Lattice field theories
Exercise sheet 9 – Phase transitions and critical phenomena

Lectures: Jan Pawlowski j.pawlowski@thphys.uni-heidelberg.de
Felipe Attanasio pyfelipe@thphys.uni-heidelberg.de

Tutorials: Felipe Attanasio pyfelipe@thphys.uni-heidelberg.de

Institut für Theoretische Physik, Uni Heidelberg

Consider d-dimensional fermionic and scalar fields coupled via a Yukawa interaction

$$S = \int d^d x \left[\frac{1}{2} \left(\partial_\mu \phi\right)^* \partial_\mu \phi + \frac{m^2}{2} |\phi|^2 + \frac{g}{4!} (|\phi|^2)^2 \right] , \tag{1}$$

where $m^2, g, \in \mathbb{R}$, $\mu \in \{1, 2, \dots, d\}$, and ϕ can be a real or complex scalar field.

Exercise 19: Reality of the action

Show that the action in eq. (1), for the case of $\phi \in \mathbb{C}$ can be written in terms of two real fields.

Exercise 20: Discrete version

Discretise eq. (1) on a d-dimensional Euclidean lattice and use the following (dimensionless) parametrisation:

$$a^{\frac{d-2}{2}} \phi \to (2\kappa)^{1/2} \varphi \,, \tag{2}$$

$$(am)^2 \to \frac{1 - 2\lambda}{\kappa} - 2d \,, \tag{3}$$

$$a^{-d+4} g \to \frac{6\lambda}{\kappa^2} \,, \tag{4}$$

Exercise 21: Symmetries, spontaneously broken or otherwise

Demonstrate that the action in eq. (1) has $\mathbb{Z}_2$ symmety in the case of $\phi \in \mathbb{R}$ and U(1) if $\phi \in \mathbb{C}$ (or O(2) if it is written as two real fields). Additionally, show that the symmetry is spontaneously broken, at the *classical* level, at

$$\kappa_{c,\text{classical}}(\lambda) = \frac{1 - 2\lambda}{2d} \,, \tag{5}$$

i.e., for $\kappa > \kappa_{c,\text{classical}}$ the classical potential no longer has a single minimum.

Exercise 22: Looking for critical behaviour

Use the Metropolis algorithm to simulate the action in eq. (1) for the following cases:

- $d = 2$ and $\phi \in \mathbb{R}$;
- $d = 2$ and $\phi \in \mathbb{C}$;
- $d = 3$ and $\phi \in \mathbb{R}$;
- $d = 3$ and $\phi \in \mathbb{C}$;

Choose a fixed value for λ, e.g., $\lambda = 0.02$ and scan the κ-axis (in some vicinity of $\kappa_{c,\text{classical}}$). Measure the average value,

$$M = \left\langle \frac{1}{V} \sum_n \varphi_n \right\rangle , \tag{6}$$

and the susceptibility,

$$\chi_2 = V \left(\left\langle \left(\frac{1}{V} \sum_n \varphi_n \right)^2 \right\rangle - \left\langle \frac{1}{V} \sum_n \varphi_n \right\rangle^2 \right) , \tag{7}$$

where V is the spacetime volume, i.e., number of lattice sites, for each of the real fields.

Ideally, you should run the simulation at multiple volumes and see how the peak of the susceptibility changes with the volume. If it grows with the volume, that is an indicator of a second order phase transition, as it should diverge in the thermodynamic limit $V \to \infty$.

For each of the cases suggested ($d \in \{2, 3\}$, $\phi \in \{\mathbb{R}, \mathbb{C}\}$):

- What is the critical value of κ?
- You should see a change from $M \approx 0$ to $M \neq 0$ as κ changes. Is that transition smooth? Do you notice evidence of non-analytic behaviour as V is increased?
- How does the qualitative behaviour of M and χ_2 compare for the different cases considered here?

Hint: The most expensive part of the simulation is computing the action twice for each Monte Carlo step: one before the proposal and one after. Keep in mind that the updates are *local*. Use this fact to come up with clever ways of computing $\Delta S = S_{\text{new}} - S_{\text{old}}$ that make the code more efficient. Note that no approximations are needed or should be used.

Hint: In order to choose how big the lattice should be, consider how long the simulation takes to run and how many data points you can get. It is important to use different volume here to see how the observabls change. Using, for example in the $d = 3$ case, $V \in \{10^3, 12^3, 14^3\}$ could be already good, provided you have good statistics. Feel free to use bigger, smaller, and more lattice sizes.