II QCD

II - 1 Introduction

1.1 Euclidean Path integral for gauge theories

Classical action of non-Abelian gauge theory:

gauge fields $A_\mu \in su(N_c)$

$$A_\mu = A_\mu^a \, t^a \qquad , t^a \text{ generators of } SU(N_c)$$

$$a = 1, \ldots, N_c^2 - 1$$

action

$$S_{YM}[A] = \frac{1}{2} \int d^d x \; \underbrace{\mathrm{tr}_f}_{\boxed{\text{fundamental}}} F_{\mu\nu} F_{\nu\mu}$$

$$= \frac{1}{4} \int d^d x \; F_{\mu\nu}^a \, F_{\mu\nu}^a$$

with field strength $F_{\mu\nu} = F_{\mu\nu}^a \, t^a$,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

and Lie algebra, traces

$$[t^a, t^b] = f^{abc} t^c$$

$$\mathrm{tr}_f \, t^a t^b = -\frac{1}{2} \delta^{ab}$$

Colour-electric/magnetic fields

$$E_i^a = F_{0i}^a \quad , \quad B_i^a = \tfrac{1}{2}\,\varepsilon_{ijk}\,F_{jk}^a$$

Gauge symmetry:

covariant derivative (adjoint representation)

$$D_\nu^{ab}(A) = \partial_\nu \delta^{ab} + g\, f^{acb} \underbrace{A_\nu^c}_{\text{connection}}$$

with $[D_\nu, D_r] = g\, F_{\nu r}$

gauge transformation $U \in SU(N_c)$

$$A_\nu \to A_\nu^U = U A_\nu U^\dagger + \frac{1}{g} U \partial_\nu U^\dagger$$

with

$$D_\nu(A^U) = U\, D_\nu(A) U^\dagger$$

$$F_{\nu r}(A^U) = \frac{1}{g}\left[D_\nu(A^U), D_r(A^U)\right]$$

$$= U \frac{1}{g}\left[D_\nu(A), D_r(A)\right] U^\dagger$$

$$= U \cdot F_{\nu r}(A)\, U^\dagger$$

$$\Rightarrow S_{YM}[A^U] = \frac{1}{2}\int d^dx \operatorname{tr} U\, F_{\nu r}(A) F_{\nu r}(A)\, U^\dagger \underset{\substack{\uparrow \\ \text{cyclicity of trace}}}{=} S_{YM}[A]$$

Infinitesimal transformation: $\mathcal{U} = e^{g\omega}$

$(\omega^{+} = -\omega)$

$$A^{\mathcal{U}} = A - [\mathcal{D}, \omega] + \mathcal{O}(\omega^2)$$

$$F(A^{\mathcal{U}}) = F - [F, \omega] + \mathcal{O}(\omega^2)$$

gauge transformations are generated by δ_ω:

$(\delta \simeq g, \mathcal{G})$

$$\delta_\omega A = -[\mathcal{D}, \omega]$$

with representation $\delta_\omega = +\int d^dx \; \omega^b \delta^b$

$$\delta^a = \mathcal{D}^{ab}_\mu \frac{\delta}{\delta A^b_\mu}$$

Example II-33a

We have

$$\delta^a_{(x)} A^b_\nu(y) = \mathcal{D}^{ab}_{\nu, x} \, \delta(x - y)$$

Gauge symmetry:

$$\delta^a S_{YM}[A] = 0$$

$$\left(= \frac{\delta}{\delta\omega^a} \delta_\omega S_{YM}\right)$$

Example:

$$\delta_\omega A_\nu(x) = \int d^d y \; \omega^b(y)\, \delta^b \, A_\nu^a(x) t^a$$

$$= \int d^d y \; \omega^b(y)\, \mathcal{D}_\nu^b \frac{\delta}{\delta A_\nu^c}(y)\, A_\nu^a(x)\; t^a$$

$$= \int d^d y \; \omega^b(y)\, \mathcal{D}_\nu^{ba}(y)\, \delta^{(d)}(y-x)\; t^a$$

$$= - \int d^d y \left(\mathcal{D}_\nu^{ab}\, \omega^b\right)(y)\; \delta^{(d)}(x-y)\; t^a$$

