I The Functional Renormalisation Group

Quantum field theories are given/ determined by a complete set of correlation functions.

Example: scalar field theory with a real field $\varphi(x)$ in d dim.

$\boxed{\text{finite}}$ correlation functions:

$$\langle \varphi(x_1) \ldots \varphi(x_n) \rangle \quad , n \in \mathbb{N}_0$$

$n=0$: $\langle 1 \rangle \overset{!}{=} 1$ normalised cor. fct.

$n=1$: $\phi(x) := \langle \varphi(x) \rangle$ mean field

$n=2$: $G(x,y) := \langle \varphi(x)\, \varphi(y) \rangle - \phi(x)\, \phi(y)$

propagator (connected 2 point fct

⋮

Generating functional: Euclidean space

finite $Z[J]$ with $\underset{\Uparrow}{\dfrac{1}{Z[J]}} \dfrac{\delta^n \; Z[J]}{\delta J(x_1)\cdots\delta J(x_n)} = \langle \varphi(x_1)\cdots\varphi(x_n)\rangle$

$Z[J]$ is the <u>renormalised</u> <u>finite</u> generating functional of normalised Green functions (correlation fcts.) of the theory.

Reminder: classical action

$$S[\varphi] = \frac{1}{2}\int d^dx\left(\partial_\mu\varphi(x)\,\partial_\mu\varphi(x) + m^2\varphi(x)^2\right)$$
$$+ \frac{1}{4}\int d^dx\;\lambda\;\varphi(x)^4$$

and

$$Z[J] = \frac{1}{N}\int[d\varphi]_{\text{ren}}\, e^{-S[\varphi] + \int d^dx\, J(x)\,\varphi(x)}$$

with e.g.

$$N = \int[d\varphi]_{\text{ren}}\, e^{-S[\varphi]} \quad , \; N = 1$$

- In the path integral representation the task is to define $\int d\varphi\, e^{-S}$.

- $Z[J]$ generates also disconnected Green functions.

  $\Rightarrow$ Schwinger functional $W[J]$:

$$W[J] = \ln Z[J] \qquad \text{finite}$$

  generates connected Green functions
  proof when deriving the flow (FRG)

- $\Gamma[\phi]$ generates 1PI Green functions

$$\Gamma[\phi] = \sup_J \left\{ \int d^dx\, J(x)\phi(x) - W[J] \right\}$$

$$\Rightarrow \quad \phi(x) = \left.\frac{\delta W}{\delta J(x)}\right|_{J_{sup}} \qquad \text{(if differentiable)}$$

$$\frac{\delta \Gamma}{\delta \phi(x)} = \int d^dx' \,\frac{\delta J_{sup}(x')}{\delta \phi(x)}\, \phi(x') + J_{sup}(x) - \int d^dx' \,\frac{\delta J_{sup}(x')}{\delta \phi(x)} \cdot \left.\frac{\delta W}{\delta J}\right|_{J_{sup}}$$

$$= J_{sup}(x)$$

1PI proof with flow

$$\int d^d x' \frac{\delta^2 W[J]}{\delta J(x)\, J(x')} \quad \frac{\delta^2 \Gamma}{\delta\phi(x')\,\delta\phi(y)} \Big[= \delta^{(d)}(x-y) \Big]$$

$$= \int d^d x' \left(\frac{\delta}{\delta J(x)} \phi(x') \right) \frac{\delta}{\delta\phi(x')} J(y)$$

$$= \int d^d x' \quad \frac{\delta}{\delta J(x)} J(y) \quad = \quad \delta^{(d)}(x-y)$$

or with $\Gamma^{(n)}(x_1, \dots, x_n) = \dfrac{\delta^n \Gamma}{\delta\phi(x_1) \cdots \delta\phi(x_n)}$

$$W^{(n)}(x_1, \dots, x_n) = \frac{\delta^n W}{\delta J(x_1) \cdots \delta J(x_n)}$$

$$\boxed{\int d^d x' \cdot W^{(2)}(x, x')\, \Gamma^{(2)}(x', y) = \delta^{(d)}(x-y)}$$

and $G(x,y) = W^{(2)}(x,y) = 1/\Gamma^{(2)}(x,y)$

The above relations are valid in the presence of non-vanishing fields/currents, e.g.

