’Functional’ RG flows for integrals

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'Functional' RG flows for integrals

generating function

$$Z[j] = \int dx \, \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

'Functional' RG flows for integrals

generating function with 'cutoff' R

$$Z[j;R] = \int dx \, \exp\left(-\frac{1}{2}(1+R)\,x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

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- $R \to \infty$: $Z[j;R] \to \int dx \exp\left(-\frac{1}{2}(1+R)x^2 + j\,x\right)$

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- $R \to 0$: $\quad Z[j;R=0] = Z[j]$

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- $R \to 0$: $\quad Z[j;R=0] = Z[j]$
- $j = 0$: $\quad Z[0;R] = \frac{2}{\sqrt{1+R}}\, e^{\frac{3(1+R)^2}{4\lambda}} \sqrt{\frac{3(1+R)^2}{4\lambda}}\, K\left(\frac{1}{4}, \frac{3(1+R)^2}{4\lambda}\right)$

'Functional' RG flows for integrals

generating function with 'cutoff' R

$$Z[j;R] = \int dx \, \exp\left(-\frac{1}{2}(1+R)\,x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

flow of $\ln Z[j;R]$ and $\Gamma[x;R] = j\,x - \ln Z[j;R] - \frac{1}{2}R\,x^2$

$$\partial_R \ln Z[j;R]\,, \qquad \partial_R \Gamma[x;R] = -\partial_R \ln Z[j;R] - \tfrac{1}{2}x^2\,,$$

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$$\partial_R \ln Z[j;R] = -\langle \frac{1}{2}x^2 \rangle_j$$

'Functional' RG flows for integrals

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flow of $\ln Z[j;R]$ and $\Gamma[x;R] = j\,x - \ln Z[j;R] - \frac{1}{2}R\,x^2$

$$\partial_R \ln Z[j;R] = -\frac{1}{2}\left(\partial_j^2 \ln Z[j] + (\partial_j \ln Z[j])^2\right)$$

'Functional' RG flows for integrals

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$$Z[j;R] = \int dx \ \exp\left(-\frac{1}{2}(1+R)\,x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

flow of $\ln Z[j;R]$ and $\Gamma[x;R] = j\,x - \ln Z[j;R] - \frac{1}{2}R\,x^2$

$$\partial_R \Gamma[x;R] = \frac{1}{2}\,\frac{1}{\partial_x^2\Gamma[x;R] + R}$$

- $\partial_j^2 \ln Z[j] = \dfrac{1}{\partial_x^2\Gamma[x;R] + R}$

'Functional' RG flows for integrals

generating function with 'cutoff' R

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flow of $\ln Z[j;R]$ and $\Gamma[x;R] = j\,x - \ln Z[j;R] - \frac{1}{2}R\,x^2$

$$\partial_R \Gamma[x;R] = \frac{1}{2}\frac{1}{\partial_x^2 \Gamma[x;R] + R}$$

- boundary condition: $\Gamma[x; R \to \infty] = \frac{1}{2}x^2 + \frac{\lambda}{4!}x^4$

'Functional' RG flows for integrals

$$\ln Z[j] = \ln \int dx \, \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$

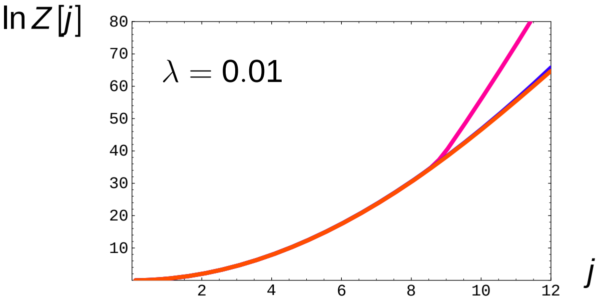
- asymptotic perturbative series with optimal order

$$n_{\rm opt}(j) \leq n_{\rm opt}(0) \sim \frac{3}{2\lambda}$$

- flow with boundary condition
  - $\Gamma[x, R = 10^3] = \frac{1}{2}x^2 + \frac{\lambda}{4!}x^4$
  - $\Gamma[x = \pm 10^2, R] = \Gamma[x = \pm 10^2, R = 10^3]$
- numerical integration of $\ln Z[j]$

'Functional' RG flows for integrals

$$\ln Z[j] = \ln \int dx\ \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$