$f^{bca} = -f^{acb}$

$$= - \mathcal{D}_\nu^{ab}\, \omega^b(x) \quad t^a$$

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$$= - \partial_\nu \omega^b t^b - g\, f^{acb}\, A_\nu^c\, \omega^b\; t^a$$

$$= -\partial_\nu \omega - g\, f^{cba}\, A_\nu^c\, \omega^b\; t^a$$

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$$= - \partial_\nu \omega - g\, [A_\nu, \omega]$$

$$= - [\mathcal{D}_\nu, \omega]$$

Generating functional

$$Z[J] = \int dA \; e^{-S_{YM}[A] + \int d^dx \, J^a_\nu A^a_\nu}$$

but : A_{gf} : $\mathcal{F}[A] = 0$

with gauge fixing condition $\mathcal{F}$

Gribov copies $\left\{ \begin{array}{l} \text{covariant gauge: } \mathcal{F}[A] = \partial_\nu A_\nu \\ \text{Coulomb gauge: } \mathcal{F}[A] = \partial_i A_i \end{array} \right.$

'no Gribov copies' $\left\{ \begin{array}{l} \text{axial gauge: } \mathcal{F}[A] = n_\nu A_\nu \\ \text{(algebraic gauge)} \\ \text{Polyakov gauge: } \partial_0 A_0 = 0 \end{array} \right.$

$A_0 = A_0^c \big|_{SU(2)}$: $\mathrm{tr}\,\sigma^2 A_0 = \mathrm{tr}\,\sigma^1 A_0 = 0$

PI measure : $A = A_{gf}^{\,\mathcal{U}}$: $S_{YM}[A] = S_{YM}[A_{gf}]$

$$dA = dA_{gf}\, d\mathcal{U} \cdot J \quad \text{with} \quad \int d\mathcal{U} = \infty$$

Haar measure

$\boxed{\text{reparameterisation: Jacobian } J[A_{gf}] + \text{const}}$

Oberservables: $\mathcal{O}[A]$, e.g. $F(x)\,F(0)$

$$\langle \mathcal{O} \rangle := \frac{\int dA\, \mathcal{O}[A]\, e^{-S_{YM}}}{\int dA\, e^{-S_{YM}}} = \frac{\int dA_{gf}\, J\mathcal{O} e^{-S_{YM}}\, \cancel{d\mathcal{U}}}{\int dA_{gf}\, J e^{-S_{YM}}\, \cancel{d\mathcal{U}}}$$

$$= \frac{\int dA_{gf}\, J[A_{gf}]\ \mathcal{O}[A_{gf}]\, e^{-S_{YM}[A_{gf}]}}{\int dA_{gf}\ J[A_{gf}]\, e^{-S_{YM}[A_{gf}]}}$$

Faddeev-Popov quantisation: (comp. of J)

Insertion of 1 in path integral

$$1 = \int d\mathcal{U}\ \delta[\mathcal{F}[A^{\mathcal{U}}]]\ \Delta_{\mathcal{F}}[A]$$

with

$$\Delta_{\mathcal{F}}[A] = \left(\int d\mathcal{U}\ \delta[\mathcal{F}[A^{u}]] \right)^{-1}$$

! gauge invariant !

Expand $\delta[\mathcal{F}[A^\mu]]$ about A_{gf} : $A = A_{gf}^{e^{g\omega}}$

$$\left(\delta(f(x)) = \sum_i \frac{1}{|f'(x)|} \delta(x - x_{0i}) \right)$$

$$\delta[\mathcal{F}[A^\mu]] = \sum_i \frac{1}{\left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right|} \delta[\omega - \omega_{0i}]$$

with Fadeev-Popov-determinant

$$\det \frac{\delta \mathcal{F}}{\delta \omega}$$

Example: $\mathcal{F}[A] = \partial_\nu A_\nu$

$$\Rightarrow \left. \frac{\delta \mathcal{F}}{\delta \omega} \right|_{\omega = 0} = - \frac{\delta \, \partial_\nu D_\nu \omega}{\delta \omega}$$

Fadeev-Popov op.

$$= - \partial_\nu D_\nu \mathbb{1}$$

It follows (assume one ω_0)

$$\Delta_{\mathcal{F}}[A] = \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right|$$

orent gauge $\quad = \det(-\partial_\nu D_\nu)[A] \leftarrow (-\partial_\nu \partial_\nu \text{ positive})$

