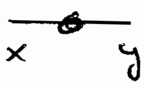
I-2 Truncation Schemes, optimisation & numerics

(i) Perturbation theory

(a) 1-loop: $\partial_t \Gamma_k^{1\text{-loop}} = \frac{1}{2}$  Loop

$\Rightarrow$ 

$= \dfrac{1}{S^{(2)}[\phi] + R_k}(x, y)$

and hence

$$\partial_t \Gamma_k^{1\text{-loop}}[\phi] = \frac{1}{2} \mathrm{Tr} \, \frac{1}{S^{(2)}[\phi] + R_k} \, \partial_t R_k$$

k-dep

$$= \frac{1}{2} \mathrm{Tr} \, \partial_t \ln\left(S^{(2)}[\phi] + R_k\right)$$

($\mathrm{Tr}\, \partial_t \neq \partial_t \mathrm{Tr}$, strictly speaking)

Integration:

$$\Gamma_k^{1\text{-loop}}[\phi] = \Gamma_\Lambda^{1\text{-loop}}[\phi] + \int_\Lambda^k \frac{dk'}{k'} \, \partial_{t'} \Gamma_{k'}^{1\text{-loop}}[\phi]$$

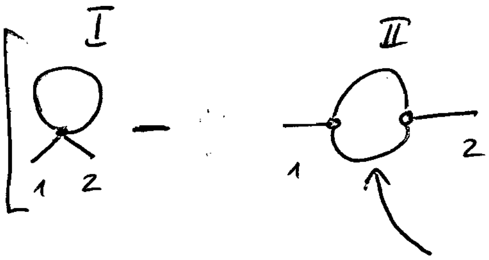
renormalisation

$$\Rightarrow \Gamma_k^{1\text{-loop}}[\phi] = \Gamma_\Lambda^{1\text{-loop}}[\phi] + \frac{1}{2}\mathrm{Tr}\left[\ln\left(S^{(2)}[\phi]+R_k\right) - \ln\left(S^{(2)} + R_\Lambda\right)\right]$$

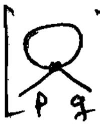
finite

e.g.:

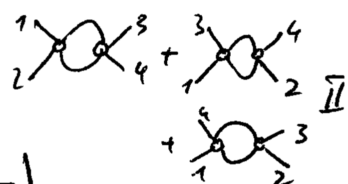
$$\Gamma_k^{1\text{-loop}\,(2)}[\phi] = \Gamma_\Lambda^{1\text{-loop}\,(2)}[\phi]$$

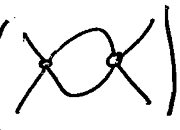
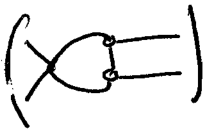
$+ \frac{1}{2}\Big[$  $\Big]_\Lambda^k$

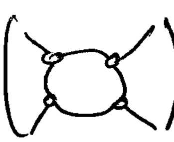
$\frac{1}{S^{(2)}[\phi]+R_k}$

$\Big(\Gamma_k^{1\text{-loop}}[0](p,q) = \frac{1}{2}\Big[$  $\Big]_\Lambda^k \Big) + \Gamma_\Lambda^{1\text{-loop}\,(2)}[0](p,q)$

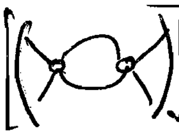
$$\Gamma_k^{1\text{-loop}\,(4)}[\phi] = \frac{\delta^2}{\delta\phi^2}\Gamma_k^{1\text{-loop}\,(2)}[\phi]$$



$= \Big[-\frac{1}{2}\Big($  $\Big) + \frac{1}{2}\Big($  $\Big)$

$- \frac{1}{2}\Big($  $\Big)\Big]_\Lambda^k + \Gamma_\Lambda^{1\text{-loop}\,(4)}[\phi]$

see pages I-14(a,b)

$\Rightarrow \Gamma_k^{1\text{-loop}\,(4)}[0](p_1,\dots,p_4) = -\frac{1}{2}\Big[$  $\Big]_\Lambda^k + \Gamma_\Lambda^{1\text{-loop}\,(4)}[0](p_1,\dots,p_4)$

Renormalisation:

(1) Γ_k indep. of Λ!

$$\Rightarrow \Lambda \partial_\Lambda \Gamma_k = 0$$

$$= \Lambda \partial_\Lambda \Gamma_\Lambda + \frac{1}{2} \mathrm{Tr} \Big[\ln(\Gamma^{(2)}[\phi] + R_k) \Big]_\Lambda^k$$

(2) Λ-dep. of Γ_Λ is fixed by Flow

⇒ Renormalisation is

(α) adjusting Λ-indep. of Γ_k, $k \neq \Lambda$
~ regularisation

(β) fixing Λ-indep parts of Γ_Λ
~ renormalisation conditions

(3) extends trivially to full flow

Example: β-function in ϕ^4-theory in 4-dim I – 19 a

$$S[\phi] = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, \phi(p)\left(p^2+m^2\right)\phi(-p)$$

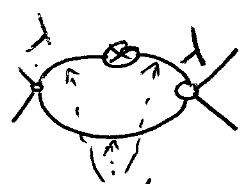
$$+\frac{\lambda}{4!}\int_{p_1,\dots,p_4} \phi(p_1)\cdots\phi(p_4) \cdot (2\pi)^4\, \delta^{(4)}(p_1+\dots+p_4)$$

with $\int_p = \int \frac{d^dp}{(2\pi)^d}$, here $d=4$.