$$\Gamma^{(2)} = \Gamma^{(2)}[\phi J](x_1, x_2)$$

- functional relations (instead of path integral)

Quantum equations of motion [Dyson-Schwinger eq. DSE]

$$\int [d\varphi]_{ren} \frac{\delta}{\delta\varphi(x)} \left\{ e^{-S[\varphi] + \int d^d x\, J(x)\varphi(x)} \right\} = 0$$

$$\Rightarrow \langle J(x) \rangle_J - \left\langle \frac{\delta S[\varphi]}{\delta\varphi(x)} \right\rangle_J = 0$$

$$\Rightarrow \boxed{J(x) = \left\langle \frac{\delta S[\varphi]}{\delta\varphi(x)} \right\rangle_J}$$

Important relation:

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_J = \left(\frac{\delta}{\delta J(x_1)} + \phi(x_1) \right) \langle \varphi(x_2) \cdots \varphi(x_n) \rangle_J$$

remember: $\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_J = \frac{1}{Z[J]} \int [d\varphi]_{ren} \varphi(x_1) \cdots \varphi(x_n) e^{-S + \int J\varphi}$

$$\Rightarrow \quad \langle \varphi(x_1) \cdots \varphi(x_n) \rangle_J = \prod_{i=1}^{n} \left(\frac{\delta}{\delta J(x_i)} + \phi(x_i) \right)$$

Use

$$\frac{\delta}{\delta J(x_i)} = \int d^d x \, \frac{\delta \phi(x')}{\delta J(x_i)} \, \frac{\delta}{\delta \phi(x')} = \int d^d x \, G(x_i, x') \frac{\delta}{\delta \phi(x')}$$

$$= G \cdot \frac{\delta}{\delta \phi}(x_i)$$

$$\Rightarrow \quad \boxed{\frac{\delta \Gamma}{\delta \phi(x)} = \frac{\delta S}{\delta \varphi(x)} \left[\varphi(x) = G \frac{\delta}{\delta \phi}(x) + \phi(x) \right]}$$

S action of real scalar field:

$$\frac{\delta S}{\delta \varphi(x)} = -\partial_\mu^2 \varphi(x) + m^2 \varphi(x) + \lambda \varphi(x)^3$$

$$\Big|_{\varphi = G \frac{\delta}{\delta \phi} + \phi} = -\partial_\mu^2 \phi(x) + m^2 \phi(x) + \lambda \phi(x)^3$$

$$+ \lambda \left[\left(G \frac{\delta}{\delta \phi} + \phi \right)^3 - \phi^3 \right]$$

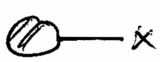
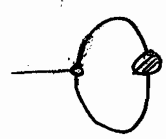
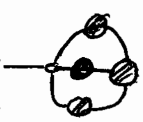
$$\Rightarrow \quad \frac{\delta S}{\delta \varphi(x)}\bigg|_{\varphi = G\frac{\delta}{\delta\phi}+\phi} = \frac{\delta S[\phi]}{\delta \phi(x)} + \lambda\left(G\frac{\delta}{\delta\phi}(x)\,\phi^2(x) + \phi(x)\, G\frac{\delta}{\delta\phi}(x)\,\phi(x)\right)$$

$$+ \lambda\left(G\frac{\delta}{\delta\phi}\right)^2 \phi$$

$$= \frac{\delta S[\phi]}{\delta\phi(x)} + 3\lambda\, G(x,x)\, \phi(x)$$

$$-\lambda \prod_i \int d^d x_i\, G(x,x_i)\, \Gamma^{(3)}(x_1,x_2,x_3)$$

Diagrammatically:

 $= \dfrac{\delta S}{\delta\phi(x)} + \dfrac{1}{2}$  $- \dfrac{1}{3!}$ 

with

 $= 1/\Gamma^{(2)}[\phi](x,y)$

 $= \Gamma^{(n)}[\phi](x_1,\dots,x_n)$

 $= S^{(n)}[\phi](x_1,\dots,x_n)$

General DSE (including symmetry ID's)

$$\int d\varphi \; \frac{\delta}{\delta\varphi(x)} \left\{ \Psi[\varphi] \, e^{-S[\varphi] + \int J \varphi \, d^d x} \right\} = 0$$

see 'Aspects of the FRG', chapter II

Repetition:

$$Z[J] = \int [d\varphi]_{ren}\; e^{-S[\varphi] + \int d^d x\, \varphi(x) J(x)}$$