Faddeev - Popov quantisation

gauge theory

$$\int dA \quad e^{-S[A]}$$

$$U = e^{g\omega} \downarrow \quad A^g = U A_\mu U^+ + \frac{i}{g} U \partial_\mu U^+$$

$$1 = \int dg \quad \delta[\mathcal{F}[A^g]] \cdot \Delta_{\mathcal{F}}[A]$$

with

$$\Delta_{\mathcal{F}}[A] = \left(\int dg \ \delta[\mathcal{F}[A^g]]\right)^{-1}$$

$$= \#\sum_{\omega_0} \left| \det \frac{\delta \mathcal{F}}{\delta \omega} \right|_{A = A^{U(\omega_0)}}$$

$$\Rightarrow \int dA \quad e^{-S[A]}$$

$$= \int dA \, dg \quad \delta[\mathcal{F}[A^g]] \Delta_{\mathcal{F}}[A] e^{-S[A]}$$

$$= \int dA \ \delta[\mathcal{F}[A]] \ \Delta_{\mathcal{F}}[A] \ e^{-S[A]} \cdot \left[\int dg\right]$$

$$= \int dA_{gf} \ \Delta_{\mathcal{F}}[A_{gf}] e^{-S[A_{gf}]} \cdot \left[\int dg\right]$$

0-dim

$$\int d^2x \quad e^{-S(r)}$$

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \downarrow \quad x_1^\theta = \underbrace{x_1 \cos\theta + x_2 \sin\theta}_{= \mathcal{F}}$$

$$1 = \int_0^{2\pi} d\theta \quad \delta(x_1^\theta) \cdot \Delta_{\mathcal{F}}(\vec{x})$$

with

$$\Delta_{\mathcal{F}}(\vec{x}) = \left(\int_0^{2\pi} d\theta \ \delta(x_1^\theta)\right)^{-1}$$

$$= \Bigg(\int_0^{2\pi} d\theta \frac{1}{|-x_1 \sin\theta + x_2 \cos\theta|} \cdot \left(\delta\left(\theta - \arctan\frac{x_2}{x_1}\right) + \delta\left(\theta - \arctan\frac{x_2}{x_1} + \pi\right)\right)\Bigg)^{-1}$$

$$= r/2 \ (\vec{x})$$

$$\Rightarrow \int d^2x \quad e^{-S(r)}$$

$$= \int d^2x \int_0^{2\pi} d\theta \quad \delta(x_1^\theta) \ r/2 \quad e^{-S(r)}$$

$$= \int d^2x \ \delta(x_1) \quad r/2 \quad e^{-S(r)} \cdot [2\pi]$$

$$= \int_0^\infty dx_2 \ x_2 \quad e^{-S(\sqrt{x_2^2})} \cdot [2\pi]$$

with $x_2 = r\left(\vec{x} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}\right)$

Gribov copies:

$$x_2^{\theta} = x_2 \cos\theta - x_1 \sin\theta \Big|_{\substack{\theta_+ = \arctan x_2/x_1 \\ \theta_- = \arctan x_2/x_1 + \pi}} : \quad x_2^{\theta_+} = -x_2^{\theta_-}$$

and $r = \pm x_2^{\theta}$ ($x_2^{\theta_+}$ can be posit./negative)

Remove absolute value:

$$|x_2^{\theta}| \to x_2^{\theta}$$

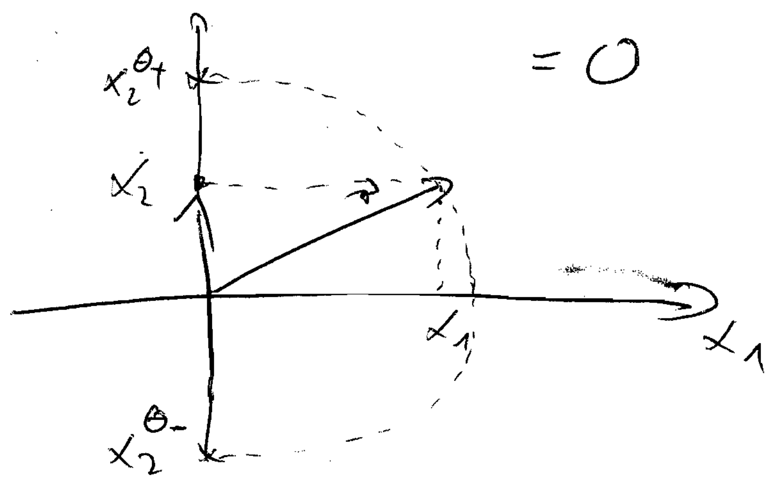
a.) $\int d\theta \, \frac{1}{x_2^{\theta}} \left(\delta(\theta - \arctan x_2/x_1) + \delta(\theta - \arctan x_2/x_1 + \pi) \right)$