Inserting into $\partial_t \Gamma^{(4)}\big|_{\phi=0} = \partial_t \lambda = \dot{\lambda}$

1- Loop: rhs: $\Gamma^{(2)} = S^{(2)} = \left(p^2+m^2\right)\delta^{(4)}(p+q)$

$$\Gamma^{(4)} = S^{(4)} = \lambda\,(2\pi)^4\,\delta^{(4)}(p_1+\dots+p_4)$$

I-14b $\Rightarrow$ $\dot{\Gamma}^{(4)} = 3\,\cdot$ 

$\frac{1}{p_i^2+m^2}$

$$= 3\lambda^2 \int \frac{d^4p}{(2\pi)^4}\left(\frac{1}{p^2+m^2+R_k}\,\dot{R}_k(p^2)\,\frac{1}{p^2+m^2}\right) \cdot \frac{1}{p^2+m^2+R_k}$$

$$dp\, p = \tfrac{1}{2} dp^2$$

$$\mathring{\Gamma}^{(4)} = \mathring{\lambda} = 3\lambda^2 \cdot \int \frac{d\Omega_4}{(2\pi)^4} \cdot \frac{1}{2} \int_0^\infty dp^2\, p^2 \left(\frac{1}{p^2 + m^2 + R_k(p^2)}\right)^3 \mathring{R}_k(p^2)$$

Introduce $x = p^2/k^2$

$$R_k(p^2) = p^2\, r(x)$$

$$\Rightarrow \mathring{R}_k(p^2) = -2p^2\, x\, r'(x) = -2k^2\, x r'(x)$$

$$\int d\Omega_4 = \Omega_4$$

$$\Rightarrow \mathring{\lambda} = -\tfrac{2}{2}\, 3\lambda^2 \frac{\Omega_4}{(2\pi)^4} \int_0^\infty dx\, x^3 \left(\frac{1}{x(1+r) + \hat{m}^2}\right)^3 r'$$

$$= -3\lambda^2 \frac{\Omega_4}{(2\pi)^4} \int_0^\infty dx \left(\frac{1}{1+r + \hat{m}^2/x}\right)^3 r'$$

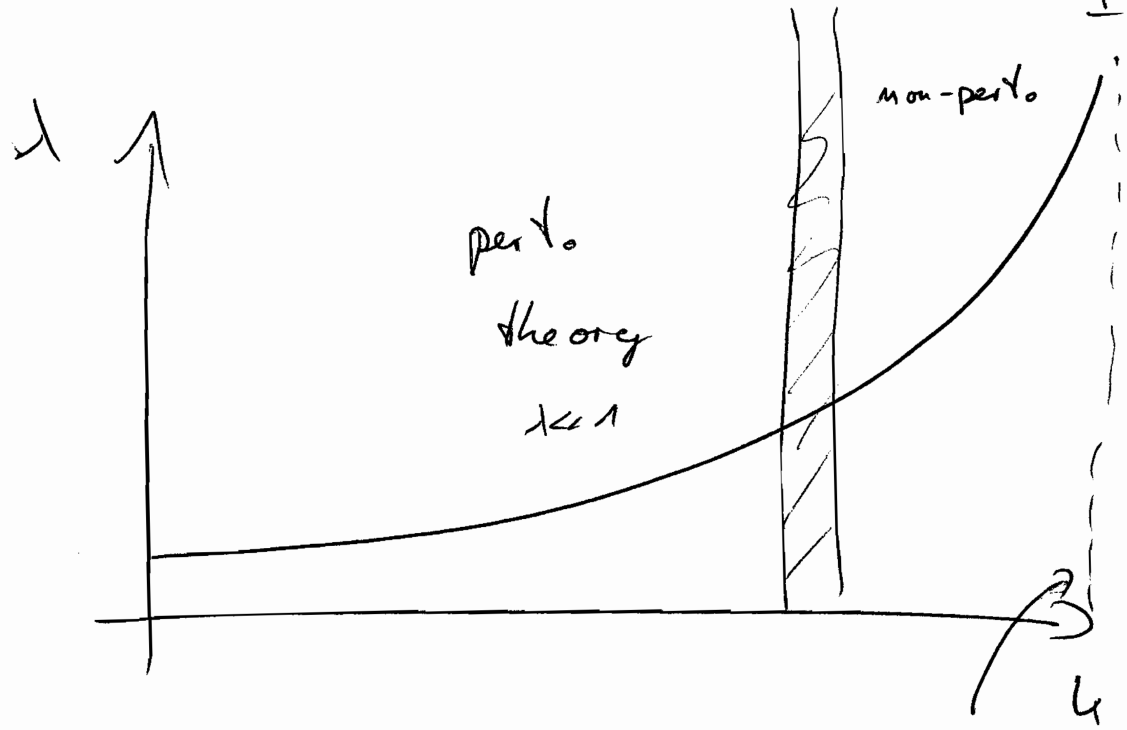
$$= \tfrac{3}{2}\lambda^2 \frac{\Omega_4}{(2\pi)^4} \int_0^\infty dx \left[\frac{\partial}{\partial x} \frac{1}{(1+r+\hat{m}^2/x)^2} - \frac{\hat{m}^2}{x^2} \frac{1}{(1+r+\hat{m}^2/x)^3}\right]$$