DSE:

$$J = \left\langle \frac{\delta S}{\delta \varphi}(x) \right\rangle_J \quad := \frac{1}{Z[J]} \int [d\varphi]_{ren} \cdot \frac{\delta S}{\delta \varphi}\, e^{-S + \int J \cdot \varphi}$$

$$\frac{\delta \Gamma}{\delta \Phi}(x) = \frac{\delta S}{\delta \varphi}\Big|_{\varphi = G \cdot \frac{\delta}{\delta \Phi} + \Phi}]$$

Scalar field: graphical: $\boxed{I-6}$

Regul.:

$Z_k[J] \leftarrow$ page I-7

$$\Delta S_k[\varphi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ldots$$

$\Longrightarrow$ I-8

I-1 Derivation

Heuristic idea: Kadanoff block-spinning in continuum

Define

$$Z_k[J] = \int [d\varphi]_{p^2 \gtrsim k^2} \, e^{-S[\varphi] + \int d^dx \, \varphi(x) J(x)}$$

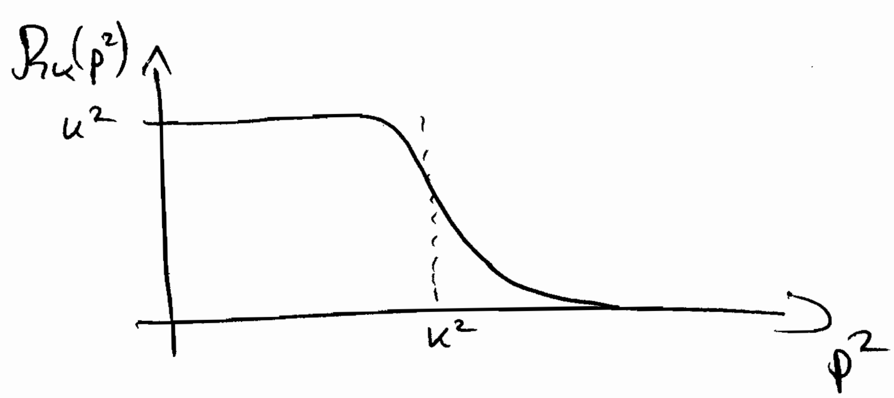
suppression of infrared (IR) modes

Practically

$$[d\varphi]_{\mathrm{ren},\, p^2 \gtrsim k^2} = [d\varphi]_{\mathrm{ren}} \, e^{-\Delta S_k[\varphi]}$$

with

$$\Delta S_k[\varphi] = \frac{1}{2} \int \frac{d^dp}{(2\pi)^d} \, \varphi(p) \, R_k(p^2) \, \varphi(-p)$$



Fourier transformation

$$\varphi(x) = \int \frac{d^d p}{(2\pi)^d} \overset{(\sim)}{\varphi}(p)\, e^{ip\cdot x}$$

$$\Rightarrow \varphi(p) = \int d^d x\, \varphi(x)\, e^{-ipx}$$

In particular:

$$\int d^d x\, d^d y\, \varphi(x)\, f(x,y)\, \varphi(y)$$

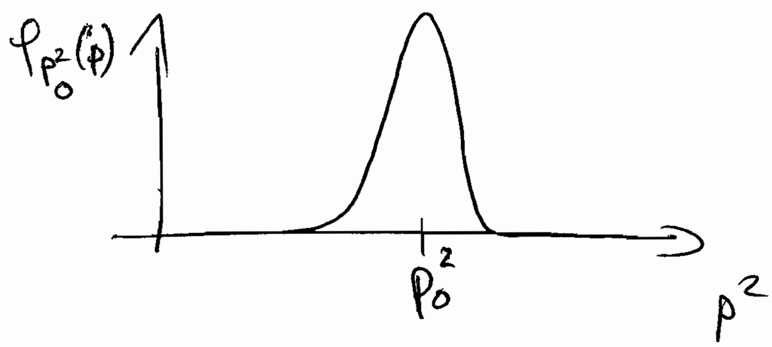
$$= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \varphi(p)\, \varphi(q) \int d^d x\, d^d y\, f(x,y)\, e^{ipx} e^{iqy}$$

$$= \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \varphi(p)\, \varphi(q) \cdot f(-p,-q)$$

