} = \mathfrak{g}\bar{A} = 0$. Note that the covariant derivative reads $D = D(a + \bar{A})$. Therefore, the mSTIs (7.28) for $\mathcal{W}_{\mathfrak{s},I}$ and $\mathcal{W}_{\mathfrak{g},I}$ persist. Within appropriate gauges, e.g. the background field gauge $\mathcal{F} = D(\bar{A})a$, there is an additional symmetry: the action (7.7) is invariant under a combined gauge transformation of the background

field $\bar{A} \rightarrow \bar{A} + D(\bar{A})\omega$ and the fluctuation field $a \rightarrow [\omega, a]$. This invariance follows by using that the fluctuation field ϕ in (7.52) as well as the covariant derivatives $D(A)$ and $D[\bar{A}]$ transform as tensors under this combined transformation. Defining background field transformations

$$\bar{\mathfrak{g}}_\omega(\varphi, \bar{A}) = (-D(\bar{A})\omega, 0, 0, D(\bar{A})\omega), \quad (7.54)$$

the transformation properties under the combined transformation are summarised in

$$(\mathfrak{g} + \bar{\mathfrak{g}})_\omega(\varphi, D(A), D(\bar{A})) = [\omega, (\varphi, D(A), D(\bar{A}))]. \quad (7.55)$$

with \mathfrak{g} defined in (7.47). As the action S in (7.7) with $\mathcal{F} = D(\bar{A})$ or similar choices can be constructed from $(\varphi, D(A), D(\bar{A}))$ this leads us to

$$(\mathfrak{g} + \bar{\mathfrak{g}})S[\phi, \bar{A}] = 0, \quad (7.56)$$

Then, the corresponding effective action $\Gamma[\phi, \bar{A}]$ is invariant under the above transformation, in particular we define a gauge-invariant effective action with

$$\Gamma[A] = \Gamma[\phi = 0, A]. \quad (7.57)$$

We have $(\mathfrak{g} + \bar{\mathfrak{g}})\Gamma[\phi, \bar{A}] = 0$, where $\mathfrak{g}, \bar{\mathfrak{g}}$ act on $\phi = \varphi$ according to (7.47) and (7.54). This implies in particular $\mathfrak{g}\Gamma[A] = 0$. The gauge invariance of $\Gamma[\phi, A]$ persists in the presence of a regulator if $\Delta S[a, R(\bar{A})]$ is invariant under the combined transformation of a and \bar{A} . This is achieved for regulators R that transform as tensors under gauge transformations $\bar{A} \rightarrow \bar{A} + D(\bar{A})\omega$. This amounts to the definition of a background field dependent $R(\bar{A})$ with

$$\bar{\mathfrak{g}}R(\bar{A}) = [\omega, R(\bar{A})]. \quad (7.58)$$

For example, standard flows follow with the regularisation $\Delta S = R^{ij}(\bar{A})a_i a_j + R^{\alpha\beta}(\bar{A})C_\alpha C_\beta$. The invariance property $(\mathfrak{g} + \bar{\mathfrak{g}})\Delta S = 0$ follows immediately from (7.55) and (7.58). The relation (7.58) is e.g. achieved for regulators in momentum space depending on covariant momentum $D(\bar{A})$. Correlators \tilde{I} still satisfy the modified STI (7.28), but additionally there is a modified STI related to the background field gauge transformations (7.54). The related generator is

$$\hat{I}_{\bar{\mathfrak{g}}} = \left(J^{\mathfrak{a}}(\bar{\mathfrak{g}}\hat{\phi})_{\mathfrak{a}} - (\bar{\mathfrak{g}}(S + \Delta S))[\hat{\phi}] \right)_{\hat{\phi} = \frac{\delta}{\delta J}}, \quad (7.59)$$

leading to mSTIs (7.22) and (7.28) for $\mathcal{W}_{\bar{\mathfrak{g}},I}$. For the effective action ($I = 1$) the mSTI reads

$$\begin{aligned} \bar{\mathfrak{g}}\Gamma[\phi, R] &= \left(\frac{1}{2\xi} \bar{\mathfrak{g}}(\mathcal{F}^\alpha \mathcal{F}^\alpha)[G_{\frac{\delta}{\delta\phi}} + \phi] \right) \\ &\quad - \left(\bar{\mathfrak{g}}(\bar{C}_\alpha \frac{\partial \mathcal{F}^\alpha}{\partial A_i} D_i^\beta C_\beta)[G_{\frac{\delta}{\delta\phi}} + \phi] \right). \end{aligned} \quad (7.60)$$

Adding (7.51) and (7.60) we arrive at

$$(\mathfrak{g} + \bar{\mathfrak{g}})\Gamma[\phi, \bar{A}, R] = 0. \quad (7.61)$$

The derivation makes clear that, despite background gauge invariance (7.61), the effective action $\Gamma[\phi, \bar{A}]$ still carries the BRST symmetry (7.28) displayed in $\mathcal{W}_{\mathfrak{s}}$ or $\mathcal{W}_{\mathfrak{g}}$, where the background field is a spectator $s\bar{A} = 0$. In other words, the non-trivial relations between N -point functions of the fluctuation field are still present. However, for N -point functions in the background field they play no rôle which has been used for simplifications within loop computations.

Therefore it is tempting to use these features for the construction of gauge-invariant flows. General flows within such a setting are still provided by (3.60). In particular with (3.63) we arrive at the flow of $\Gamma_k[A]$ as [48, 49, 108–110, 115, 116, 118, 119, 121–123, 148–151]

$$\dot{\Gamma}_k[A] = (\Delta S[G \frac{\delta}{\delta \phi} + \phi, \dot{R}(A)])_{\phi=0} - \Delta S'[0, \dot{R}(A)]. \quad (7.62)$$

It has already been discussed in [48, 49, 54, 55, 122] that the flow (7.62) is not closed as it depends on

$$\frac{\delta^2 \Gamma_k[0, \bar{A}, R]}{\delta \phi^2}, \quad (7.63)$$

the propagator of the fluctuation field, and possibly higher derivatives w.r.t. ϕ evaluated at vanishing fluctuation field $\phi = 0$. The lhs of (7.62) cannot be used to compute this input as it only depends on $\bar{A} = A$. Moreover, as has been stated above, these N -point functions still satisfy the modified Slavnov-Taylor identities discussed in the last section. The differences between $\Gamma^{(2)}[A]$ and the fluctuation propagator (7.63) become important already at two loop, and are crucial for confinement [145]. The correct input (7.63) at one loop was used to compute the universal two loop β -function which cannot be reproduced by using $\Gamma^{(2)}[A]$ [48, 49]. Still one can hope that qualitative features of the theory are maintained in such a truncation. Then, a measure for the quality of such a truncation is given by the difference between a derivative w.r.t. \bar{A} and one w.r.t. a of the effective action. This relation reads [48, 49, 122]

$$\begin{aligned} \left(\frac{\delta}{\delta \bar{A}} - \frac{\delta}{\delta a} \right) \Gamma[\phi, \bar{A}, R] &= \left\langle \left(\frac{\delta}{\delta \bar{A}} - \frac{\delta}{\delta \hat{a}} \right) (S + \Delta S) \right\rangle \\ &= \left(\left(\frac{\delta}{\delta \bar{A}} - \frac{\delta}{\delta \hat{a}} \right) (S + \Delta S) \right) [G \frac{\delta}{\delta a} + \phi, \bar{A}], \end{aligned} \quad (7.64)$$

and can be understood as a Nielsen identity. Eq. (7.64) relates Green functions of the background field with that

of the fluctuation field. The latter satisfy mSTIs whereas the former transform as tensors under gauge transformations reflecting gauge invariance. Hence, (7.64) encodes the mSTIs. Note also that the background field dependence stemming from the regulator should be understood as a parameter dependence and not as a field dependence¹⁴. An improvement of the current results in gauge theories [48, 49, 108–110, 115, 116, 118, 119, 121–123, 148–151] requires an implementation of the Nielsen identity (7.64) beyond perturbation theory.

It is possible to enhance background field flows to fully gauge-invariant flows with standard STIs by identifying the background field with a dynamical field. There are two natural choices: $\bar{A} = \hat{A}$ ¹⁵ and $\bar{A} = \langle \hat{A} \rangle = A$. The latter leads to the definition of the effective action as a higher order Legendre transform. Then we have additional terms to those (3.60) as

$$\Gamma^{,a} = J^a - \left\langle \frac{\delta(S + \Delta S)}{\delta \bar{A}} \right\rangle. \quad (7.65)$$

With (7.65) we get additional terms in the relations between ϕ -derivatives of Γ and J -derivatives of W . Eq. (7.65) is actually implementing the Nielsen identity (7.64) on the level of the Legendre transformation. This entails that in particular the basic relations (3.45) and (3.46) receive modifications originating in (7.65). As an example we study the standard flow for the effective action which reads

$$\begin{aligned} \dot{\Gamma}[A, \phi] &= \dot{R}^{ab} W_{,ab} \\ &= \dot{R}^{ab} G_{ab} + \left(\frac{\delta(S + \Delta S)}{\delta \bar{A}} \text{-terms} \right), \end{aligned} \quad (7.66)$$

where the propagator G is defined with $G = 1/(\Gamma^{(2)} + \Delta S^{(2)})$. The propagator G of the dynamical field transforms as a tensor under gauge transformations reflecting gauge invariance. However, it can be shown in a perturbative loop expansion that effectively the flow equation can be rewritten as that in the background field formalism: the effective propagator $W^{(2)} + (W^{(1)})^2$ behaves as that of the fluctuation field in the background field formulation. This is already indicated in (7.65). The correction terms involve the same correlation functions already relevant in the Nielsen identity (7.64). So still

¹⁴ For infrared diverging regulator $R(\bar{A})$ even the computation of the one loop β -function requires a subtraction of the field dependence of $R(\bar{A})$ [48, 49, 122].