$$= \tfrac{3}{2}\lambda^2 \frac{\Omega_4}{(2\pi)^4} + \hat{m}^2\text{-corrections}$$

$$\Omega_4 = 2\pi^2 \qquad\qquad \hat{m}^2 = m_0^2/k^2 + \lambda^2\text{-terms}$$

$$\Rightarrow \boxed{\partial_t \lambda \simeq \frac{3}{16\pi^2}\lambda^2} \rightarrow \lambda(t) = \frac{\lambda_0}{1 + \frac{3}{16\pi^2}(t_0 - t)\lambda_0}$$

Landau pole



Landau-pole

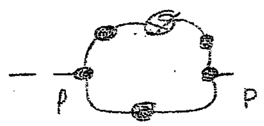
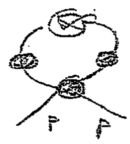
triviality: demand finite λ for $t \to \infty$

- ϕ^4-theory valid at all scales
- no Landau pole

$\Rightarrow$ $\boxed{\lambda_{phys} = \lambda_{k=0} \hat{=} 0}$

[as $t_0 \to \infty$]

2-Punktfkt für ϕ^3-Theorie in 6 dim

$\dot\Gamma^{(2)}(p^2) = -$  $+ \frac{1}{2}$  $e v_0$

$= \Big|_{1\text{-Loop}} \Big(-$  $\Big)$

$$= -g^2 \int \frac{d^6 q}{(2\pi)^6} \left[\frac{1}{q^2+m^2+R(q^2)}\right]^2 \dot R(q^2) \frac{1}{(q+p)^2+m^2+R((q+p)^2)}$$

$$= -g^2 k^2 \int \frac{d^6 \hat q}{(2\pi)^6} \left[\frac{1}{\hat q^2+\hat m^2+R(\hat q^2)}\right]^2 \dot R(\hat q^2) \frac{1}{(\hat q+\hat p)^2+\hat m^2+R((\hat q+\hat p)^2)}$$

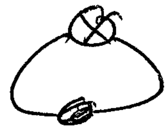
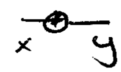
mit $\hat x = x/k^2$

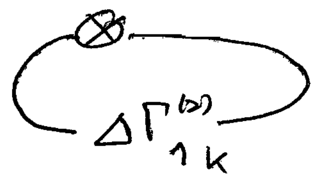
$$\Rightarrow \dot\Gamma^{(2)}(0) = g^2 k^2 f_0(\hat m^2)$$

$$(\partial_{p^2}\dot\Gamma^{(2)})(0) = -g^2 f_1(\hat m^2)$$

mit $f_1(0) = -\frac{1}{(2\pi)^3}\frac{1}{48}$

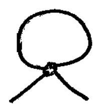
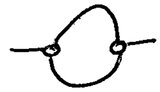
(b) 2-loop:

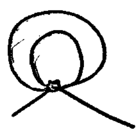
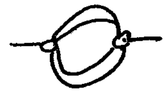
$\partial_t \Gamma_k^{2\text{-loop}} = \frac{1}{2}$  with  $= \frac{1}{\Gamma^{1\text{-loop}(2)}[\phi]+R_k}$

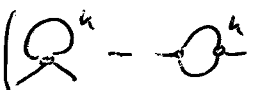
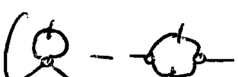
$= \frac{1}{2}$  $- \frac{1}{2}$ 

1-loop

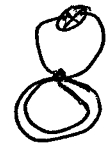
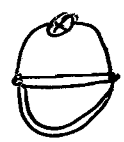
with $\Delta\Gamma^{(2)}_{1k} = \Gamma_k^{1\text{-loop}(2)} - S^{(2)}$

$= \frac{1}{2}\Big[$  $-$  $\Big]^k_\Lambda + \left(\Gamma_\Lambda^{1\text{-loop}(2)} - S^{(2)}\right)$

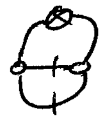
$= \frac{1}{2}\Big[$  $-$  $\Big]$

$= \frac{1}{2}\Big($  $\Big) - \frac{1}{2}\Big($  $\Big)$

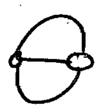
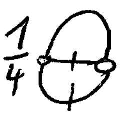
It follows

$\partial_t \Gamma_k^{2\text{-loop}} = \frac{1}{2}$  $- \frac{1}{4}$  $+ \frac{1}{4}$ 