Regulator: $\quad R_k(x,y) = R_k(-\partial_x^2)\, \delta^{(d)}(x-y)$

$$\Rightarrow R_k(-p,-q) = R_k(p^2)\, \delta^{(d)}(p+q)$$

$\Rightarrow$ page I-7



$$\Delta S_k[\varphi_{p_0^2 \ll k^2}] \simeq \frac{1}{2}\left(\int d^d p \; \varphi_{p_0^2}(p)\, \varphi_{p_0^2}(-p)\right) k^2$$

mass-suppression

$$\Delta S_k[\varphi_{p_0^2 \gg k^2}] \simeq 0$$

$$\Rightarrow \boxed{\text{IR suppressed}}$$

2nd lecture

limits:

- UV: $k \to \infty$, all momentum modes suppressed

  ΔS_k dominates S

  $\Rightarrow$ Gaussian path integral

- IR: $k \to 0$, no modes suppressed

  $\Delta S_k \to 0$

  $Z_k \to Z$

Question: $[d\varphi]_{ren}\, e^{-\Delta S_k[\varphi]}$

renormalised measure ?

Not necessarily naively

Formally correct:

(1) $Z[J]$ finite renormalised gen. funct.

of " " Green fcts.

(2) $\frac{\delta^n Z}{\delta J^n}$ exists for all n

((1) assumes existence of theory and

(2) 'good' choice of field variable $\varphi(x)$)

(3)

$$\boxed{Z_k[J] = e^{-\Delta S_k[\frac{\delta}{\delta J}]}\, Z[J]}$$

$$\Big[\simeq e^{-\Delta S_k[\frac{\delta}{\delta J}]} \int d\varphi\, e^{-S[\varphi] + \int J\varphi}$$

$$= \int d\varphi\, e^{-S[\varphi] - \Delta S_k[\varphi] + \int J\varphi}$$

$$e^{f[\frac{\delta}{\delta J}]}\, e^{\int J\varphi} = e^{f[\varphi]} \Big]$$

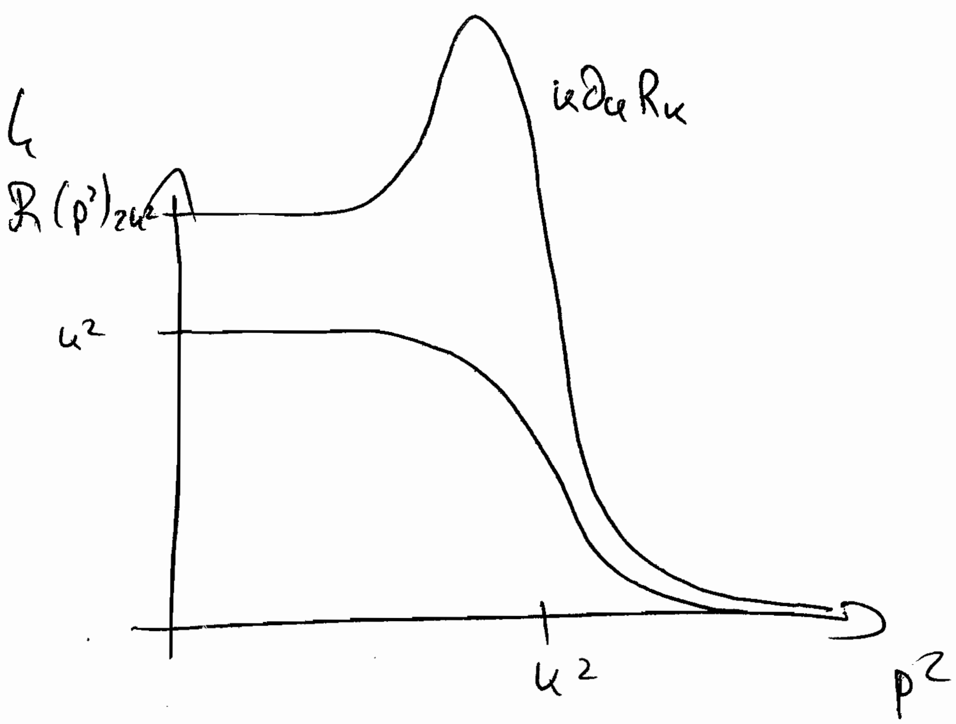
Flow equation:

$$k\,\partial_k Z_k[J] = -\left(k\partial_k \Delta S_k\left[\frac{\delta}{\delta J}\right]\right)\underbrace{e^{-\Delta S_k\left[\frac{\delta}{\delta J}\right]} Z[J]}_{Z_k[J]}$$

$$= -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\,\frac{\delta}{\delta J(p)}\,k\partial_k R(p^2)\,\frac{\delta}{\delta J(-p)}\,Z_k[J]$$

$$\Rightarrow \boxed{k\,\partial_k Z_k[J] = -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\,\frac{\delta^2 Z_k[J]}{\delta J(p)\,\delta J(-p)}\,k\partial_k R_k(p^2)}$$

with



Flow of Schwinger functional $\boxed{t = \ln k}$

- $\frac{1}{Z_k} k\partial_k Z_k = k\partial_k \ln Z_k = \partial_t W_k$

- $$\frac{1}{Z_k[J]} \frac{\delta^2 Z_k[J]}{\delta J(p)\delta J(-p)} = \frac{\delta^2 W_k}{\delta J(p)\delta J(-p)} + \phi(p)\phi(-p)$$

with $\phi(p) = \frac{\delta W}{\delta J(p)}$

and $$\frac{\partial^2 \ln Z_k}{\delta J(p)\delta J(-p)} = \frac{\delta}{\delta J(p)} \frac{1}{Z_k}\frac{\delta Z_k}{\delta J(-p)} = \frac{1}{Z}\frac{\delta^2 Z_k}{\delta J(p)\delta J(-p)} - \frac{1}{Z_k}\frac{\delta Z_k}{\delta J(p)}\frac{1}{Z_k}\frac{\delta Z_k}{\delta J(-p)}$$

$\Longrightarrow$

$$\boxed{\partial_t W_k[J] = -\frac{1}{2}\int\frac{d^d p}{(2\pi)^d}\left[W_k^{(2)}(p,-p) + \phi(p)\phi(-p)\right] \cdot \partial_t R_k}$$

Flow of effective action

$$\Gamma_k[\phi] = \sup_J \left\{ \int d^dx \, J(x)\phi(x) - W_k[J] \right\}$$

$$- \Delta S_k[\phi]$$

Flow: $(J = J_{\sup}[\phi])$

$$\partial_t\Big|_\phi \Gamma_k[\phi] = \int d^dx \, \partial_t\Big|_\phi J(x) \underbrace{\left[\phi(x) - \overset{\phi(x)}{\overset{/\!/}{\frac{\delta W_k[J]}{\delta J(x)}}} \right]}_{0}$$

$$- \partial_t\Big|_J W_k[J] - \partial_t\Big|_\phi \Delta S_k[\phi]$$

$$= \frac{1}{2} \int \frac{d^dp}{(2\pi)^d} \left[W_k^{(2)}(p,-p) + \phi(p)\phi(-p) - \phi(p)\phi(-p) \right]$$

$$\cdot \, \partial_t R_k$$

$$\Rightarrow \quad \boxed{\partial_t \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^dp}{(2\pi)^d} \left[W_k^{(2)}(p,-p) \right] \partial_t R_k(p)}$$

Relation between ϕ- der. of Γ_k

and J- der. of W :

(i) $\Gamma_k[\phi] + \Delta S_k[\phi]$ Legendre trafo of W_k

$\Rightarrow$ I-3 – I-4 :

$$\frac{\delta(\Gamma_k + \Delta S_k)}{\delta\phi(x)} = J_{\sup_k}(x)$$

$$\frac{\delta W_k}{\delta J(x)} = \phi(x)$$

I-3a :

$$\int d^dx' \, \frac{\delta^2 W_k[J]}{\delta J(x)\,\delta J(x')} \, \frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\phi(x')\,\delta\phi(y)} = \delta^{(d)}(x-y)$$

$$\boxed{\Rightarrow \int d^dx' \, W_k^{(2)}(x, x')\,(\Gamma^{(2)} + R_k)(x', y) = \delta^{(d)}(x-y)}$$