¹⁵ This choice can be only used in the regulator.

we deal with non-trivial symmetry identities. Nonetheless the above formulation furthers the knowledge about truncation schemes that expand about $\Gamma^{,a} = J^a$, or alternatively about $(\frac{\delta}{\delta A} - \frac{\delta}{\delta a})\Gamma[\phi, \bar{A}, R] = 0$. Details shall be provided elsewhere.

The other suggestion $\bar{A} = A$ relates to the use of a regulator term $\Delta S[A, R(A)]$. Such a regulator term can be written as $\widehat{\Delta S}[A, \hat{R}] = \Delta S[A, R(A)]$, where $\hat{R}^{a_1 \dots a_n} = \Delta S^{a_1 \dots a_n}[0, R(0)]/(n!)$ is the n th expansion coefficient in a Taylor expansion of $\Delta S[A, R(A)]$ in the gauge field A . This flow is covered by the general flow (3.60) and involves all loop orders in the full propagator. Again this effectively reduces to the background field flow and comes at the expense of an infinite series of loop terms in the flow. In this context we remark that the latter set-back is avoided within the Polchinski equation. This follows in the present setting with (3.40) and the flow (3.28) for the Schwinger functional.

2. Geometrical effective action

We have seen in the last section that the flow of the gauge-invariant effective action within the background field formulation is not closed. In the process of curing this problem we encounter the persistence of non-trivial symmetry relations, conveniently summarised in (7.64). Both aspects originate in the fact that the sources are coupled to fields that do not transform trivially under gauge or BRST transformations. Hence the question arises whether one can do better. Within the framework of the geometrical or Vilkovisky-DeWitt effective action the fields ϕ coupled to the sources are scalars under gauge transformations.

Then, gauge-invariant flows can be formulated [159, 160]. We do not want to go in the details of the general construction that can be found in [160]. The configuration space is provided with a connection Γ_V (Vilkovisky connection) which is constructed such that the disentanglement between gauge fibre and base space is maximal. The gauge fields A_i are substituted by geodesic normal fields ϕ_i that are tangent vectors at a base point (background field) \bar{A} . As a consequence the geodesic fields ϕ_α tangential to the fibre drop out of the path integral, only the fields ϕ_A tangential to the base space remain and are gauge-invariant. This construction is lifting up the relation between fluctuation field and background field (7.53). The linear background relation can be read as the limit in which the connection Γ_V is neglected. The

full relation reads schematically

$$\phi_i = A_i - \bar{A}_i + \Gamma_{V,i}{}^{jk} \phi_j \phi_k + O(\phi_i^3), \quad (7.67)$$

with $\mathfrak{g}\phi^A = 0 = \bar{\mathfrak{g}}\phi^A$. This is used to construct a gauge-invariant effective action $\Gamma[\phi^A, \bar{A}, R]$ which is gauge-invariant under both sets of gauge transformations \mathfrak{g} and $\bar{\mathfrak{g}}$ [160]. Again a gauge-invariant effective action in one field can be defined as $\Gamma[A, R] = \Gamma[\phi = 0, A, R]$. The flows of $\Gamma[\phi, \bar{A}, R]$ and $\Gamma[A, R]$ are given by (3.63) and (7.62) respectively, both being gauge-invariant flows. We still have a Nielsen identity equivalently to (7.64). In the underlying theory without regulator term it reads

$$\Gamma_{,i} + \Gamma_{,a} \langle \phi^a{}_{;i} \rangle = 0, \quad (7.68)$$

where $\phi^a{}_{;i}$ stands for the covariant derivative with the Vilkovisky connection Γ_V . The related symmetry operator is provided by

$$\hat{I}_n = \frac{\delta}{\delta A} + J^a \hat{\phi}_{a;i} \left[\frac{\delta}{\delta J} \right]. \quad (7.69)$$

With (3.8b) this turns into

$$\begin{aligned} \hat{I}_n &= \frac{\delta}{\delta A} - \frac{\delta \Delta S}{\delta A} [G_{\frac{\delta}{\delta \phi}} + \phi] + \frac{\delta \Delta S}{\delta A} [\phi] \\ &+ \left(J^a - \Delta S^{,ab} [G_{\frac{\delta}{\delta \phi}}] G_{bc} \frac{\delta}{\delta \phi} \right) \hat{\phi}_{a;i} [G_{\frac{\delta}{\delta \phi}} + \phi], \end{aligned} \quad (7.70)$$

in the presence of the regulator term. For standard flows the choice $\mathcal{W}_{n,1}$ in (7.30) reproduces the Nielsen identity derived in [160],

$$\Gamma_{k,i} = \frac{1}{2} G^{ab} R_{ba,i} - \left(\Gamma_{k,a} - R_{ab} G^{bc} \frac{\delta}{\delta \phi^c} \right) \langle \phi^a{}_{;i} \rangle. \quad (7.71)$$

For more details and its use within truncation schemes we refer to [160]. The formalism discussed above provides gauge-invariant flows that are closely linked to the background field formalism (in the Landau-DeWitt gauge) as well as to standard Landau gauge. This comes with the benefit that results obtained in the latter can be partially used within the present formalism. Indeed the present setting can be used to improve the gauge consistency of these results. We hope to report on results for infrared QCD as well as gravity in near future.

To conclude, we have discussed the various possibility of defining gauge-invariant flows and their relations to gauge-fixed formulations. These relations come with the benefit that it allows to start an analysis in the gauge-invariant formulations on the basis of non-trivial results already achieved in gauge-fixed settings, one does not have to start from scratch.

D. Chiral symmetry and anomalies

We want to close this chapter with a brief discussion of FRG flows in theories with symmetries that are flawed by anomalies on the quantum level, e.g. [116–118, 124]. A more detailed account shall be given elsewhere. In particular a discussion of the chiral symmetry breaking requires a careful investigation of chiral anomalies. The deformation of the chiral symmetry from a general RG transformation has already been considered in [35], and leads to the Ginsparg-Wilson relation¹⁶. This has been emphasised in [124–126]. A discussion of chiral symmetry breaking requires a careful investigation of chiral anomalies. Integrated anomalies are tightly linked to topological degrees of freedom like instantons via the index theorem. FRG methods have been shown to be sensitive to topological degrees of freedom [115, 116], an interesting quantum mechanical example can be found in [179]. In the present section we consider the gauge field action (7.7) together with a Dirac action

$$S_D[\phi] = \bar{\psi}_a (\not{D} + m)^{ab} \psi_b, \quad (7.72)$$

with a possible mass term and $\phi = (A, C, \bar{C}, \psi, \bar{\psi})$. The Dirac operator \not{D} reads

$$\not{D}^{ab} = (\not{\partial} + \mathcal{P}A)^{ab} \quad (7.73)$$

with the free Dirac operator $\not{\partial}$ and a coupling to the gauge field with a possible projection \mathcal{P} either proportional to the identity $\mathcal{P} = \mathbb{1}$, or projecting on right- or left-handed Weyl fermions $\mathcal{P}_{\pm} = \frac{1 \pm \gamma_5}{2}$. Here we consider

$$\mathcal{P} = \mathcal{P}_+ = \frac{1 + \gamma_5}{2}, \quad m = 0, \quad (7.74)$$

a theory with left-handed Weyl fermions coupled to a gauge field, and free right-handed Weyl fermions. The symmetry transformation that leaves the action (7.72) invariant is given by

$$\mathfrak{g}_+ \phi = (\mathfrak{g}_+ A, \mathfrak{g}_+ C, \mathfrak{g}_+ \bar{C}, \omega \mathcal{P}_- \psi, -\bar{\psi} \mathcal{P}_+ \omega). \quad (7.75)$$

The transformations (7.75) cover both, BRST transformations with $\mathfrak{g}_+ = \mathfrak{s}$ with $\omega = C$, and $\mathfrak{g}_+ = \mathfrak{g}$ with gauge transformation parameter ω . Here we stick to $\mathfrak{g}_+ = \mathfrak{g}$. The chiral anomaly comes into play since the fermionic path integral measure $d\psi d\bar{\psi}$ is not left invariant under

the transformation (7.75). In other words, (7.75) is not unitary. We quote the result

$$\mathfrak{g}(d\psi d\bar{\psi}) = \omega^\alpha \mathcal{A}_\alpha d\psi d\bar{\psi}, \quad (7.76)$$

with infinitesimal variation $\omega^\alpha \mathcal{A}_\alpha$. The non-Abelian anomaly \mathcal{A} reads

$$\mathcal{A}^\alpha(x) = \frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr} t^\alpha (\partial_\mu A_\nu F_{\rho\sigma} - \frac{1}{2} A_\nu A_\rho A_\sigma). \quad (7.77)$$

Then, the generator of gauge transformations $\hat{I}_\mathfrak{g}$ in (7.48) receives a further contribution and reads

$$\hat{I}_\mathfrak{g} = \left(J^\mathfrak{a}(\mathfrak{g}\hat{\phi})_\mathfrak{a} - (\mathfrak{g}S[\hat{\phi}]) - \mathcal{A}[\hat{\phi}] \right)_{\hat{\phi}=\frac{\delta}{\delta J}}. \quad (7.78)$$

and with (3.8b) in the presence of the regulator term we arrive at

$$\hat{I}_\mathfrak{g} = \left(J^\mathfrak{a}(\mathfrak{g}\hat{\phi})_\mathfrak{a} - (\mathfrak{g}(S[\hat{\phi}] + \Delta S)) - \mathcal{A}[\hat{\phi}] \right)_{\hat{\phi}=\frac{\delta}{\delta J}}. \quad (7.79)$$

Eq. (7.79) can also be read off from (7.50) since the anomaly term $\mathcal{A}[\hat{\phi}]$ commutes with ΔS .