$\Rightarrow \partial_t \Gamma_k^{2\text{-loop}} = \partial_t(1\text{-loop}) - \frac{1}{4}\Big[$  $-$  $\Big]$

$+ \frac{1}{4}\Big[$  $-$  $\Big]$

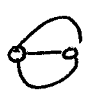
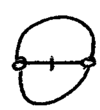
$= \partial_t(1\text{-loop}) + \partial_t \Big\{ \frac{1}{8}$  $- \frac{1}{4}$ 

$- \frac{1}{12}$  $+ \frac{1}{4}$  $\Big\}$

$$\Rightarrow \Gamma_k^{2\text{-loop}} = \Gamma_\Lambda^{2\text{-loop}} + \int_\Lambda^k \frac{dk'}{k'}\, \partial_{t'} \Gamma_{k'}$$

$= S_{cl} + (1\text{-loop})_{ren}$

$+ \frac{1}{8}$  $- \frac{1}{4}$ 

$- \frac{1}{12}$  $+ \frac{1}{4}$  $+ \left(\Gamma_\Lambda^{2\text{-loop}} - S - \Gamma_\Lambda^{1\text{-loop}}\right)$

$\Rightarrow \Delta\Gamma_k^{2\text{-loop}\,(2)} \Rightarrow 3\text{-loop}$

(ii) Effective Potential approximation

(zeroth order derivat. expansion)

Effective Potential: Φ_c constant

examples: I-22a

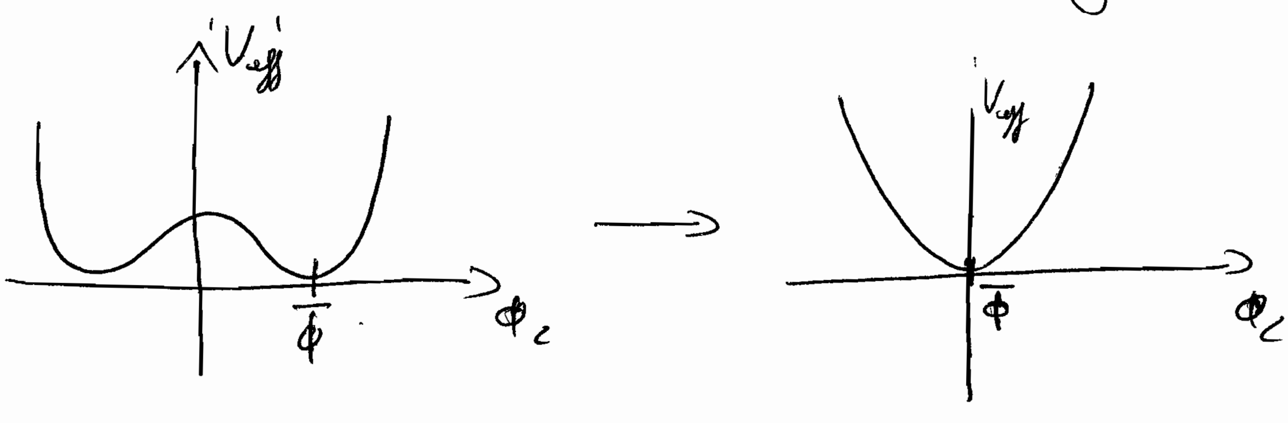
$$\mathrm{Vol}_d \cdot V_k[\Phi_c] := \Gamma_k[\Phi_c] \qquad (V_{\mathrm{eff}\,k})$$

$\uparrow$ $\dim V_k = d$

quantum equ. of classical pot.

$$\left.\frac{\partial V_k}{\partial \Phi_c}\right|_{\bar\Phi} = 0 \quad \text{approximates ground state}$$

e.g. order parameter of symmetry breaking



broken phase   symmetric phase

$$V_{\mathrm{eff}} = V_{k=0}$$

Examples:

(a) classical action

$$S_{cl}[\phi] = \frac{1}{2}\int d^dx\, \partial_\nu\phi\,\partial_\nu\phi + \int d^dx \left\{ \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right\}$$

$$\Rightarrow S_{cl}[\phi_c] = \left\{ \frac{m}{2}\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 \right\} \underbrace{\int d^dx}_{\mathrm{Vol}_d}$$

(b) local potential approximation (LPA)

[0th derivative expansion]

$$\Gamma_k[\phi] = \frac{1}{2}\int d^dx\, \partial_\nu\phi\,\partial_\nu\phi + \int d^dx\, V_k[\phi(x)]$$

$$\Rightarrow \Gamma_k[\phi_c] = \mathrm{Vol}_d\, V_k[\phi_c]$$

full flow for $V_k[\phi_c]$:

rhs requires $\Gamma_k^{(2)}[\phi_c](p,q)$:

$$\Gamma_k^{(2)}[\phi_c](p,q) = \left(Z_k(p^2,\phi_c)\,p^2 + \partial_{\phi_c}^2 V_k[\phi_c]\right)(2\pi)^d \delta^{(d)}(p+q)$$