- perturbative expansion: $n = 26$
- flow
- numerical integration

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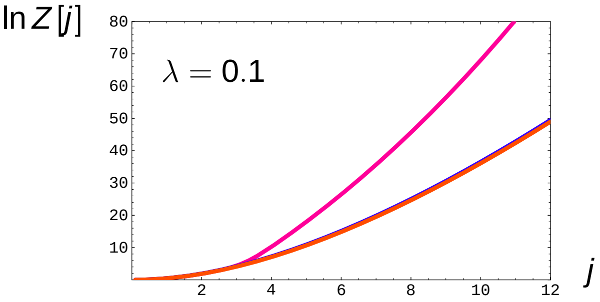

- perturbative expansion: $n = 6$
- flow
- numerical integration

'Functional' RG flows for integrals

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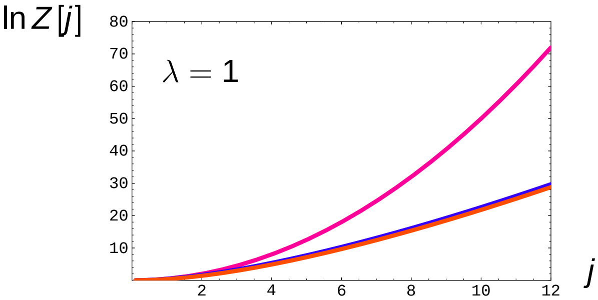

- perturbative expansion: $n = 0$
- flow
- numerical integration

'Functional' RG flows for integrals: truncations

$$\partial_R \Gamma[x;R] = \frac{1}{2}\frac{1}{\partial_x^2\Gamma[x;R]+R}$$

- parameterisation

requires convergence of Taylor expansion

$$\Gamma[x;R] = \frac{1}{2}\alpha[R]\,x^2 + \frac{1}{4!}\lambda[R]x^4 + \frac{1}{4!}\lambda_6[R]x^6 + \sum_{n=3}^{\infty}\frac{1}{(2n)!}\lambda_{2n}[R]x^{2n}$$

- initial conditions at $R_{\rm in}$

$$\alpha[R_{\rm in}] = 1\,, \qquad \lambda[R_{\rm in}] = \lambda\,, \qquad \lambda_{2n}[R_{\rm in}] = 0 \quad \forall n > 2\,.$$

- truncation

(i) $\lambda_{2n>4}[R] \equiv 0$

(ii) $\lambda_{2n>6}[R] \equiv 0$

'Functional' RG flows for integrals: truncations

$$\partial_R \Gamma[x;R] = \frac{1}{2}\frac{1}{\partial_x^2\Gamma[x;R]+R}$$

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$$\alpha[R_{\rm in}] = 1\,, \qquad \lambda[R_{\rm in}] = \lambda\,, \qquad \lambda_{2n}[R_{\rm in}] = 0 \quad \forall n > 2\,.$$

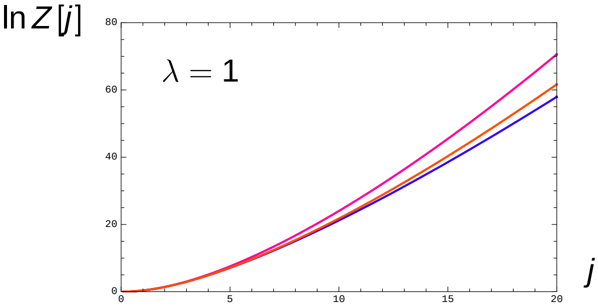
- truncation (i): $\lambda_{2n>4}[R] \equiv 0$ (ii): $\lambda_{2n>6}[R] \equiv 0$

- flows for coefficients

$$\partial_x^n\Gamma[0,R] = \partial_x^n\left[\frac{1}{2}\frac{1}{\partial_x^2\Gamma[x;R]+R}\right]_{x=0}$$

'Functional' RG flows for integrals: truncations

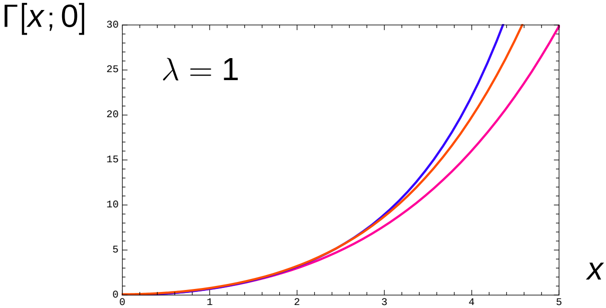
$$\ln Z[j] = \ln \int dx\ \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + j\,x\right)$$



- truncation (i): $\lambda_6 \equiv 0, \quad \lambda_{2n>6} \equiv 0$
- truncation (ii): $\lambda_6 \neq 0, \quad \lambda_{2n>6} \equiv 0$
- numerical integration