We conclude with briefly discussing the $U_A(1)$ -anomaly relevant for anomalous chiral symmetry breaking. We restrict ourselves to standard flows with quadratic regulator. The Dirac action (7.72) with $\mathcal{P} = \mathbb{1}$ is invariant under global axial $U_A(1)$ -transformations. The related Noether current is derived from the $U_A(1)$ transformations of the fermions

$$\mathfrak{g}_A \psi = \omega \gamma_5 \psi, \quad \mathfrak{g}_A \bar{\psi} = \bar{\psi} \gamma_5 \omega. \quad (7.80)$$

The rest of the fields transforms trivially with $\mathfrak{g}_A A = \mathfrak{g}_A C = \mathfrak{g}_A \bar{C} = 0$. The related anomaly reads

$$\mathcal{A} = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma}. \quad (7.81)$$

The anomalous Ward identity for the effective action, $\tilde{\mathcal{W}}_{\mathfrak{g}_A,1}$, in the presence of the regulator reads

$$\begin{aligned} & (\mathfrak{g}_A \phi)_a \Gamma^{,a} + (\mathfrak{g}_A(S_D + \Delta S))[G \frac{\delta}{\delta \phi} + \phi] \\ & = (\mathcal{A}[G \frac{\delta}{\delta \phi} + \phi]). \end{aligned} \quad (7.82)$$

The space time integral of (7.82) produces the (expectation value of the) topological charge on the rhs, as well as the analytical index of the modified Dirac operator on the lhs. In [116] it has been shown that the number of zero modes stays the same for regulators with chiral symmetry. The chiral anomaly has been investigated in [117]. In general the lhs of (7.82) is computed directly from the effective action. Accordingly we can use (7.82) for testing

¹⁶ The derivation in [35] makes no use of the lattice.

the potential of given truncations to the effective action for incorporating the important topological effects. Additionally it provides non-trivial relations between the couplings. For example, the leading order effective action derived in [116] satisfies (7.82) up to sub-leading terms (in $1/k$). Eq. (7.82) can be used to determine coefficients and form of these sub-leading terms, in particular in view of CP-violating effects.

VIII. TRUNCATION SCHEMES AND OPTIMISATION

The reliability of results obtained within the functional RG depends on the appropriate choice of a truncation scheme for the physics under investigation, as well as an optimisation of the truncation with the methods introduced in section V. The truncation has to take into account all relevant operators or vertices. In theories with a complicated phase structure this might necessitate introducing a large number of vertices to the effective action in terms of the fundamental fields. A way to avoid such a drawback is to reparameterise the theory in terms of the relevant degrees of freedom [47, 80–90].

Fixed point quantities like critical exponents and general anomalous dimensions have very successfully been derived within the flow equation approach, mostly in the derivative expansion, see reviews [16–27, 29]. For the evaluation of these results in view of quantitative reliability one has to assess the problem of optimisation. To that end we evaluate the consequences of the relation between RG scaling and flow for an appropriate choice of classes of regulators. As an example for the optimisation criterion developed in section V, we discuss functional optimisation within the zeroth order derivative expansion. The unique optimised regulator is derived and its extension to higher order of the truncation scheme is discussed. For explicit results we refer the reader to the literature, in particular [76].

A. Field reparameterisations

The derivation of the flow in section III was based on a bootstrap approach in which the existence of a renormalised Schwinger functional in terms of the possibly composite fields ϕ was assumed. This already took into account that the fundamental fields φ may not be suitable degrees of freedom for all regimes of the theory under

investigation. For example, we could consider fields $\phi(\varphi)$ that tend towards the fundamental fields in the perturbative regime for large momenta,

$$\phi(\varphi) \xrightarrow{p^2 \rightarrow \infty} \varphi, \quad (8.1)$$

while being a non-trivial function of φ in the infrared. This includes the bosonisation of fermionic degrees of freedom [80, 81, 86–88], e.g. in low-energy QCD, where the relevant degrees of freedom are mesons and baryons instead of quarks. More generally such a situation applies to all condensation effects.

In such a case the Green functions of φ will show a highly non-trivial momentum dependence or even run into singularities. Moreover, physically sensible truncations to the effective action in terms of φ could be rather complicated. These problems can be at least softened with an appropriate choice of ϕ that mimics the relevant degrees of freedom in all regimes. Such a choice may be adjusted to the flow by implementing the transition from φ to $\phi(\varphi)$ in a k -dependent way [80, 81]. This can be either done by coupling the current and the regulator to a k -dependent field $\hat{\phi}_k$, or by choosing a k -dependent argument ϕ_k of the effective action Γ_k :

The former option leads to additional loop-terms in the flow. The relation (3.18) is modified as the full Schwinger functional $W[J]$ couples to a k -dependent field $\hat{\phi}_k$ with $\partial_t W[J] = J^{\mathbf{a}} \langle \partial_t \hat{\phi}_{k\mathbf{a}} \rangle$, and the flow operator ΔS_2 changes as the regulator term has an additional k -dependence via the field, $\Delta S_2[\phi, \dot{R}] \rightarrow \Delta S_2[\phi, \dot{R}']$ where R' is defined with

$$\Delta S[\hat{\phi}_k, \dot{R}'] = \Delta S[\hat{\phi}_k, \dot{R}] + \partial_t \hat{\phi}_{k\mathbf{a}} \Delta S^{\mathbf{a}}[\hat{\phi}_k, \dot{R}], \quad (8.2)$$

where $\partial_t \hat{\phi}_{k\mathbf{a}} = \partial_t \hat{\phi}_{k\mathbf{a}}(\hat{\phi}_k)$. With these modifications the derivation of the flow can straightforwardly be redone.

The latter option keeps the flow (3.60) as the partial derivative is taken at fixed argument ϕ : $\partial_t \tilde{I}_k = \partial_t|_{\phi} \tilde{I}_k$. For integrating the flow the total derivative is required,

$$\frac{d\tilde{I}_k[\phi_k]}{dt} = -\Delta S_2[\phi_k, \dot{R}] \tilde{I}_k[\phi_k] + \partial_t \phi_{k\mathbf{a}} \tilde{I}_k^{\mathbf{a}}[\phi_k]. \quad (8.3)$$

We can also combine the above options. For the sake of simplicity we restrict ourselves to the flow of the effective action which reads in this general case

$$\begin{aligned} \frac{d\Gamma_k[\phi]}{dt} &= (\Delta S[G \frac{\delta}{\delta\phi} + \phi, \dot{R}']) - \Delta S'[\phi, \dot{R}'] \\ &+ \left(\partial_t \phi_{\mathbf{a}} - \langle \partial_t \hat{\phi}_{\mathbf{a}} \rangle \right) \Gamma_k^{\mathbf{a}}. \end{aligned} \quad (8.4)$$

In (8.4) we dropped the subscript k with $\phi = \phi_k$. The first term on the rhs is the expectation value of $\Delta S[\hat{\phi}, \dot{R}']$ defined in (8.2). The second term originates in the definition of Γ_k in (3.43). The expectation value in the second line in (8.4) can be written as $\langle \partial_t \hat{\phi} \rangle = ((\partial_t \hat{\phi})[G_{\delta\phi}^{\delta} + \phi])$, and \dot{R}' is defined in (8.2). We remark that (8.4) is finite for k -dependences of $\hat{\phi}$ that are local in momentum space. General k -dependences may require additional renormalisation. The flow (8.4) can be used in several ways to improve truncations.

A given truncation scheme can be further simplified in a controlled way by expanding the effective action about a stable solution $\bar{\phi}$ of the truncated equations of motion, $\Gamma_k^{\mathbf{a}}[\bar{\phi}] = 0$. Then the second line in (8.4) is sub-leading for $\phi - \bar{\phi}$ small and can be dropped if restricting the flow to the vicinity of $\bar{\phi}$. As this is an expansion about a minimum of the effective action, such a truncation has particular stability.