with $\Delta S_k[\phi] = \frac{1}{2}\int d^dx \; \phi(x)\, R_k(x,y)\, \phi(y)$

$$\Rightarrow \boxed{G_k(x,y) = W_k^{(2)}(x,y) = \frac{1}{\Gamma_k^{(2)} + R_k}(x,y)}$$

e.g. $R_k(x,y) = R_k(-\partial_x^2)\,\delta^{(d)}(x-y)$

Summary: $t = \ln k/k_0$

before II-14 – II-16

$$W_k[J] \simeq \log \int [d\varphi]_{\text{ren}}\, e^{-S[\varphi] - \Delta S_k[\varphi] + \int \varphi \cdot J}$$

with

$$\Delta S_k[\varphi] = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\, \varphi(p)\, R_k(p^2)\, \varphi(-p)$$

$$\Rightarrow \partial_t W_k[J] = \langle \partial_t \Delta S_k[\varphi] \rangle_J$$

$$= \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\, \underbrace{\langle \varphi(-p)\, \varphi(p) \rangle_J}_{\left(W_k^{(2)}(p,-p) + \phi(-p)\,\phi(p)\right)\partial_t R_k}\, \partial_t R_k(p^2)$$

$$\Rightarrow \boxed{\partial_t \Gamma_k[\phi] = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\, \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}(p,-p)\, \partial_t R_k(p^2)}$$

IR-finiteness

UV-finiteness

also

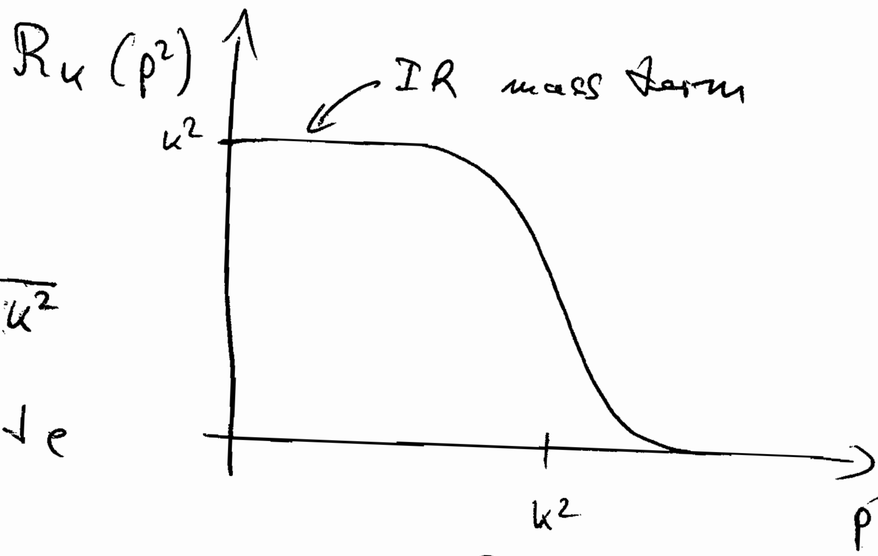
$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\, \underset{\int \frac{d^d p}{(2\pi)^d}}{\mathrm{Tr}}\, \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}\, \partial_t R_k$$

final flow:

$$\boxed{\partial_t \Gamma_k[\phi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}(p,-p)\,\partial_t R_k(p^2)}$$

Finiteness

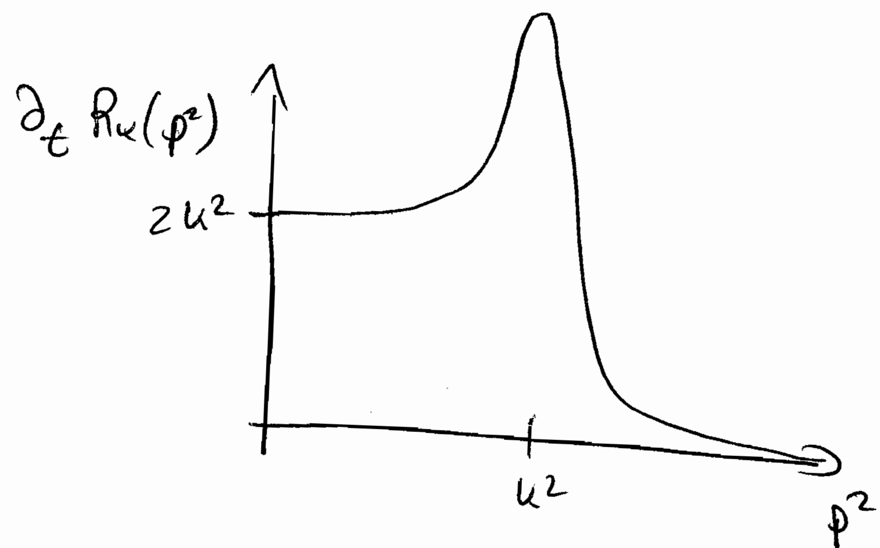
(i) IR:



$$\frac{1}{\Gamma_k^{(2)}[\phi] + R_k} \xrightarrow{p^2/k^2 \to 0} \frac{1}{\Gamma_k^{(2)} + k^2}$$

$\Rightarrow \partial_t \Gamma_k$ IR-finite

(ii) UV:



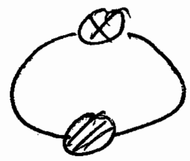
$$p^d \frac{1}{\Gamma_k^{(2)}[\phi]} R_k(p^2) \xrightarrow{p^2/k^2 \to \infty} 0$$

$\boxed{\mathrm{Tr}, \int d^d p}$

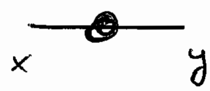
$\Rightarrow \partial_t \Gamma_k$ UV-finite

3-rd lecture

Diagrammatically: (DSE I-6)

$$\partial_t \Gamma_k = \frac{1}{2}$$ 

with

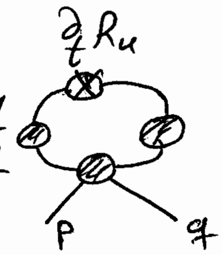


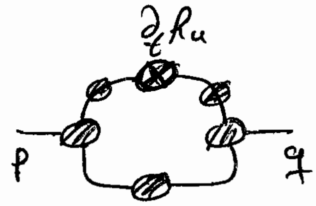
$$= \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}(x,y)$$

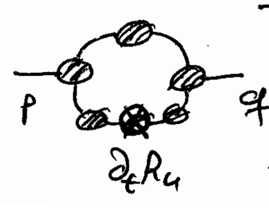
$$\otimes = \partial_t R_k$$

$$= \Gamma^{(n)}[\phi](x_1, \ldots, x_n)$$

Examples:

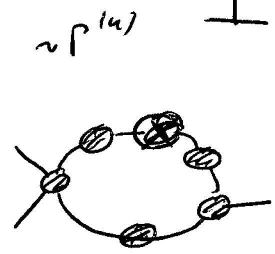
$$\frac{\delta}{\delta\phi(p)} \frac{\delta}{\delta\phi(q)} \partial_t \Gamma_k[\phi] = \partial_t \Gamma_k^{(2)}[\phi](p,q) = -\frac{1}{2}$$ 

$$+ \frac{1}{2}\Big[$$ 

$+$ 

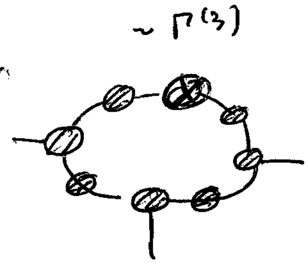
$\Big]$

$\partial_t \Gamma^{(3)} = -\frac{1}{2}$ 

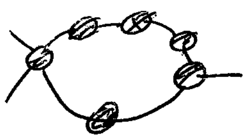
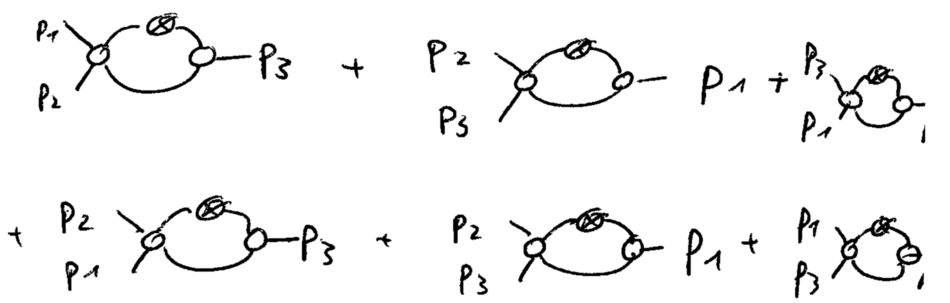
$+ \frac{1}{2}\Big($ 

