Lattice field theories
Exercise sheet 1 – Derivative Discretisations

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Consider the action that was derived on the fist lecture,

$$S_E[q] = \int dt \left\{ \frac{1}{2} m \dot{q}^2 + V(q) \right\} , \tag{1}$$

with $q = q(t)$ and $t \in [0, T]$. For numerical simulations typically q will be stored as an array, whose elements will be accessed like `q[n]`, where n is an integer. Therefore, one needs to discretise the time coordinate. Usually this is done by partitioning the interval $[0, T]$ in N parts of equal size a with the discrete times $t \to t_n = na$.

In conseuqence, the derivatives in the kinetic term in S_E have to be substituted with derivatives, as discussed in detail in the lecture from 26th of October. This leads us, e.g., to

$$\dot{q} \to \frac{q(t_n + a) - q(t_n)}{a} , \tag{2}$$

the right derivative discussed in the lecture for a scalar field, or

$$\dot{q} \to \frac{q(t_n + a) - q(t_n - a)}{2a} , \tag{3}$$

the symmetric derivative. In the limit $a \to 0$ the differences between these definitions disappear, if the limit is well-defined. The differences in eq. (2) and eq. (3) can be computed numerically as `(q[n+1] - q[n])/a` or `(q[n+1] - q[n-1])/2*a`, respectively.

Exercise 1: Taylor expansions

- Assume now that $q(t)$ is defined for $t \in \mathbb{R}$. Use the right-hand side of the two expressions above, and compute the Taylor expansion about t_n up to the second order.
- How does each of them differ from the actual $\dot{q}$?

Bonus: Can you think of other discretisations? How do they compare with those above?

Exercise 2: Dispersion relation and continuum limit

Consider now the action of the quantum harmonic oscillator

$$S_E = \int dt \left[\dot{q}^2 + \frac{1}{2} m q^2 \right] , \tag{4}$$

and its discretised version

$$S_E = a \sum_n \left(\frac{1}{a} \left[\frac{1+\beta}{2} q(t_n + a) - \frac{1-\beta}{2} q(t_n - a) - \beta q(t_n) \right] \right)^2 + \frac{m}{2} q^2 , \tag{5}$$

where $\beta \in [0, 1]$. This interpolates the kinetic term between the two discretisations shown above: for $\beta = 0$ one has eq. (3) and for $\beta = 1$ one has eq. (2).

Since the action is quadratic, it may be written in momentum (or, more accurately, frequency) space by using Fourier transforms

$$q(t_n) = \sum_j e^{i p_j t_n} \hat{q}(p_j) , \tag{6}$$

$p_j = j\pi/aN$ and $j = 0, 1, \cdots, N-1$. The action in momentum space then has the form

$$S_E = \frac{1}{2a} \sum_j \hat{q}^*(p_j) D(p_j) \hat{q}(p_j) . \tag{7}$$

- Compute $D(p_j)$.
- It is instructive to plot $D(p_j)$ as a function of p_j for different values of β. How do the different curves compare to each other?
- Using Fourier transforms to write eq. (4) also in momentum space and find the continuum version of $D(p)$.
- Consider now the limit of $a \to 0$ (the *continuum limit*) of $D(p_j)$, and compare that with $D(p)$. For which values of p_j do both agree?