The second line also vanishes for $\partial_t \phi - \langle \partial_t \hat{\phi} \rangle = 0$. Subject to a given ϕ we demand $\hat{\phi}$ to satisfy

$$\langle \partial_t \hat{\phi} \rangle = \partial_t \phi. \quad (8.5)$$

With (8.5) the second line in (8.4) vanishes identically and the flow reduces to the first line. The construction of \dot{R}' requires the knowledge $\partial_t \hat{\phi}(\hat{\phi})$. Within given truncations (8.5) turns into a set of loop constraints that accompany the flow. These constraints resolve the dependences of the flowing composite fields ϕ_k on the microscopic degrees of freedom. This is more information than required for solving the flow. Indeed, we also can use (8.5) to circumvent the necessity of finding $\partial_t \hat{\phi}(\hat{\phi})$. We write for the expectation value of the second term in (8.2)

$$\begin{aligned} \Delta S^{\mathbf{b}}[G_{\delta\phi}^{\delta} + \phi, R] \gamma^{\mathbf{a}}_{\mathbf{b}} \langle \partial_t \hat{\phi}_{\mathbf{a}} \rangle \\ = (\Delta S^{\mathbf{b}}[G_{\delta\phi}^{\delta} + \phi, R] \gamma^{\mathbf{a}}_{\mathbf{b}} \partial_t \phi), \end{aligned} \quad (8.6)$$

where we have used (3.50) and (8.5). With (8.5) and (8.6) we can substitute all dependences on $\hat{\phi}, \partial_t \hat{\phi}$ in the flow (8.4) by that on $\phi_k, \partial_t \phi_k$. We are led to a closed flow for the effective action

$$\begin{aligned} \partial_t \Gamma_k[\phi] = (\Delta S[G_{\delta\phi}^{\delta} + \phi, \dot{R}]) - \Delta S'[\phi, \dot{R}'] \\ + (\Delta S^{\mathbf{b}}[G_{\delta\phi}^{\delta} + \phi, R] \gamma^{\mathbf{a}}_{\mathbf{b}} \partial_t \phi) - \partial_t \phi_{\mathbf{a}} \Gamma_k^{\mathbf{a}}. \end{aligned} \quad (8.7)$$

The first term in the second line keeps track of the k -dependence in $\hat{\phi}_k$ necessary to satisfy (8.5). The last term carries the k -dependence of ϕ_k . For the standard quadratic regulator (8.7) reads

$$\partial_t \Gamma_k[\phi] = G_{\mathbf{bc}} \dot{R}^{\mathbf{bc}} + 2R^{\mathbf{ab}} G_{\mathbf{ac}} \dot{\phi}_{\mathbf{b}}{}^{\mathbf{c}} - \dot{\phi}_{\mathbf{a}} \Gamma_k^{\mathbf{a}}. \quad (8.8)$$

We illustrate the above considerations within simple examples for quadratic regulator terms (8.8). Furthermore the examples are based on linear relations between $\partial_t \phi$ and ϕ . Then (8.5) can be resolved explicitly and up to rescalings (8.8) simplifies to the standard case: we absorb a t -dependent wave function renormalisation $Z_{\phi}^{1/2}$ into the field: $\phi_k = Z_{\phi}^{1/2} \phi_0$ with $\partial_t \phi_k = \gamma_{\phi} \phi_k$ with $\gamma_{\phi} = \frac{1}{2} \partial_t \ln Z_{\phi}$. Eq. (8.5) is satisfied with $\hat{\phi}_k = Z_{\phi}^{1/2} \hat{\phi}_0$. Then (8.8) reduces to

$$(\partial_t + \gamma_{\phi} \phi_{\mathbf{a}} \frac{\delta}{\delta \phi_{\mathbf{a}}}) \Gamma_k[\phi] = G_{\mathbf{bc}} (\partial_t + 2\gamma_{\phi}) R^{\mathbf{bc}}, \quad (8.9)$$

which also can be obtained by explicitly using $\hat{\phi}_k = Z_{\phi}^{1/2} \hat{\phi}$. The flow (8.9) also makes explicit that the transformation $\phi \rightarrow Z_{\phi}^{1/2} \phi$ is a RG rescaling. This procedure can be used to fix the flow of vertices.

Another simple example is the expansion of the effective action $\Gamma_k[\phi]$ about its minimum at $\phi_{\min}(k)$, implying $\phi \rightarrow \phi_k = \phi - \phi_{\min}(k)$. Such a reparameterisation guarantees that the minimum is always achieved for $\phi_k = 0$. The flow (3.60) only constitutes a partial t -derivative, as it is defined at fixed fields ϕ . With $\hat{\phi}_k = \hat{\phi} - \hat{\phi}_{\min}(k)$ we satisfy (8.5) and we are led to (8.7) with $\partial_t \phi_k = -\partial_t \phi_{\min}$ with $\partial_t \phi_{\mathbf{b}}{}^{\mathbf{c}} = 0$. The flow (8.7) reduces to the standard flow,

$$\begin{aligned} \partial_t \Gamma_k[\phi] = (\Delta S[G_{\delta\phi}^{\delta} + \phi, \dot{R}]) - \Delta S'[\phi, \dot{R}'] \\ + \Gamma_k^{\mathbf{a}}[\phi] (\partial_t \phi_{\min})_{\mathbf{a}}, \end{aligned} \quad (8.10)$$

now describing a total t -derivative of the effective action Γ_k . For quadratic regulators $R^{\mathbf{ab}}$ it reads

$$\partial_t \Gamma_k[\phi] = \dot{R}^{\mathbf{ab}} G_{\mathbf{ab}} + \Gamma_k^{\mathbf{a}}[\phi] (\partial_t \phi_{\min})_{\mathbf{a}}. \quad (8.11)$$

The flow of the minimum ϕ_{\min} can be resolved with help of $\frac{d}{dt} (\Gamma_k^{\mathbf{a}}[\phi_{\min}]) = 0$, and reads

$$(\partial_t \phi_{\min})_{\mathbf{a}} = \left(\frac{1}{\Gamma_k^{(2)}[\phi_{\min}]_{\mathbf{ab}}} \right) \partial_t \Gamma_k^{\mathbf{b}}[\phi_{\min}]. \quad (8.12)$$

As mentioned before, the examples used linear dependences of $\partial_t \phi$ on ϕ . Then (8.8) can also be derived explicitly as $\hat{\phi}$ is known. In the general case this is not possible, and (8.7) or (8.8) are the fundamental flows.

B. RG scaling and optimisation

The reliability of results obtained within functional RG flows hinges on an appropriately chosen truncation

scheme and a regulator choice that optimises the given truncation scheme. Without specifying the truncation scheme the following observation can be made: the renormalisation group analysis in section IV relates the RG equation for the full theory with that in the presence of a regulator. In particular we deduce from (4.25) and by identifying s with the RG scale μ , that the RG equation for the regularised effective action reads

$$D_\mu \Gamma_k = G_{\mathbf{bc}}[(D_\mu + \gamma_\phi)R]^{\mathbf{bc}}. \quad (8.13)$$

The right hand side of (8.13) entails the modification of the RG properties in the presence of the regulator. In (8.13) we have restricted ourselves to quadratic regulators. As explained in detail in the context of optimisation in chapter V, for full flows without truncations different choices of regulators, in particular those with different RG properties, lead to a RG rescaling of fields and coupling in the full effective action Γ . However, within truncations this modification usually leads to a physical change of the end-point of the flow. In turn, this problem is softened if restricting the class of regulators to those with [48, 49]

$$(D_\mu + \gamma_\phi)R = 0, \quad (8.14)$$

where $(\gamma_\phi R)^{\mathbf{ab}} = 2\gamma_\phi^{\mathbf{a}} R^{\mathbf{cb}}$. The constraint (8.14) leads to

$$D_\mu \Gamma_k = 0. \quad (8.15)$$

For the class of regulators with (8.14) the regularised correlation functions satisfy the same RG equation as in the underlying full theory, in particular this holds for the effective action, (8.15). Apart from the general optimisation arguments made above this facilitates the identification of anomalous dimensions and critical exponents. Indeed, the choice (8.14) with the additional identification $t = \ln \mu$ allows for the straightforward identification of t -running and RG running within fixed point solutions at all orders of the truncation scheme.

An explicit example for a class of regulators in standard flows that satisfy (8.14) is provided by [48, 49]

$$R^{\mathbf{ab}} = \frac{1}{2} \hat{\Gamma}^{\mathbf{ac}}[\bar{\phi}] r^{\mathbf{cb}}, \quad (8.16a)$$

with

$$D_\mu r = 0, \quad (8.16b)$$

where $\hat{\Gamma}^{\mathbf{ab}}$ is $\Gamma^{\mathbf{ab}}$ evaluated at some background field $\bar{\phi}$, with a possible subtraction. The subtraction can be used

to normalise $\hat{\Gamma}^{\mathbf{ab}}$. It could be proportional to $\Gamma^{\mathbf{ab}}$ evaluated at some momentum, e.g. at vanishing momentum. By construction (8.16) satisfies (8.14) as the two-point function does, $(D_s + \gamma_\phi)_{\mathbf{ac}} \hat{\Gamma}^{\mathbf{cb}} = 0$. If evaluating the standard flow (3.74) for the effective action at the background field $\bar{\phi}$, it takes the simple form

$$\begin{aligned} \dot{\Gamma}_k[\bar{\phi}] &= \frac{1}{2} \left(\frac{1}{1+r} \right)_{\mathbf{bc}} \dot{r}^{\mathbf{bc}} \\ &+ \frac{1}{2} \left(\frac{r}{1+r} \right)_{\mathbf{bc}} \left(\frac{1}{\Gamma^{(2)}} \partial_t \Gamma^{(2)} \right)^{\mathbf{bc}}, \end{aligned} \quad (8.17)$$

where for the sake of simplicity we have taken $\hat{\Gamma}^{\mathbf{ab}}[\bar{\phi}] = \Gamma^{\mathbf{ab}}[\bar{\phi}]$, that is no subtraction. The first term on the rhs of (8.17) can be integrated explicitly and contributes to the effective action only at perturbative one loop order. The second term gives non-trivial contributions if the spectral density changes. Eq. (8.17) is a spectrally adjusted flow.