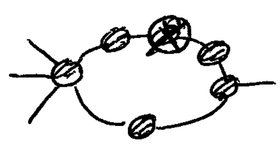
$\Big)$

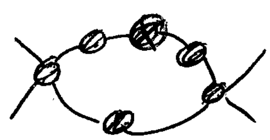
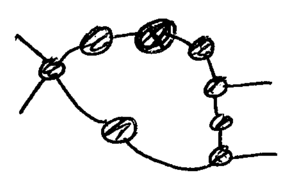
permut.

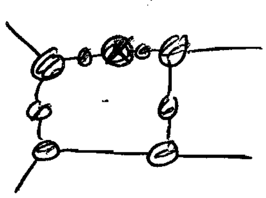
$-\frac{1}{2}\Big($ 

$\Big)$

e.g.: $\Big($  $\Big) =$ 

$\partial_t \Gamma^{(4)} = -\frac{1}{2}$  $+ \frac{1}{2}\Big($  $\Big)$

$+ \frac{1}{2}\Big($  $\Big)$ $\frac{1}{2}\Big($  $\Big)$

$+ \frac{1}{2}\Big($  $\Big)$

Remarks:

(1) Finiteness, see I-14

(2) Flow equation for $\partial_t \Gamma_k[\phi]$ + initial condition $\Gamma_\Lambda[\phi]$ at some (UV/IR) scale Λ provide definition of the quantum field theory related to Γ_Λ.

(a) perturbative renormalisability

$$\lim_{\Lambda\to\infty} \Gamma_\Lambda \sim \underset{\substack{\uparrow \\ \text{bare action}}}{S_{cl}} \quad \text{(all perturb. relevant terms)}$$

(b) non-perturbative renormalisability

$$\lim_{\Lambda\to\infty} \Gamma_\Lambda = \Gamma_{\text{fixed-point}} \quad \text{(includes (a))}$$

(3) No restriction to momentum cut-off:

$$\underset{\text{time evol.}}{R_k(t,t')}\,, \quad \Delta S_k \sim \underset{\text{vertex/coupl. reg., nPI-reg}}{\int R_k(x_1,\dots,x_n)\,\phi(x_1)\dots\phi(x_n)}$$

'Interlude': Integro-diff. eq. for Γ_k

$$e^{-\Gamma_k[\phi]} = \int d\varphi \, e^{-(S[\varphi] + \Delta S_k[\varphi]) + \int J\cdot(\varphi-\phi)} \cdot e^{+\Delta S_k[\phi]}$$

$$\Leftrightarrow \quad e^{-\Gamma_k[\phi]} = \int d\varphi \, e^{-S[\varphi] - \Delta S_k[\varphi] + \int\left(\frac{\delta \Gamma_k}{\delta\phi} + \frac{\delta \Delta S_k}{\delta \phi}\right)(\varphi-\phi)} \cdot e^{\Delta S_k[\phi]}$$

$$= \int d\varphi \, e^{-S[\varphi] + \Delta S_k[\varphi-\phi] + \int \frac{\delta\Gamma_k}{\delta\phi}(\varphi-\phi)}$$

$$\Rightarrow \quad \boxed{e^{-\Gamma_k[\phi]} = \int d\varphi \, e^{-S[\varphi+\phi] + \Delta S_k[\varphi] + \int \frac{\delta \Gamma_k}{\delta\phi}\cdot\varphi}}$$

$k \to \infty$: $\quad \Delta S_k[\varphi] \longrightarrow \infty$: Saddlepoint exp. becomes exact

$$\Rightarrow \quad e^{-\Gamma_k[\phi]} \simeq e^{-S[\phi] + \text{ren.} + \mathcal{O}(1/k)}$$

(4) regulator term ($\sim \phi^2$)

$$\Delta S_k[\phi]$$

may break symmetries; in particular non-linear symmetries like

(a) non-Abelian gauge symmetries
QCD

(b) diffeomorphisms
gravity