see p. I-23a

$$\Rightarrow \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \frac{1}{\Gamma_k^{(2)}[\phi_c]+R_k}(p,-p)\,\partial_t R_k(p^2)$$

see p. I-23a

$$= \underbrace{\mathrm{Vol}_d}_{(2\pi)^d \delta^{(d)}(p+q)} \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \frac{1}{Z_k(p^2,\phi_c)\,p^2 + \partial_{\phi_c}^2 V_k[\phi_c] + R_k(p^2)}\,\partial_t R_k(p^2)$$

lhs : $\partial_t \Gamma_k[\phi_c] = \mathrm{Vol}_d \cdot \partial_t V_k[\phi_c]$

$$\Rightarrow \boxed{\partial_t V_k[\phi_c] = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \frac{1}{Z_k(p^2,\phi_c)\,p^2 + \partial_{\phi_c}^2 V_k[\phi_c] + R_k(p^2)}\,\partial_t R_k}$$

full flow, not closed

because of Z_k

$$\Gamma_k^{(2)}[\phi_c](p,q) = \left(Z_k(p^2,\phi_c)\, p^2 + \partial_{\phi_c}^2 V_k[\phi_c]\right)(2\pi)^d\, \delta^{(d)}(x-y)$$

e.g. from $\Gamma_k[\phi] = \frac{1}{2}\int d^dx\; Z_k(-\partial^2, \phi(x))\,\partial_\mu\phi(x)\,\partial_\mu\phi(x)$

$$+ \int d^dx\; V_k[\phi(x)]$$

with

$$\left.\frac{\delta^2 \Gamma_k}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\phi_c} = -Z_k(-\partial_x^2, \phi_c)\;\partial_x^2\;\delta(x-y)$$

$$+ \partial_{\phi_c}^2 V_k[\phi_c]\;\delta^{(d)}(x-y)$$

$$\frac{1}{\Gamma_k^{(2)}[\phi_c] + R_k}(p,q) = \frac{1}{p^2 Z_k(p^2,\phi_c) + V_k[\phi_c] + R_k(p^2)}\,\delta^{(d)}(p+q)$$

$$R_k(p,q) = R_k(p^2)\,\delta^{(d)}(p+q)$$

$$\mathring{R}_k(p,q) = \mathring{R}_k(p^2)\,\delta^{(d)}(p+q)$$

$$(2\pi)^d\,\delta^{(d)}(p=0) = \left.\int d^dx\; e^{ipx}\right|_{p=0} = Vol_d$$

0th order deriv. expansion:

$$Z_k(p^2, \phi_c) = 1 \quad \leftarrow \text{flow closed}$$

- good low energy (momentum) approximatio

$\Leftrightarrow$ requires mass-scales!

regulator choise:

$$R_k = (k^2 - p^2)\,\Theta(k^2 - p^2)$$
$$\dot{R}_k = 2k^2\,\Theta(k^2 - p^2)$$

optimised cut-off

! for 0th order der. expans.!

$$\Rightarrow \partial_t V_k[\phi_c] = \underbrace{\frac{\Omega_d}{(2\pi)^d}}_{\int d\Omega_d} \cdot \int_0^k dp\, p^{d-1} \frac{k^2}{k^2 + \partial_{\phi_c}^2 V_k}$$

$$= \frac{1}{d} \cdot \frac{\Omega_d}{(2\pi)^d} \; \frac{k^{d+2}}{k^2 + \partial_{\phi_c}^2 V_k[\phi_c]}$$

with

$$\Omega_d = 2\pi^{d/2} / \Gamma[d/2]$$

Example: flow of λ in $d=4$:

$$V_k = \frac{1}{2} m_k^2 \phi_c^2 + \frac{\lambda_k}{4!} \phi_c^4 + \frac{\lambda_{6k}}{6!} \phi_c^6 + \dots$$

$$\Rightarrow \partial_{\phi_c}^2 V_k = m_k^2 + \lambda_k/2\, \phi_c^2 + \frac{\lambda_{6k}}{4!} \phi_c^4 + \dots$$

$$\partial_{\phi_c}^4 \dot{V}_k \Big|_{\phi_c=0} = \dot{\lambda}_k = \partial_{\phi_c}^4 \Big|_{\phi_c=0} \frac{1}{2} \frac{1}{16\pi^2} \frac{k^6}{\left(k^2 + m_k^2 + \frac{\lambda_k}{2}\phi_c^2 + \frac{\lambda_{6k}}{4!}\phi_c^4 + \dots\right)}$$