In most truncation schemes used in the literature (8.16) simply amounts to the multiplication of the wave function renormalisation Z_ϕ . Then the propagator factorises $G[Z_\phi] = Z_\phi^{-1} G[1]$ which facilitates the computations. It is for the latter reason that (8.14) is a standard choice for regulators and it is a fortunate fact that the simple structure of flows for the choice (8.14) goes hand in hand with better convergence towards physics.

C. Integrated flows and fixed points

An optimisation with (5.29b) requires the minimisation of the norm of the difference between the regularised propagator and the full propagator with the constraint of keeping a fixed gap, see (5.32a). This implies a fine-tuning of the regulator in dependence of the two-point function, $\Gamma^{\mathbf{ab}}$. Here we outline a way of solving the flow equation which naturally incorporates such a task and hence minimises the additional numerical effort. First we turn the flow (3.60) into an integral equation

$$\tilde{I}_0 = \tilde{I}_\Lambda + \int_\Lambda^0 dt \Delta S_2 \tilde{I}_k, \quad (8.18a)$$

where Λ is the initial cut-off scale and the integrated flow for the effective action derives from (3.63) as

$$\begin{aligned} \Gamma_0 &= \Gamma_\Lambda + \int_\Lambda^0 dt \left((\Delta S[G_{\frac{\delta}{\delta\phi}} + \phi, \dot{R}]) \right. \\ &\quad \left. + \Delta S'[\phi, \dot{R}] \right). \end{aligned} \quad (8.18b)$$

Eq. (8.18) constitutes DSEs as already explained in section VI A. As distinguished to standard DSEs they only involve full vertices and propagators. Such a set of equations can be solved within an iteration about an ansatz for the full flow trajectory $\tilde{I}^{(0)}[\phi, R(k)]$. The better such an ansatz fits the result, the less iterations are needed for convergence towards the full result $\tilde{I}^{(\infty)}[\phi, R(k)]$. A benefit of such an approach is that it facilitates an implementation of the optimisation criterion (5.26) in its form (5.32a). After each iteration step we can prepare our regulator according to (5.32a) for the next step. Such a preparation is in particular interesting for truncations with a non-trivial momentum dependence for propagators and vertices. Furthermore the integral equations (8.18) are likely to be more stable in the vicinity of poles of the propagator.

The integral form (8.18) also is of use for an analysis of asymptotic regimes and in particular fixed point solutions. In general functional RG methods have been very successfully used within computations of physics at a phase transition. In particular critical exponents can be accessed easily.

At $k = 0$ the flows (3.60) have a trivial fixed point, $\partial_t \tilde{I}|_{k=0} \equiv 0$. In case the theory admits a mass-gap Λ_{gap} , this can be used to resolve the theory below this scale hence getting access to the deep infrared behaviour. For the sake of simplicity we further assume dimensionless couplings. The dimensionful case will be discussed elsewhere. Then, in the regime

$$k^2 \ll \Lambda_{\text{gap}}^2, \quad (8.19)$$

the flow of correlation functions \tilde{I}_k is parametrically suppressed by powers of k/Λ_{gap} ,

$$\partial_t \tilde{I}_k = O(k/\Lambda_{\text{gap}}). \quad (8.20)$$

Eq. (8.20) applies in particular to the effective action and its derivatives. It is convenient to parameterise the correlation functions \tilde{I}_k as

$$\tilde{I}_k = \tilde{I}_0(1 + \delta\tilde{I}_k). \quad (8.21)$$

Inserting this parameterisation into the integrated flow (8.18) we arrive at an integral equation for $\delta\tilde{I}_k$,

$$\delta\tilde{I}_k = - \int_k^0 dt' \Delta S_2 \left(\tilde{I}_0(1 + \delta\tilde{I}_{k'}) \right), \quad (8.22)$$

where ΔS_2 depends on Γ_k^{ab} (and its derivatives) that admit the same parameterisation (8.21). Assume for the

moment that $\delta\tilde{I}_{k'}$ and $\delta\Gamma_k^{\text{ab}}$ on the rhs of (8.22) only depend on dimensionless ratios

$$\hat{p}_i = \frac{p_i}{k}, \quad (8.23)$$

where the p_i are momenta of the correlation functions \tilde{I}_k , e.g. external momenta of n -point vertices. This assumption reads

$$\delta\tilde{I}_k = \delta\tilde{I}[\hat{p}_1, \dots, \hat{p}_n] + O(k/\Lambda_{\text{gap}}). \quad (8.24)$$

Inserting (8.24) into the rhs of the integrated flow (8.22) we deduce from a scale analysis that the resulting δI_k on the lhs can only depend on dimensionless ratios \hat{p}_i . A good starting point for the iteration is $\tilde{I}_k = \tilde{I}_0$ with $\delta\tilde{I} \equiv 0$. Such a choice trivially only depends on the ratios (8.23). Hence this holds true for each iteration step, and we have proven (8.24).

Now we invoke the optimisation (5.26) with $D_{R_\perp} \tilde{I} = \tilde{I}_0 D_{R_\perp} \delta\tilde{I}$, and we are led to the constraint

$$\int_k^0 dt' \Delta S_2 \left(\tilde{I}_0(D_{R_\perp} \delta\tilde{I}_{k'}) \right)_{R_{\text{stab}}} = 0. \quad (8.25)$$

For positive definite $\delta\tilde{I}$ a solution to (8.25) is given by $\Delta S_2 \delta\tilde{I} = 0$. In this context we remark that $\delta\tilde{I}$ is not a correlation function \tilde{I} , and the above resolution does not imply a vanishing flow of $\delta\tilde{I}$. An optimisation along these lines was put forward in the infrared regime of QCD, for details see [142, 143].

D. Optimisation in LPA

We continue with a detailed analysis of the optimisation (5.26), (5.29) in the LPA of a scalar theory with a single scalar field $a = x$. We shall show that within the LPA the regulator (5.11) follows as the unique solution to (5.28), see also the more explicit form without RG scaling, (5.17). For the sake of simplicity we use the standard flow (3.72) with $\Delta S_2 \tilde{I}_k = (G \hat{R} G)_{bc} \tilde{I}_k^{cb}$. In the LPA we have to evaluate (5.28) for constant fields. Moreover we consider correlation functions \tilde{I}_k that are functionals of ϕ and not operators. For example, in the present truncation scheme relevant correlation functions are provided by

$$\int d^d x \tilde{I}_{k, \text{diag}}^{(n)}[\phi] = \langle \int d^d x \phi^n(x) \rangle_{J^a = \Gamma_k^a + 2R^{ab} \phi_b}, \quad (8.26)$$

and combinations thereof. In LPA all quantities are evaluated for constant fields $\bar{\phi}$. On the rhs of the standard

flow (3.72) the second derivatives \tilde{I}_k^{ab} are required. In LPA they are parameterised as

$$\tilde{I}_k^{(2)}[\bar{\phi}](p, q) = \mathcal{I}_k(\bar{\phi}, p^2)\delta(p - q). \quad (8.27)$$

We also need the full propagator $G(p, q) = (\tilde{I}_k^{(2)} - (\tilde{I}_k^{(1)})^2)[\bar{\phi}](p, q)$, which reads $1/(\Gamma_k^{(2)}[\bar{\phi}] + 2R)(p, q) = 1/(p^2 + V''[\bar{\phi}] + R(p^2))\delta(p - q)$. Here we have normalised in the standard way, such that $R(p, q) = 1/2R(p)\delta(p - q)$. Inserting these objects into (5.17) we arrive at

$$\delta R_{\perp}^{a_1 a_2} \frac{\delta(G \dot{R} G)_{bc}}{\delta R^{a_1 a_2}} \left(\frac{\partial^2 \tilde{I}_k[\bar{\phi}]}{(\partial \bar{\phi})^2} \right)^{cb} \Bigg|_{R=R_{\text{stab}}} = 0, \quad (8.28)$$

which we recast in a more explicit form

$$\int \frac{d^d p}{(2\pi)^d} \delta R_{\perp}(p^2) \frac{\delta}{\delta R(p^2)} \Bigg|_{\mathcal{I}_k} \quad (8.29)$$

$$\times \int \frac{d^d q}{(2\pi)^d} \frac{\mathcal{I}_k(\bar{\phi}, q^2)}{(q^2 + R(q^2) + V''[\bar{\phi}])^2} \partial_t R(q^2) \Bigg|_{R=R_{\text{stab}}} = 0.$$