'Functional' RG flows for integrals: truncations

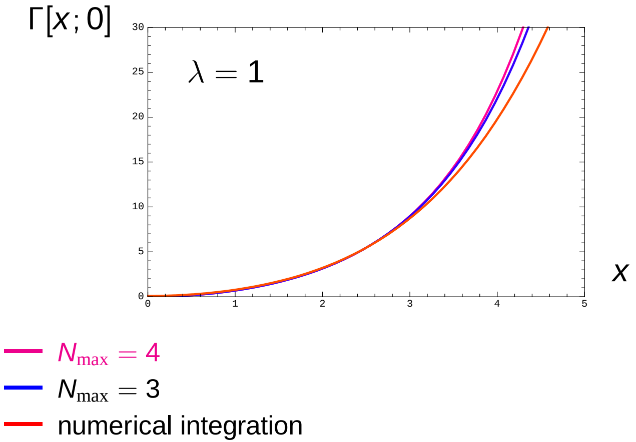
$$\Gamma[x;R] = \frac{1}{2}\alpha[R]\,x^2 + \frac{1}{4!}\lambda[R]x^4 + \sum_{n=3}^{N_{\max}} \frac{1}{(2n)!}\lambda_{2n}[R]x^{2n}$$



- $N_{\max} = 2$: $\lambda_{2n>4} \equiv 0$
- $N_{\max} = 3$: $\lambda_{2n>6} \equiv 0$
- numerical integration

'Functional' RG flows for integrals: truncations

$$\Gamma[x;R] = \frac{1}{2}\alpha[R]\,x^2 + \frac{1}{4!}\lambda[R]x^4 + \sum_{n=3}^{N_{\max}} \frac{1}{(2n)!}\lambda_{2n}[R]x^{2n}$$



'Functional' RG flows for integrals: truncations

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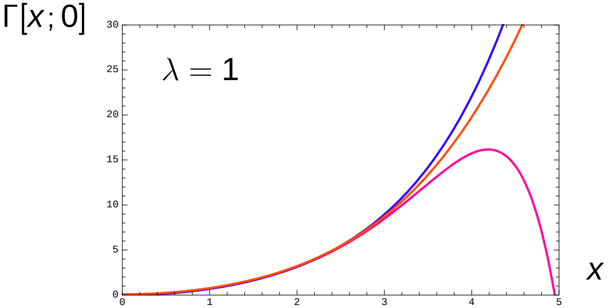
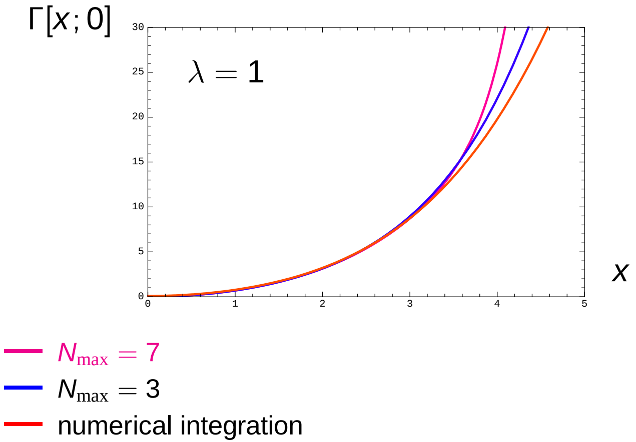


- $N_{\rm max} = 5$
- $N_{\rm max} = 3$
- numerical integration

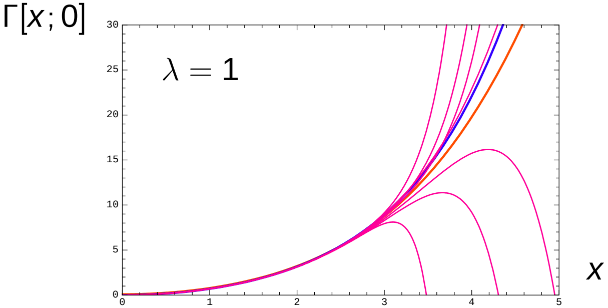
'Functional' RG flows for integrals: truncations

$$\Gamma[x;R] = \frac{1}{2}\alpha[R]\,x^2 + \frac{1}{4!}\lambda[R]x^4 + \sum_{n=3}^{N_{\max}} \frac{1}{(2n)!}\lambda_{2n}[R]x^{2n}$$



'Functional' RG flows for integrals: truncations

$$\Gamma[x;R] = \frac{1}{2}\alpha[R]\,x^2 + \frac{1}{4!}\lambda[R]x^4 + \sum_{n=3}^{N_{\rm max}} \frac{1}{(2n)!}\lambda_{2n}[R]x^{2n}$$