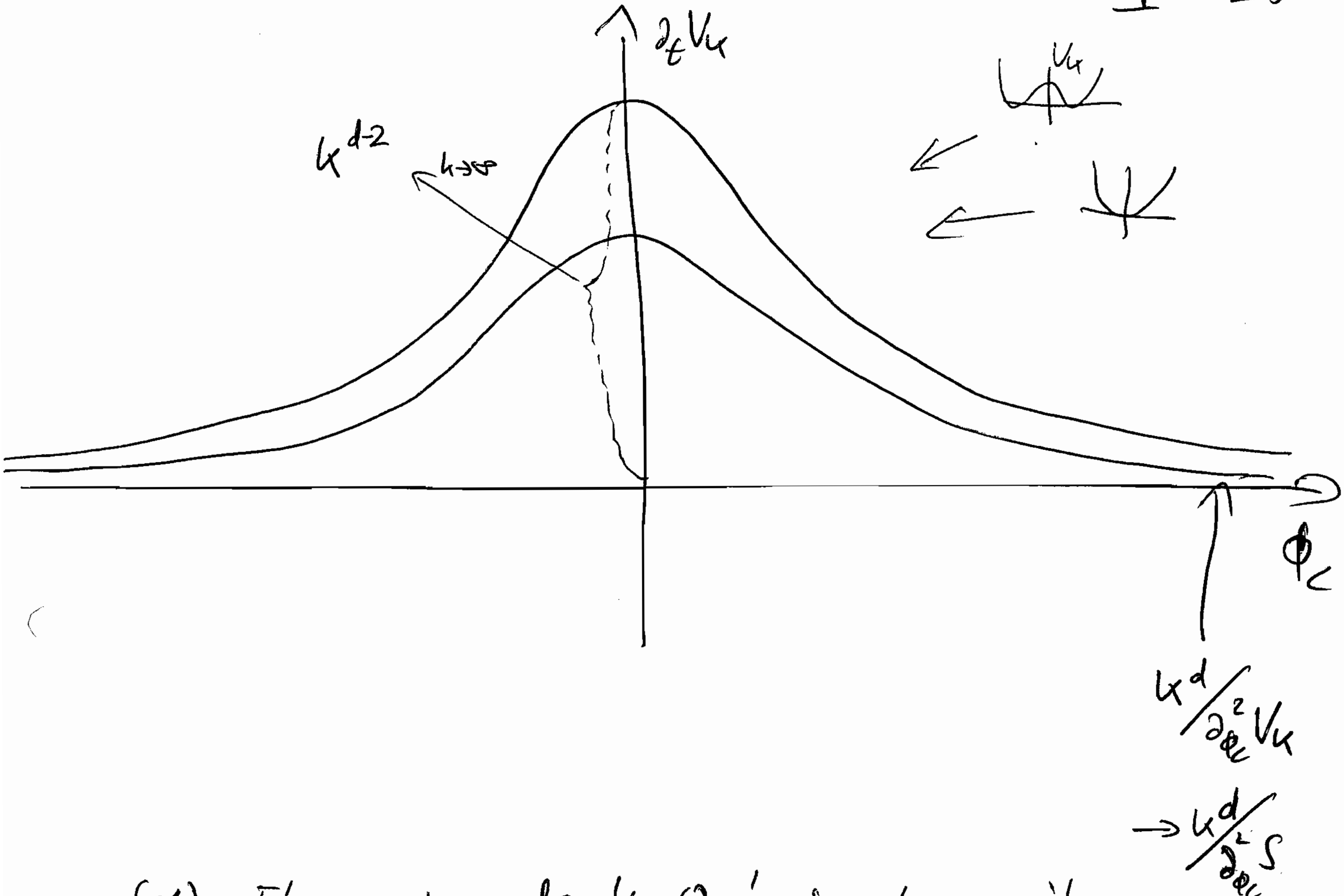
$$= \frac{6}{2} \frac{1}{16\pi^2} \lambda_k^2 \frac{1}{(1+\underbrace{\hat{m}_k^2}_{= m_k^2/k^2})^3}$$

$$- \frac{1}{2} \frac{1}{16\pi^2} \lambda_{6k} k^2 \frac{1}{(1+\hat{m}_k^2)^2}$$

$\hat{m}_k^2 \sim 0$, $k^2 \lambda_{6k} \sim 0$:

$$\boxed{\dot{\lambda}_k = 3 \frac{1}{16\pi^2} \lambda_k^2}$$

see page I-19b
pert. theory



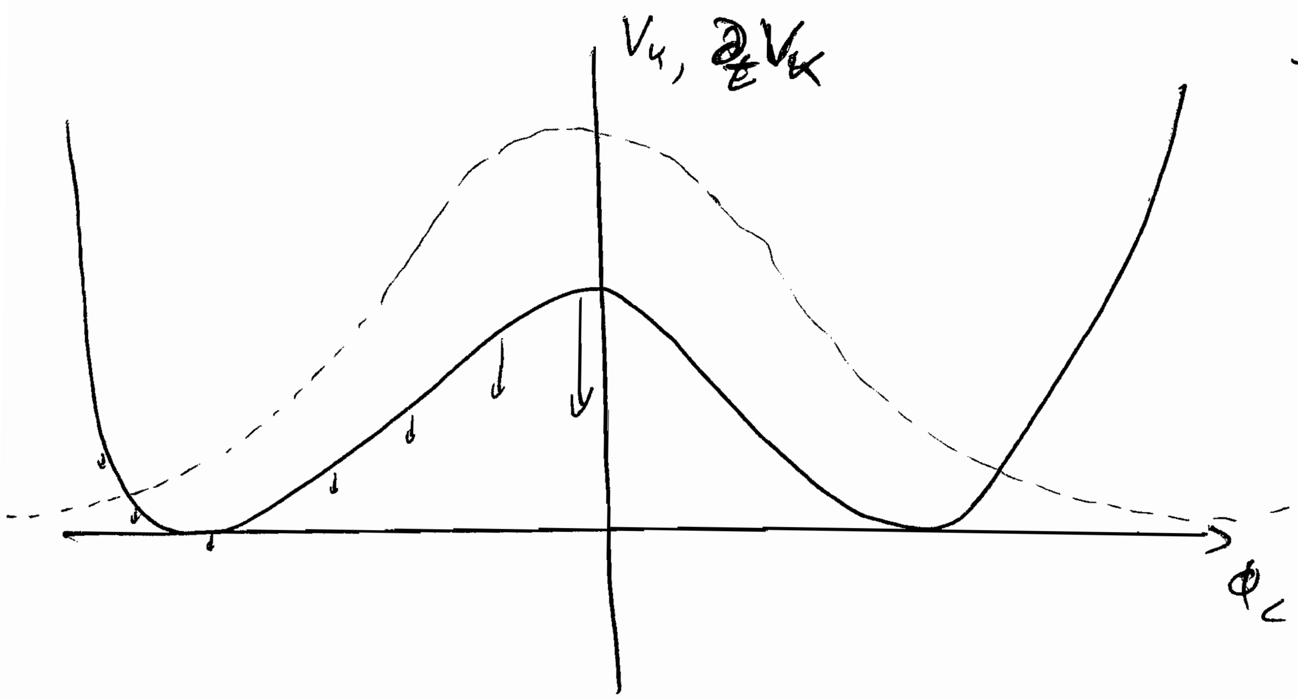
(α) Flow towards $k=0$ 'enforces' convexity:

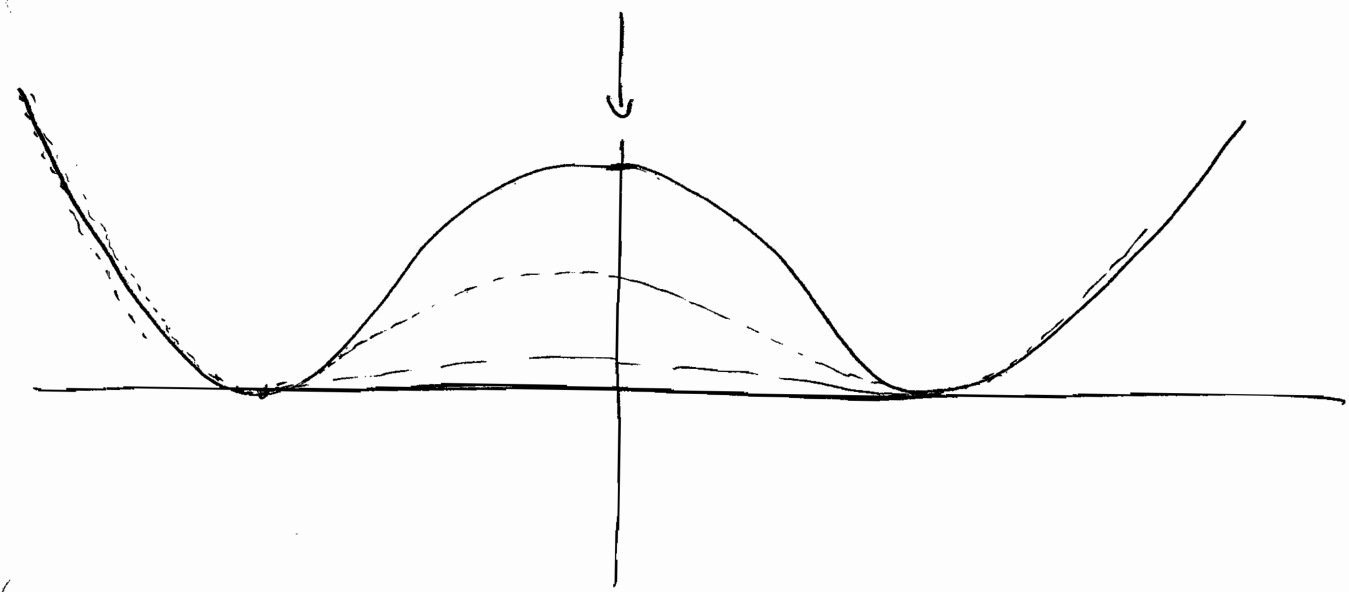
(i) $$\partial_{\phi_c}^2 V[\phi_{c_1}] < \partial_{\phi_c}^2 V[\phi_{c_2}]$$

$$\Rightarrow \quad \frac{1}{k^2 + \partial_{\phi_c}^2 V[\phi_{c_1}]} > \frac{1}{k^2 + \partial_{\phi_c}^2 V[\phi_{c_2}]}$$

(ii) $$\partial_{\phi_c}^2 V[\phi_{c\,\mathrm{sing}}] + k^2 \to 0$$

$$\Rightarrow \quad \frac{1}{k^2 + \partial_{\phi_c}^2 V[\phi_{c\,\mathrm{sing}}]} \to \infty$$





