Lattice field theories
Exercise sheet 5 – Simple gauge theories on the lattice

Lectures: Jan Pawlowski j.pawlowski@thphys.uni-heidelberg.de
Felipe Attanasio pyfelipe@thphys.uni-heidelberg.de

Tutorials: Felipe Attanasio pyfelipe@thphys.uni-heidelberg.de

Institut für Theoretische Physik, Uni Heidelberg

Please see the supplemental material at the end of this exercise sheet for further explanations and hints.

Exercise 10: The SU(2) Polyakov Chain

Consider the action of the SU(2) Polyakov chain,

$$S = -\frac{\beta}{2}\mathrm{Tr}\left[U_1 U_2 \cdots U_{N_{\mathrm{links}}}\right] , \tag{1}$$

where $U_n \in \mathrm{SU}(2)$ for $1 \leq n \leq N_{\mathrm{links}}$ are known as *gauge links*, or just links, and $\beta \in \mathbb{R}$. Use the Metropolis algorithm from exercise sheet 2, reproduced and adapted below, to compute the expectation value of the action,

$$\langle S \rangle = \frac{1}{Z} \int \mathcal{D}U \, S \, e^{-S} \, . \tag{2}$$

You can compare your numerical results with the exact one,

$$\langle S \rangle_{\mathrm{exact}} = \beta \frac{I_2(\beta)}{I_1(\beta)} \, , \tag{3}$$

where I_m are the mth order modified Bessel function of the first kind. Note that $\langle S \rangle$ is independent of N_{links}. You should also observe this in the simulations.

Suggested parameters (but by no means an exclusive or exhaustive list):

- $|\beta| < 1$
- $1 \leq N_{\mathrm{links}} \leq 30$

but feel free to exceed these bounds!

Supplemental material

In this exercise we are dealing with a field whose elements belong to the group SU(2), and the updating procedure should be such that update proposals also belong to the group. Since every element of SU(2) group can be written as the exponential of a member of the $su(2)$ Lie algebra, updating a particular link U_n can be written as

$$U_n \to R\,U_n\,, \quad R = \exp\left[i\sum_{a=1}^{3}\alpha^a\sigma^a\right]\,, \tag{4}$$

with σ^a being the Pauli matrices and α^a real numbers. Note that the update may equally well be written as $U_n \to U_n R$. Because of the commutation properties of the Pauli matrices the computation of R can be simplified as

$$R = \cos(|\vec{\alpha}|)\mathbb{1} + i\frac{\alpha^a}{|\vec{\alpha}|}\sigma^a\sin(|\vec{\alpha}|)\,, \tag{5}$$

with $\mathbb{1}$ being the 2×2 identity matrix and $|\vec{\alpha}| = \sqrt{(\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2}$. Thus it requires only simple arithmetic and trigonometric functions.

```
matrix U[Nlinks];
for n ← 0 to N_steps do
    for 0 ≤ n < N_links do                              // sweep over the lattice
        oldAction ← S(U) ;                                // save old action
        oldValue ←U[n] ;                                    // save old U_n
        R ← randomSU2() ;                               // random SU(2) matrix
        U[n] ← R * U[n] ;                // random change to U_n within SU(2)
        ΔS = S(U) - oldAction;
        if ΔS > 0 then
            r ← UniformReal(0,1);
            if r > exp(−ΔS) then
                U[n] ← oldValue ;                          // reject proposal
            end
        end
    end
    avgAction ← avgAction + S(U) ;                    // update avg.  action
end
Obs ← Obs / N_steps;
```

As always, you should skip a number of steps at the beginning, so the system can thermalise. And remember that consecutive configurations in the Markov chain are likely to be correlated, so it is also a good idea to skip some between consecutive measurements.