Now we use that a general regulator R can be written as $R(q^2) = q^2 r(x)$ with $x = q^2/k^2$, if no further scale is present in R . This entails that $\partial_t R = q^2 \partial_t r(x) = q^2(-2x)\partial_x r(x)$. Furthermore we can rewrite the integration over q as one over x : $d^d q/(2\pi^2)^d = d\Omega_d dx x^{(d/2-1)}/2$. With these identifications we get for the q -integral in (8.29) after partial integration

$$\Omega_d \mathcal{I}_k(\bar{\phi}, 0) \delta_{2d} + \Omega_d \int_0^\infty dx x^{d/2-2} \mathcal{I}_k \left\{ \left((d/2 - 1) + x \partial_x \ln \mathcal{I}_k \right) \frac{r + V''/x}{1 + r + V''/x} - \frac{V''/x}{(1 + r + V''/x)^2} \right\}. \quad (8.30)$$

Now we are in a position to discuss the extrema (8.28). Searching for minimal flows is equivalent to searching for r that minimise the absolute value of the integrand in (8.30)

$$\min_r \left| \left((d/2 - 1) + x \partial_x \ln \mathcal{I}_k \right) \frac{r + V''/x}{1 + r + V''/x} - \frac{V''/x}{(1 + r + V''/x)^2} \right|, \quad (8.31)$$

where we have left out the overall factor $x^{d/2-2}\mathcal{I}_k$. A simpler condition is achieved by neglecting the model-dependent second term proportional to V''/x leading to

$$\min_r \frac{r + V''/x}{1 + r + V''/x}. \quad (8.32)$$

We proceed with the extremisation of the full integrand by taking the r -derivative at fixed \mathcal{I}_k of the function in (8.31). We arrive at

$$\frac{((d/2 - 1) + x \partial_x \ln \mathcal{I}_k)(1 + r + V''/x) + V''/x}{(1 + r + V''/x)^3}. \quad (8.33)$$

We remark that subject to $((d/2 - 1) + x \partial_x \ln \mathcal{I}_k) > 0$ and $V''/x > 0$ the r -derivative (8.32) is positive. Note also that $r + V''/x > 0$ cannot be obtained for all x and $\bar{\phi}$ if the potential V is not convex yet. This statement holds for all regulators¹⁷. However, for optimised r the region $V''/x < 0$ for x should have small impact on (8.30). If $d \geq 4$ we regain positivity for vanishing or positive $\partial_x \ln \mathcal{I}_k$. Leaving aside this subtlety we solve (8.31) for positive regulators r . As its derivative is positive, (8.33), this amounts to minimising r

$$r_{\text{stab}} \leq r, \quad \forall r, x. \quad (8.34)$$

So far we have not used the definition of $\{R_{\perp}\}$ in (5.26). With its use we are straightforwardly led to (8.36). Still we would like to evaluate how unique or natural the choice R_{\perp} is. If r was an arbitrary positive function of x , (8.34) leads to $r(x) \equiv 0$. However, as r has been introduced as an IR-regularisation it is inevitably constrained: it entails an IR-regularisation in momentum space only with

$$x + x r(x) \geq c \quad (8.35)$$

for some positive constant c . For a proper IR-regularisation the full propagator G has to display a maximum $G \leq 1/c_{\text{min}}$ with $c_{\text{min}} = c + V''_{\text{min}} > 0$, where V''_{min} is the minimal value of V'' , possibly negative. For momenta $x > c$ the solution of (8.34) with (8.35) is $r(x > c) \equiv 0$. For $x < c$ we saturate the inequality (8.35) with $r(x) = c/x - 1$. This leads to a unique solution r_{stab} of (8.34) for $r \in \{r_{\perp}\}$ defined by (8.35).

$$r_{\text{stab}}(x) = (c - x)\theta(c - x), \quad (8.36)$$

which is equivalent to (5.11). Note that in between (8.35) and (8.36) we have implicitly introduced the set $\{R_{\perp}\}$ of (5.26) by keeping c fixed while minimising r . Still, such a procedure was naturally suggested by the computation.

Above we have restricted ourselves to correlation functions \tilde{I}_k with $((d/2 - 1) + x \partial_x \ln \mathcal{I}_k) > 0$. If we discuss

¹⁷ All regulator functions have to decay with more than $1/x$, the exception being the mass regulator with $r = 1/x$.

optimisation on the set of $\int d^d x \tilde{I}_{k,\text{diag}}^{(n)}$, (8.26) they lead to $\mathcal{I}_k^{(n)} \propto 1/(q^2 + R + V'')^n$. For large n the contributions of $x\partial_x \ln \mathcal{I}_k$ will dominate the x -integral in (8.30). Minimising the absolute value of the integral then amounts to solving (8.32), so we still have to minimise r . Note also that this does not extremise the flow of all correlation functions $\int d^d x \tilde{I}_{k,\text{diag}}^{(n)}$.

It is also interesting to speculate about the most unstable regulator. It is found by maximising the integrand in (8.30) in the regularised momentum regime. This is achieved for $r_{\text{instab}} = \infty$. If we also demand that r is monotonicity and that the gap (8.35) is saturated at some momentum, this leads to

$$r_{\text{instab}}(x) = 1/\theta(x - c) - 1, \quad (8.37)$$

the sharp cut-off. Note that this argument concentrates on instability of the low momentum region of the flow. For high momenta maximal instability is obtained for the regulator $R_{\text{CS}} = k_{\text{eff}}^2$, the mass cut-off. The related flow equation is an un-renormalised Callan-Symanzik equation. Indeed, the results for critical exponents for scalar models in LPA are worse for the mass regulator [76] than that for the sharp cut-off.

The stable and unstable regulators (8.36) and (8.37) have been derived from (5.26) by dropping correlator-dependent terms. The regulators (8.36) and (8.37) can also be derived from (5.32c) in a very simple manner. In the present truncation (5.32c) has to be evaluated on L_2 and boils down to

$$\frac{1}{x + xr_{\text{stab}}(x) + V''} \geq \frac{1}{x + xr_{\perp}(x) + V''}, \quad (8.38)$$

which can be converted into (8.34). This nicely shows the advantage of a simple functional criterion.

Beyond LPA we are led to integrals as in (8.29) that also contain derivatives w.r.t. q . Then r also has to be differentiable to the given order. Such regulators exist, they are simply differentiable enhancements of (8.36).

E. Optimisation in general truncation schemes

In a general truncation and higher truncation order the correlation functions \tilde{I}_k resolve more structure of the flow operator ΔS_2 . Roughly speaking, a solution to the functional optimisation criterion (5.26) minimises the expansion coefficients of ΔS_2 for a given truncation scheme. For example, in higher order derivative expansion the flow $\Delta S_2 \tilde{I}_k$ is projected on the part that contains higher

order space-time derivatives. In momentum space and resorting to the representation (5.32) of the functional optimisation criterion (5.26), this amounts to differentiability of $G\psi(p)$ w.r.t. momentum at the given order. Consequently the norm has to be taken in the space of differentiable functions with

$$\|\psi\|_n^2 = \sum_{|\alpha| \leq n} \frac{n!}{(n - |\alpha|)! \alpha_1! \cdots \alpha_d!} \left\| \frac{\partial^{|\alpha|} \psi(p)}{\partial p^{\alpha_1} \cdots \partial p^{\alpha_d}} \right\|_{L_2}^2 \quad (8.39)$$

where $\alpha \in \mathbb{N}^d$ and $|\alpha| = \sum \alpha_i$. Eq. (8.39) defines the norm on Sobolev-spaces H^n with $n \in \mathbb{N}$. Applied to the functional optimisation criterion, and leaving aside the intricacies discussed in section VD 3 we arrive at the following optimisation in n th order derivative expansion:

$$\begin{aligned} & \|\theta_\lambda(G[\phi_0, R_{\text{stab}}]) - \theta_\lambda(G[\phi_0, 0])\|_n \\ &= \min_{R_\perp} \|\theta_\lambda(G[\phi_0, R_\perp]) - \theta_\lambda(G[\phi_0, 0])\|_n, \end{aligned} \quad (8.40)$$

for all $\lambda \in \mathbb{R}^+$. Here ϕ_0 is either defined by the minimum of the potential or it maximises the propagator. θ_λ has to meet the requirement of boundedness w.r.t the norm $\|\cdot\|_n$, as already discussed below (5.32b). This is achieved by using a n th-order differentiable version of (5.32b). We emphasise that the form of θ_λ is of no importance for the present purpose. The optimisation with (8.40) seems to depend on the full two-point function $\Gamma^{(2)}[\phi_0, R = 0]$. Now we proceed with the specific norm $\|\cdot\|_n$ as indicated in section VD 3 below (5.32a). The constraint (8.40) entails that the spectral values of $G[\phi_0, R_{\text{stab}}]$ are as close as possible to that of the full propagator $G[\phi_0, 0]$. Moreover it entails maximal smoothness. Hence (5.32a) can be reformulated as

$$\begin{aligned} & \|\theta_\lambda(\Gamma^{(2)}[\phi_0, R_{\text{stab}}] + R_{\text{stab}})\|_n \\ &= \min_{R_\perp} \|\theta_\lambda(\Gamma^{(2)}[\phi_0, R_\perp] + R_\perp)\|_n, \end{aligned} \quad (8.41)$$

for all $\lambda \in \mathbb{R}^+$. A solution of (8.41) provides a propagator $G[\phi_0, R_{\text{stab}}]$ which is as close as possible to the full propagator $G[\phi_0, 0]$ as well as having minimal derivatives of order $i \leq n$. Eq. (8.41) also leads to the supplementary constraint for the stability criterion (5.10). The maximisation of the gap has to be supplemented by the minimisation of

$$\|\Gamma^{(2)}[\phi_0, R_{\text{stab}}](p_0^2) + R_{\text{stab}}(p_0^2)\|_n, \quad (8.42)$$

within the class of R_{stab} singled out by (5.10). Here p_0 is the momentum at which the propagator takes its maximum. For an implementation of (8.42) see [79].

