Lattice field theories
Exercise sheet 7 – Gauged scalar fields

Lectures: Jan Pawlowski j.pawlowski@thphys.uni-heidelberg.de
Felipe Attanasio pyfelipe@thphys.uni-heidelberg.de

Tutorials: Felipe Attanasio pyfelipe@thphys.uni-heidelberg.de

Institut für Theoretische Physik, Uni Heidelberg

Consider a 2-dimensional pair of neutral scalar fields with quartic self-interactions

$$S = \int d^2x \left[\frac{1}{2} (\partial_\mu \Phi)^\dagger \partial_\mu \Phi + \frac{m^2}{2} \Phi^\dagger \Phi + \frac{g}{4!} \left(\Phi^\dagger \Phi \right)^2 \right] , \tag{1}$$

where $\Phi = [\phi_1 \quad \phi_2]^\intercal$, ϕ_1, ϕ_2 are real fields, $m^2, g \in \mathbb{R}$, and $\mu \in \{1, 2\}$. This action is invariant under global SU(2) transformations, $\Phi(x) \to \Phi'(x) = \Omega \Phi(x)$, $\Omega \in \mathrm{SU}(2)$. If we now assume the transformations to be local, the derivatives have to be made covariant in order to preserve gauge invariance: $\partial_\mu \to D_\mu = \partial_\mu + igA_\mu(x)$, where g is the gauge coupling strength. The lattice version of the action of eq. (1) reads,

$$\begin{aligned} S_{\text{lat}} = \sum_n \Bigg\{ & \frac{-1}{2a^2} \sum_\mu \left[\Phi^\dagger(n) U_\mu(n) \Phi(n+\hat{\mu}) + \Phi^\dagger(n+\hat{\mu}) U_\mu^\dagger(n) \Phi(n) - 2\Phi^\dagger(n)\Phi(n) \right] \\ & + \frac{m^2}{2} \Phi^\dagger(n)\Phi(n) + \frac{g}{4!} \left[\Phi^\dagger(n)\Phi(n) \right]^2 \Bigg\} , \end{aligned} \tag{2}$$

with $U_\mu(n)$ being the gauge link connecting sites n and $n+\hat{\mu}$ and we are using periodic boundary conditions.

Exercise 13: Connecting the lattice and the continuum

Show that, in the naïve continuum limit $a \to 0$, eq. (2) reduces to the **gauge invariant** version of eq. (1), i.e., with the replacement $\partial_\mu \to D_\mu$.

Hint: Remember that $U_\mu(n) = \exp[iagA_\mu(n)]$. You will need to expand the exponential to second order.

Hint: To first order in the lattice spacing a, one may write

$$f(n \pm \hat{\mu}) \approx f(x) \pm a\partial_\mu f(x) . \tag{3}$$

Hint: Remember that, due to the periodic boundary conditions, the sum over the entire lattice of any funciton $f(n)$ can be shifted without altering the result. In other words, for an arbitrary shift m,

$$\sum_n f(n+m) = \sum_n f(n) . \tag{4}$$

Exercise 14: The reality of the action

Show that the lattice action shown in eq. (2) is real, i.e., that $(S_{\mathrm{lat}})^\dagger = S_{\mathrm{lat}}$.

Exercise 15: A convenient (re)parametrisation (again)

Use the transformations

$$a^{\frac{d-2}{2}}\phi_i \to (2\kappa)^{1/2}\varphi_i\,, \tag{5}$$

$$(am)^2 \to \frac{1-2\lambda}{\kappa} - 2d\,, \tag{6}$$

$$a^{-d+4}g \to \frac{6\lambda}{\kappa^2}\,, \tag{7}$$

with $i = 1, 2$, and d being the number of spacetime dimensions, to write the action of eq. (2) in terms of the dimensionless parameters κ and λ.

Exercise 16: Actual simulations

Consider the combined action of eq. (2) and that of a SU(2) Yang-Mills field on the lattice, written below,

$$S_{\mathrm{YM}} = \beta \sum_n \left[1 - \frac{1}{2}\mathrm{ReTr}U_{01}(n)\right]\,. \tag{8}$$

Using the Metropolis algorithm, simulate this theory of a neutral pair of scalar fields coupled to a dynamic gauge field in two dimensions. Use a volume of $V = 10 \times 10$ and $\lambda = 0.02$.