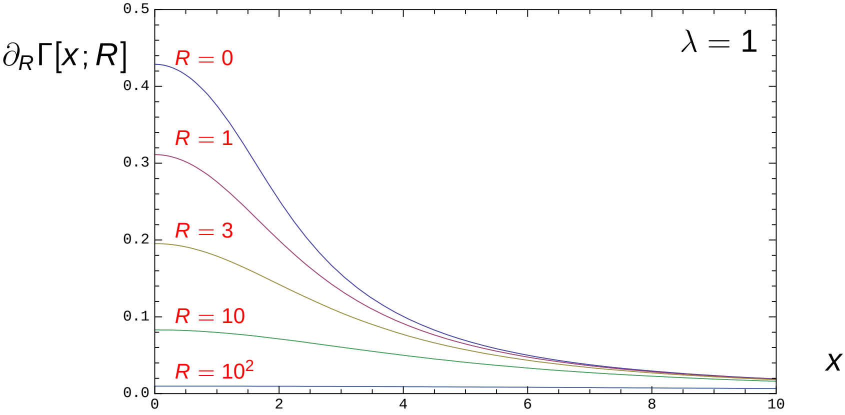


- $N_{\rm max} = 4 - 10$
- $N_{\rm max} = 3$
- numerical integration

'Functional' RG flows for integrals: truncations

full flow

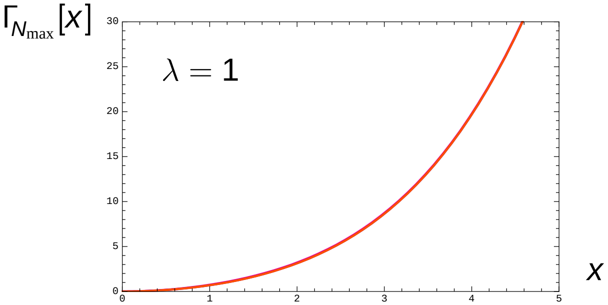
$$\partial_R \Gamma[x;R] = \frac{1}{2}\frac{1}{\partial_x^2 \Gamma[x;R] + R}$$



'Functional' RG flows for integrals: truncations

- rapid convergence for large x:

$$\Gamma_{N_{\max}}[x;R] = \frac{1}{2}x^2 + \frac{1}{4!}\lambda x^4 + \frac{1}{2}\log(1+\tfrac{1}{2}\lambda x^2 + R) + \sum_{n=0}^{N_{\max}} \frac{1}{(2n)!}\frac{\Delta\lambda_{2n}[R]\,x^{2n}}{(1+\frac{1}{2}\lambda x^2+R)^{n+2}}$$



- 1-loop perturbation theory: $\Delta\lambda_{2n} \equiv 0, \quad \forall n$
- $N_{\max} = 0$
- numerical integration

'Functional' RG flows for integrals: truncations

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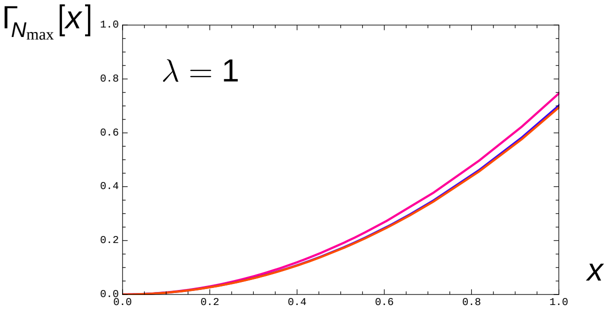

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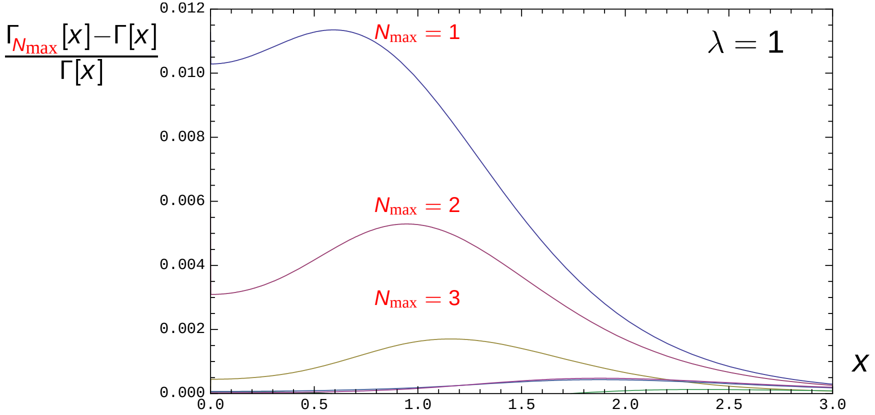

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