In truncation schemes that carry a non-trivial momentum and field dependence [104–107, 109, 110, 122, 139, 140, 142, 143, 146–149], functional optimisation suggests the use of background field dependent regulators, or even regulators with a non-trivial dependence on the full field. Evidently in the latter case structural truncations of the flows are inevitable, see also [54, 55]. In case momentum and field dependence are intertwined, as happens in the interesting truncation scheme put forward in [104–107], functional optimisation directly implies the use of (background) field dependent regulators.

We continue with a brief discussion of a peculiar case relevant for the optimisation of QCD-flows in Landau gauge QCD as initiated in [142, 143]. In case the spectral values $\lambda(p^2)$ of the full propagator are not monotonic in momentum, an optimised regulator does not resolve the theory successively in momentum. This happens for the gluon propagator in Landau gauge QCD [142, 147, 180, 182]. A propagator that is monotonic in momentum violates the condition $-\partial_t G \geq 0$ for some interval in t and some spectral values. This implies that the flow trajectory is not minimised for these spectral values. In turn, a general optimised regulator can be constructed from

$$R_{\phi,\text{stab}}(p^2) = (Z_\phi k_{\text{eff}}^2 - \Gamma_k^{(2)}(p^2))\theta[Z_\phi k_{\text{eff}}^2 - \Gamma_0^{(2)}(p^2)] \\ + (\Gamma_0^{(2)}(p^2) - \Gamma_k^{(2)}(p^2))\theta[\Gamma_0^{(2)}(p^2) - Z_\phi k_{\text{eff}}^2], \quad (8.43)$$

where $\Gamma_0^{(2)}(p^2)$ is the full two-point function at vanishing regulator, and a possibly smoothed step-function θ . An optimisation with (8.43) also requires k -dependent fields with

$$(D_t \Gamma_k^{(2)})\theta[\Gamma_0^{(2)}(p^2) - Z_\phi k_{\text{eff}}^2] = 0. \quad (8.44)$$

Then the two point function $\Gamma_k^{(2)}$ already agrees with the full two point function $\Gamma_0^{(2)}$ for spectral values $\Gamma_0^{(2)}(p^2) \geq Z_\phi k_{\text{eff}}^2$, and the second line in (8.43) vanishes. Within LPA, (8.43) boils down to the regulator (8.35), and encompasses the above mentioned flow in Landau gauge QCD with $R = (R_A, R_C, R_q)$. The practical use of the suggestion (8.43) calls for an iterative solution of the flow about a suggestion $\Gamma_0^{(2)}$ as described in section VIII C. In QCD, a combined optimisation in (R_A, R_C, R_q) with (8.43) may lead to a successive integrating out of fields as found already in the IR-optimisation in [142, 143]¹⁸. More details will be provided elsewhere [144].

In the light of the above results we add a further brief comment on the physical interpretation of optimisation as introduced in chapter V. The optimisation criterion is constructed from stability considerations. Stability implies minimal integrated flows and hence quickest convergence towards physics. At each order of the given truncation scheme the optimised propagators and correlation functions are as close as possible to the full propagator and correlation functions respectively. This minimises regulator artefacts, and triggers a most rapid approach towards the full theory. Moreover, optimised flows preserve the RG properties of the full theory within the regularisation as well as gradient flows, see (5.12). The above arguments emphasise the close structural relation of the optimisation criterion to the construction of both improved and perfect actions in lattice theory¹⁹. We emphasise that the optimisation can be implemented within an iterative procedure which leads to small additional computational costs.

IX. CONCLUSIONS

The present work provides some structural results in the functional RG which may prove useful in further applications, in particular in gauge theories. We have derived flows (3.86) and their one-parameter reductions (3.28), (3.60) and (4.20) valid for a general class of correlation functions \tilde{I}_k defined in (3.14) with (3.51). This class of correlation functions \tilde{I}_k includes N -point functions as well as Dyson-Schwinger equations, symmetry relations such as Slavnov-Taylor identities, and flows in the presence of composite operators, e.g. N -particle irreducible flows. The present formulation also allows us to directly compute the evolution of observables in gauge theories. For example, the flows (3.60), (4.20) hold for the Wilson loop and correlation functions of the Polyakov loop, see section VII A. This is a very promising approach to the direct computations of observables in the non-perturbative regime of QCD, e.g. the order parameter of the confinement-deconfinement phase transition. In section VIII A we derived closed flows in the presence of general scale-dependent reparameterisations of the theory. This extends the options for scale-adapted param-

¹⁸ The proof of an extremum being global is intricate

¹⁹ An adaptation of the criterion (5.26) for lattice regularisations leads to improved actions and operators at lowest order of an expansion scheme based on the lattice spacing.

eterisations of the theory, and is particularly relevant in the context of rebosonisation.

The functional framework developed here was used to systematically address the important issue of optimisation, and to derive a functional optimisation criterion, see section V D 3 (5.26),(5.29),(5.32). Optimal regulators are those, that lead to correlation functions as close as possible to that in the full theory for a given effective cut-off scale. The criterion allows for a constructive use, and it is applicable to general truncation schemes. It can be also used for devising new optimised schemes, for examples see section VIII, in particular section VIII D,VIII E. The use of optimisation methods becomes crucial in more intricate physical problems such as the infrared sector of QCD, and can be used to resolve the pending problem of full UV-IR flows in QCD.

Another important structural application concerns renormalisation schemes for general functional equations, e.g. DSEs and NPI effective actions. The functional flows (3.86) can be used for setting up of generalised BHPZ-type renormalisation schemes that are, by construction, consistent within general truncation schemes, see sections VI A 2,VII B 3. Moreover, such subtraction schemes are very well adapted for numerical applications.

The present setting also allows for a concise and flexible representation of symmetry constraints, which is particularly relevant in gauge theories. So far, the practical implementation of modified Slavnov-Taylor identities was restricted to their evaluation for specific momentum values. The present setting allows for a functional implementation that possibly adapts more of the symmetry, see section VII B 2,VII B 3. This opens a path towards improved truncation schemes in gauge theories relevant for a more quantitative computation in strongly interacting sectors of QCD. The above analysis also applies the Nielsen identities for gauge invariant flows of the geometrical effective action.

In summary we have presented structural results that further our understanding of the Functional Renormalisation Group. These results can be used to qualitatively and quantitatively improve FRG applications.

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APPENDIX

A. Metric

This appendix deals with the non-trivial metric γ in field space in the presence of fermions. The ultra-local metric γ is diagonal in field space for scalars and gauge fields and is given by the ϵ -tensor in fermionic space. For $\varphi_a = (\psi, \bar{\psi})_a$ the fermionic metric reads

$$(\gamma^{ab}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (\text{A.1})$$

For raising and lowering indices we use the Northwest-Southeast convention,

$$\begin{aligned} \phi^a &= \gamma^{ab} \phi_b, \\ \phi_a &= \phi^b \gamma_{ba}. \end{aligned} \quad (\text{A.2})$$

The metric has the properties

$$\begin{aligned} \gamma_b^a &= \gamma^{ac} \gamma_{bc} = \delta_b^a, \\ \gamma^a_b &= \gamma^{ac} \gamma_{cb} = (-1)^{ab} \delta_b^a, \end{aligned} \quad (\text{A.3})$$

where

$$(-1)^{ab} = \begin{cases} -1, & a \text{ and } b \text{ fermionic} \\ 1 & \text{otherwise.} \end{cases} \quad (\text{A.4})$$

Eq. (A.3) extends to indices $\mathbf{a} = a_1 \cdots a_n$ and $\mathbf{b} = b_1 \cdots b_m$ with

$$(-1)^{\mathbf{a}\mathbf{b}} = \begin{cases} -1 & \mathbf{a} \text{ and } \mathbf{b} \text{ contain odd \# of} \\ & \text{fermionic indices,} \\ 1 & \text{otherwise.} \end{cases} \quad (\text{A.5})$$

For arbitrary vectors $\phi, \tilde{\phi}$ the properties (A.3) lead to

$$\tilde{\phi}^a \phi_a = \phi^a \tilde{\phi}_a = \tilde{\phi}_a \phi^b \gamma^a_b = \phi_b \tilde{\phi}^a \gamma^b_a. \quad (\text{A.6})$$

Due to the Grassmann nature of the fermionic variables $\psi, \tilde{\psi}$ the order is important $\psi^i \tilde{\psi}_i = -\tilde{\psi}_i \psi^i$.