The scalar sector of this system is similar to that of exercise sheet 2, but with the $\mathbb{Z}_2$ symmetry replaced by an O(2) symmetry. In other words, depending on the values of κ and λ the scalar potential will have either the shape of a single-well or of a Mexican hat. Therefore, it is useful to consider as an observable

$$\rho = \frac{1}{V}\sum_n \langle \Phi^\dagger(n)\Phi(n)\rangle = \frac{1}{V}\sum_n \langle \varphi_1^2(n) + \varphi_2^2(n)\rangle\,. \tag{9}$$

For different value of the (inverse) gauge coupling β, perform a scan over κ measuring:

- the average norm of Φ, ρ; and
- the string tension $\hat{\sigma}$.

In the supplemental material you can find two plots, one for each of the observables above. You can use the same values of β and κ, but also feel free to try any you want.

Note: In the limit of $g \to 0$, i.e., $\beta \to \infty$, the system consists only of the φ fields.

Supplemental material

In this exercise we are dealing with a field whose elements belong to the group SU(2), and the updating procedure should be such that update proposals also belong to the group. Since every element of SU(2) group can be written as the exponential of a member of the $\mathfrak{su}(2)$ Lie algebra, updating a particular link U can be written as

$$U \to RU\,, \quad R = \exp\left[i\sum_{a=1}^{3}\alpha^a\sigma^a\right]\,, \tag{10}$$

with σ^a being the Pauli matrices and α^a real numbers. Note that the update may equally well be written as $U \to UR$. Because of the commutation properties of the Pauli matrices the computation of R can be simplified as

$$R = \cos(|\vec{\alpha}|)\mathbb{1} + i\frac{\alpha^a}{|\vec{\alpha}|}\sigma^a\sin(|\vec{\alpha}|)\,, \tag{11}$$

with $\mathbb{1}$ being the 2×2 identity matrix and $|\vec{\alpha}| = \sqrt{(\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2}$. Thus it requires only simple arithmetic and trigonometric functions.

As always, you should skip a number of steps at the beginning, so the system can thermalise. And remember that consecutive configurations in the Markov chain are likely to be correlated, so it is also a good idea to skip some between consecutive measurements.

Below, you can find a plot of the string tension as a function of β to use for cross-checking.

```
matrix U[Nt][Nx][d];
double phi[Nt][Nx][2];
for n ← 0 to N_steps do
    for 0 ≤ t < N_t and 0 ≤ x < N_x do                 // sweep over the lattice
        for 0 ≤ c < d do                               // go over all components
            oldAction ← S(U,phi) ;                           // save old action
            oldValue ←U[t][x][c] ;                               // save old U
            R ← randomSU2() ;                             // random SU(2) matrix
            U[t][x][c] ← R * U[t][x][c];
            ΔS = S(U,phi) - oldAction;
            if ΔS > 0 then
                r ← UniformReal(0,1);
                if r > exp(−ΔS) then
                    U[t][x][c] ← oldValue ;                  // reject proposal
                end
            end
        end
        for 0 ≤ i < 2 do                               // go over all components
            oldAction ← S(U,phi) ;                           // save old action
            oldValue ←phi[t][x][i] ;                          // save old φ(x)
            phi[t][x][i] ← phi[t][x][i] + 𝒩(0,1) ;
            ΔS = S(U,phi) - oldAction;
            if ΔS > 0 then
                r ← UniformReal(0,1);
                if r > exp(−ΔS) then
                    phi[t][x][i] ← oldValue ;                // reject proposal
                end
            end
        end
    end
    Obs ← Obs + O(U,phi) ;              // update computation of observable
end
Obs ← Obs / N_steps;
```