(β) Flow towards $k \to \infty$ leads to non-convexity:

$$\lim_{k\to\infty} m_k^2 < 0$$

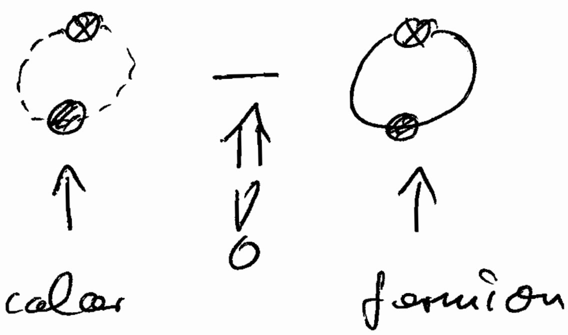
(i) $\Gamma_k + \Delta S_k$ has to be convex ("Legendre trafo")

$$\boxed{(\Gamma_k^{(2)} + R_k)(p, q) \geq 0}$$

(ii) $\det (\Gamma_k^{(2)} + R_k)$ gets singular:

$$\boxed{\text{Theory has no UV-completion}}$$

(iii) a glimpse at fermions: ψ

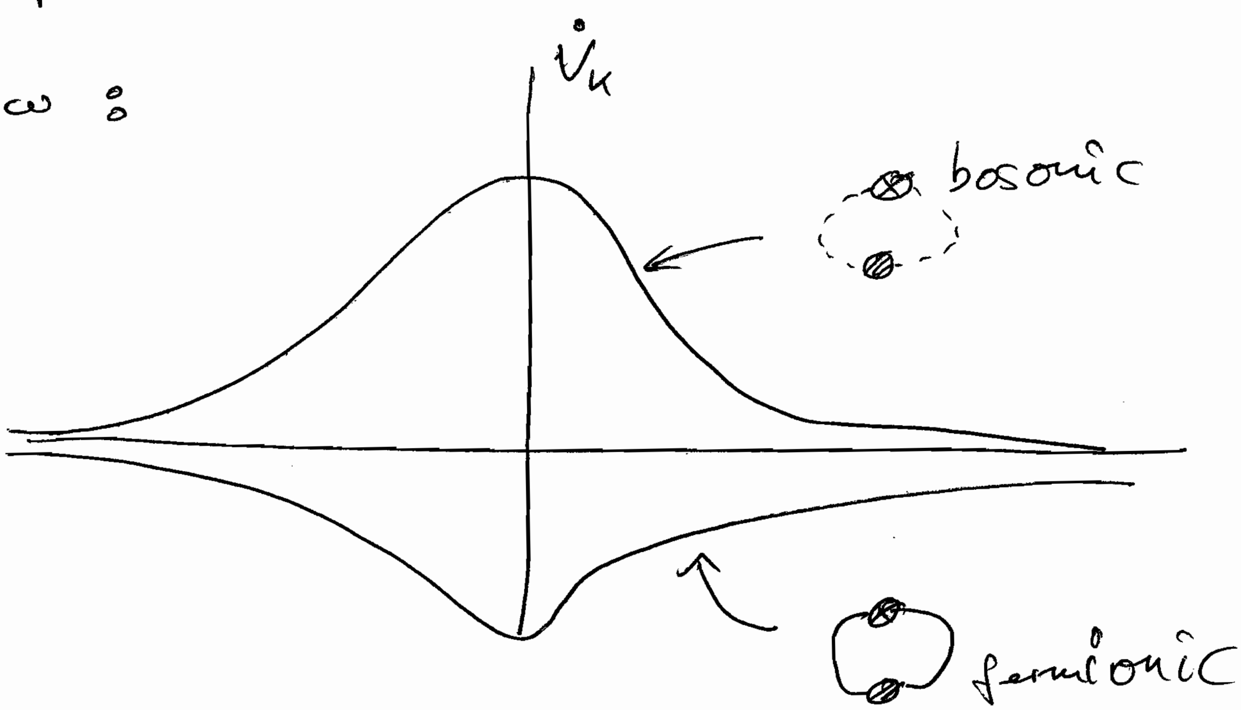
$\partial_t \Gamma_k[\phi,\psi] = \frac{1}{2}$ 

$\Rightarrow$ adds to flow of effective potential

$$\Gamma_k \sim \bar{\Psi}\, F[\phi]\, \Psi \Rightarrow \frac{\delta}{\delta\psi}\frac{\delta}{\delta\bar{\psi}}\Gamma_k\Big|_{\psi=\bar{\psi}=0} = F[\phi]$$

see p. I-27a

Flow:



- scalar (bosonic) flow is symmetry-restoring
- fermionic flow is symmetry-breaking

$$\Gamma_\psi[\phi, \psi] = \int \frac{d^d p}{(2\pi)^d} \; \bar{\psi}\,(\not{p} + m)\,\psi$$

$$+ \int d^d x \; \bar{\psi}\, F(\phi)\, \psi$$

+ bosonic - terms