We close this appendix with an example. In general a (composite) field ϕ consists of scalar components, gauge fields and fermions, the fundamental field reads in components

$$\begin{aligned} (\phi_i) &= (\varphi, A, \psi, \bar{\psi}), \\ (\phi^i) &= (\varphi, A, \bar{\psi}, -\psi). \end{aligned} \quad (\text{A.7})$$

The contraction of the fundamental ϕ with itself leads to

$$\begin{aligned} \phi^a \phi_a &= \phi_b \gamma^{ab} \phi_a = \int d^d x \left(\varphi_{\alpha_n}(x) \varphi_{\alpha_n}(x) \right. \\ &\quad \left. + A_\alpha^\mu(x) A_\mu^\alpha(x) + 2\bar{\psi}_\alpha^\xi(x) \psi_\xi^\alpha(x) \right) \end{aligned} \quad (\text{A.8})$$

where n labels the number of scalar fields, α the gauge group, and ξ sums over spinor indices and flavours. The current J related to ϕ is given by

$$\begin{aligned} (J_a) &= (J_\varphi, J_A, J_{\bar{\psi}}, J_\psi) \\ (J^a) &= (J_\varphi, J_A, J_\psi, -J_{\bar{\psi}}), \end{aligned} \quad (\text{A.9})$$

which implies schematically

$$J^a \phi_a = (J_\varphi \varphi + J_A A + J_\psi \psi + \bar{\psi} J_{\bar{\psi}}). \quad (\text{A.10})$$

Moreover

$$J^a \phi_a = \phi^a J_a = J_a \phi^b \gamma^a_b = \phi_b J^a \gamma^b_a. \quad (\text{A.11})$$

B. Derivatives

We deal with derivatives of functionals $F[f]$ w.r.t. $f(x) = \phi(x)$ or $f(x) = J(x)$. Derivatives are denoted as

$$F_{,a}[f] := \frac{\delta F[f]}{\delta f^a}, \quad F^{,a}[f] := \frac{\delta F[f]}{\delta f_a}, \quad (\text{B.1})$$

that is, derivatives are always taken w.r.t. the argument of the functional F . Eq. (B.1) implies

$$F^{,a}[f] = \gamma^{ba} F_{,b}[\phi], \quad F_{,a}[f] = \gamma_{ab} F^{,b}[f], \quad (\text{B.2})$$

which has to be compared with (A.6). We also take derivatives w.r.t. some (logarithmic) scale s , e.g. $s = t = \ln k$. The total derivative of some functional F splits into

$$\begin{aligned} \frac{dF[J]}{ds} &= \partial_s F[J] + \partial_s J^a F_{,a}[J], \\ \frac{dF[\phi]}{ds} &= \partial_s F[\phi] + \partial_s \phi_a F^{,a}[\phi], \end{aligned} \quad (\text{B.3})$$

i.e., $\partial_s F[\phi] = \partial_s|_\phi F[\phi]$ and $\partial_s F[J] = \partial_s|_J F[J]$. Partial derivatives w.r.t. the logarithmic infrared scale $t = \ln k$ we abbreviate with

$$\dot{F} = \partial_t F. \quad (\text{B.4})$$

General differential operators are similarly defined as

$$\begin{aligned} D_s F[J] &= (\partial_s + \gamma_g^i j g_i \partial_{g_j} + \gamma_J^{\mathbf{a}} \mathbf{J}^{\mathbf{b}} \frac{\delta}{\delta J^{\mathbf{a}}}) F[J], \\ D_s F[\phi] &= (\partial_s + \gamma_g^i j g_i \partial_{g_j} + \gamma_\phi^{\mathbf{a}} \phi_{\mathbf{b}} \frac{\delta}{\delta \phi_{\mathbf{a}}}) F[\phi], \end{aligned} \quad (\text{B.5})$$

with partial derivatives according to (B.3). The definitions of this appendix directly carry over to the case of multi-indices \mathbf{a}, \mathbf{b} .

C. Definition of ΔS_n

The part of $\Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]$ that contains at least $n \geq 1$ derivatives w.r.t. the variable x , e.g. $x[J] = J, \phi$, acting to the right, is given by

$$\Delta S_n[x, \dot{R}] = \Delta S_{\mathbf{a}_1 \dots \mathbf{a}_n}[x, \dot{R}] \frac{\delta}{\delta x_{\mathbf{a}_1}} \dots \frac{\delta}{\delta x_{\mathbf{a}_n}}, \quad (\text{C.1a})$$

with coefficient

$$\begin{aligned} \Delta S_{\mathbf{a}_1 \dots \mathbf{a}_n}[x, \dot{R}] \\ = \sum_{i \geq n} (\Delta S_{\mathbf{a}_1 \dots \mathbf{a}_i}[x, \dot{R}]) \frac{\delta}{\delta x_{\mathbf{a}_1}} \dots \frac{\delta}{\delta x_{\mathbf{a}_{i-n}}}. \end{aligned} \quad (\text{C.1b})$$

The coefficients $\Delta S_{\mathbf{a}_1 \dots \mathbf{a}_n}$ are operators. The functionals $(\Delta S_{\mathbf{a}_n \dots \mathbf{a}_1})$ are the coefficients in a Taylor expansion of the operator ΔS in powers of $\frac{\delta}{\delta x}$, absorbing n derivatives w.r.t. x of $\Delta S[\frac{\delta}{\delta J} + \phi, \dot{R}]$. We emphasise that $(\Delta S_{\mathbf{a}_1 \dots \mathbf{a}_n}[x, \dot{R}])$ is a functional, it contains no derivative operators. If interested in $x = J$, the expansion coefficients $(\Delta S^{\mathbf{a}_1 \dots \mathbf{a}_n}[x, \dot{R}])$ boil down to the Taylor coefficients in an expansion of ΔS in $\phi_{\mathbf{a}}$. They are the n th right derivatives of $\Delta S[x, \dot{R}]$ w.r.t. $x_{\mathbf{a}}$, evaluated at $x = \frac{\delta}{\delta J} + \phi$.

D. Standard 1PI flows

For quadratic regulators (3.3) and $\mathbf{a} = a$ the flow (3.55) reads more explicitly

$$\partial_t \tilde{I}_k + \frac{1}{2} (G \dot{R} G)_{bc} \tilde{I}_k^{,cb} - \left((\partial_t J^a) \right) \quad (\text{D.1})$$

$$+ \frac{1}{2} (G \dot{R} G)_{bc} \Gamma_k^{,cbd} \gamma^a_d - \phi_b \dot{R}^{ba} \Big) G_{ad} \tilde{I}_k^{,d} = 0,$$

where

$$(G \dot{R} G)_{bc} = G_{ba} \dot{R}^{ad} G_{dc}.$$

For the derivation of (D.1) we have to express $\Delta S[\frac{\delta}{\delta J}, \dot{R}]$ in terms of derivatives w.r.t. ϕ with the help of (3.50). For bosonic variables this is straightforwardly done. If fermionic variables are involved the ordering of terms becomes important. We shall argue that

$$\begin{aligned} \dot{R}^{ab} \frac{\delta}{\delta J^a} \frac{\delta}{\delta J^b} \\ = G_{ac} \frac{\delta}{\delta \phi_c} \dot{R}^{ab} G_{bd} \frac{\delta}{\delta \phi_d} \\ = G_{ac} \dot{R}^{ab} \left(\frac{\delta}{\delta \phi_c} G_{bd} \right) \frac{\delta}{\delta \phi_d} + G_{ca} \dot{R}^{ab} G_{bd} \frac{\delta}{\delta \phi_d} \frac{\delta}{\delta \phi_c}. \end{aligned} \quad (\text{D.2})$$

The only non-trivial term is the last one on the right hand side. Eq. (3.4) entails that for a being bosonic

(fermionic), b is bosonic (fermionic). If either a or c or both are bosonic we conclude $G_{ac} = G_{ca}$. Moreover either $\frac{\delta}{\delta\phi_c}$ and $G_{bd}\frac{\delta}{\delta\phi_d}$ or both are bosonic and (D.2) follows. If a, c both are fermionic, $\frac{\delta}{\delta\phi_c}$ and $G_{bd}\frac{\delta}{\delta\phi_d}$ are fermionic (as b is fermionic) and we have $G_{ac} = -G_{ca}$. It follows that

$$\frac{\delta}{\delta\phi_c}G_{bd}\frac{\delta}{\delta\phi_d} = \left(\frac{\delta}{\delta\phi_c}G_{bd}\right)\frac{\delta}{\delta\phi_d} - G_{bd}\frac{\delta}{\delta\phi_d}\frac{\delta}{\delta\phi_c}. \quad (\text{D.3})$$

Inserting (D.3) into (D.2) the right hand side follows. We also conclude that for b, c fermionic

$$\frac{\delta}{\delta\phi_c}G_{bd} = G_{be}\Gamma_k{}^{ecf}G_{gd}\gamma^g{}_f. \quad (\text{D.4})$$

The factor $\gamma^g{}_f$ originates in (3.46), $G_{ac}(\Gamma_k{}^{cb} + R^{bc}) = \gamma^b{}_a$. Inserting (D.4) into (D.2) we arrive at

$$\begin{aligned} \dot{R}^{ab}\frac{\delta}{\delta J^a}\frac{\delta}{\delta J^b} &= G_{ab}\dot{R}^{bc}G_{cd}\frac{\delta}{\delta\phi_d}\frac{\delta}{\delta\phi_a} \\ &\quad - (G\dot{R}G)_{ad}\Gamma_k{}^{daf}\gamma^g{}_fG_{ge}\frac{\delta}{\delta\phi_e}. \end{aligned} \quad (\text{D.5})$$

with $(G\dot{R}G)_{ad} = G_{ab}\dot{R}^{bc}G_{cd}$.

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