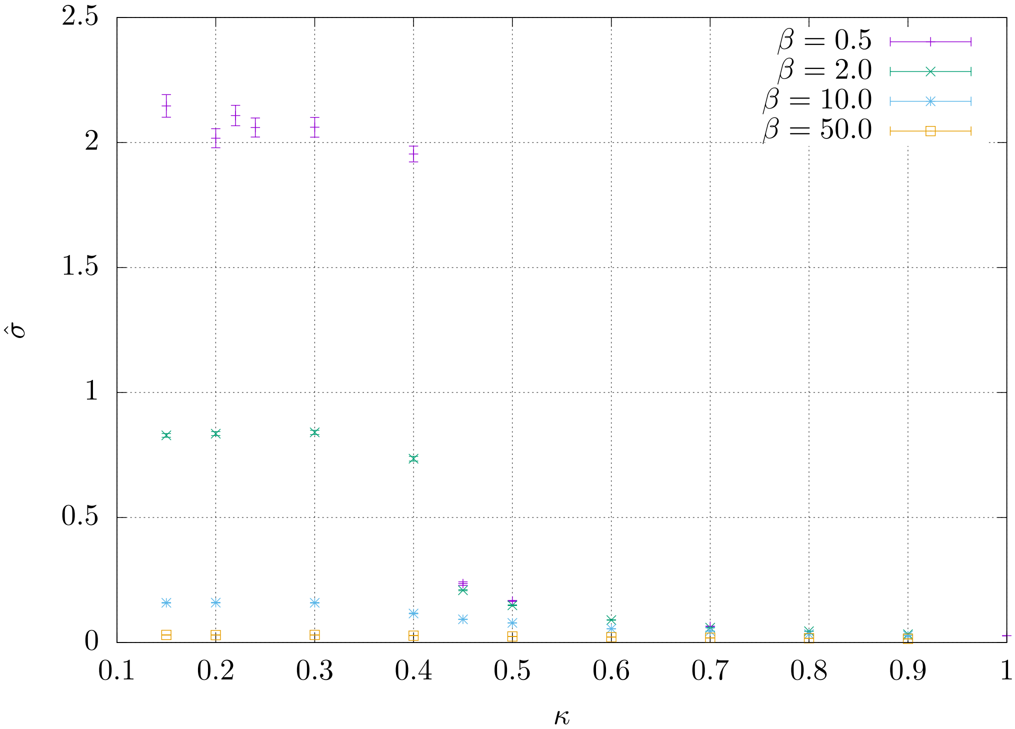

Figure 1: String tension, in lattice units, as a function of κ for different values of the (inverse) coupling. Computed on a 10×10 lattice.

Please note that the runs used in these plots may suffer from "low" statistics and/or ergodicity problems. In other words, they were rather short. Take these plots with a grain of salt.

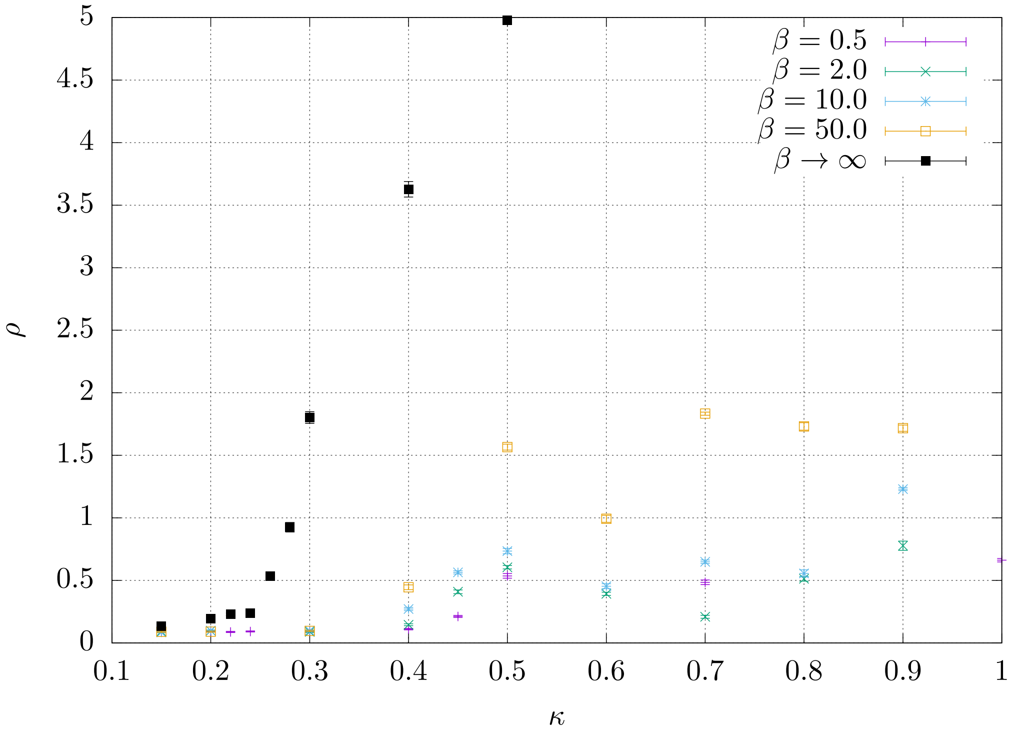

Figure 2: Norm of Φ as a function of κ for different values of the (inverse) coupling. Computed on a 10×10 lattice.

Please note that the runs used in these plots may suffer from "low" statistics and/or ergodicity problems. In other words, they were rather short. Take these plots with a grain of salt.