$= 0$

b.) $\int d^2x \; x_2^{\theta}/2 \; \delta(x_1^{\theta}) \, e^{-S(r)}$

$= \int d^2x \; x_2/2 \; \delta(x_1) \, e^{-S(r)} = \int dx_2 \; x_2/2 \; e^{-S(\sqrt{x_2^2})}$

$= 0$



ghosts: assume exactly $\underline{1}$ copy

$\boxed{\frac{\delta \mathcal{F}}{\delta \omega} > 0}$

$$\det \frac{\delta \mathcal{F}}{\delta \omega} = \int d\bar{c}\, dc\, e^{-\int d^d x\, \bar{c} \frac{\delta \mathcal{F}}{\delta \omega} c}$$

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$$= \int d\bar{c}\, dc\, e^{-S_{gh}[A, c, \bar{c}]}$$

with

$$S_{gh}[A, c, \bar{c}] = \int d^d x\, \bar{c}\, \frac{\delta \mathcal{F}}{\delta \omega} c$$

cov. gauge $\qquad = -\int d^d x\, \bar{c}^a \partial_\nu D_\nu^{ab} c^b$

Generating functional: $\mathcal{F}^a \to \mathcal{F}^a - \mathcal{C}^a$

$\Delta_{\mathcal{F}} \to \Delta_{\mathcal{F}}$

$$Z[J, \eta, \bar{\eta}] = \frac{1}{N} \int dA \int d\mathcal{C}\, e^{-\frac{1}{2\xi} \int d^d x\, \mathcal{C}^a(x)^2}$$

$$\int d\bar{c}\, dc\, e^{-S_{gh}[A, c, \bar{c}]}$$

$$\cdot \int d\mathcal{U}\, \delta[\mathcal{F} - \mathcal{C}]\, e^{-S_{YM}[A]}$$

$$\cdot\, e^{\int d^d x \{J_\nu^a A_\nu^a + \bar{\eta}^a c^a - \bar{c}^a \eta^a\}}$$

Remark II-37b Neuberger problem

Fermions - reminder:

Grassmann variables: (1) $c^2 = \bar{c}^2 = 0$

(2) $c\bar{c} = -\bar{c}c$

$$\int dc\, c^n = \delta_{n1}$$

$$\Rightarrow \int dc\, f(c) = \left.\frac{\partial f}{\partial c}\right|_{c=0}$$

$$\Rightarrow \int \prod_n d\bar{c}_n\, dc_n\; e^{-\vec{\bar{c}}\, \mathcal{M}\, \vec{c}}$$

$$= \int \prod_{n=1}^{N} d\bar{c}_n\, dc_n\; e^{-\bar{c}_i \mathcal{M}_{ij} c_j}$$

$$= \int \prod_{n=1}^{N} d\bar{c}_n\, dc_n\; \left(-\bar{c}_i \mathcal{M}_{ij} c_j\right)^N \frac{1}{N!}$$

$$= \sum_\sigma (-1)^\sigma \prod_i \mathcal{M}_{i\sigma(i)} \qquad = \det \mathcal{M}$$

Neuberger problem:

Fadeev-Popov (BRST) gauge-fixed lattice gen. functional vanishes identically:

$$\int dA_{gf} \det \frac{\delta \mathcal{F}}{\delta \omega} \; e^{-S[A_{gf}]} \stackrel{\wedge}{=} 0$$

Resolutions:

(1) take absolute value:
difficult as gauge fixing is done as extremisation

(2) gauge fixing as topo field theory (Morse theory ...)

(3) do not gauge-fix

$$\boxed{Z[J,\eta,\bar\eta] \simeq \int dA\, d\bar c\, dc\; e^{-S[A,c,\bar c]} \cdot e^{\int d^dx \{ J^a_\nu A^a_\nu + \bar\eta^a c^a - \bar c^a \eta^a \}}}$$

with action

$$S[c,\bar c,A] = S_{YM}[A] + S_{gh}[A,c,\bar c] + S_{gf}[A]$$

where $S_{gf}[A] = \frac{1}{2\xi}\int d^dx\, \mathcal{F}^a[A]\, \mathcal{F}^a[A]$

cov. gauge $= \frac{1}{2\xi}\int d^dx\, (\partial_\nu A^a_\nu)^2$

$= -\frac{1}{2\xi}\int d^dx\, A^a_\nu\, \partial_\nu \partial_\nu A^a_\nu$

Questions:

(i) manifestation of gauge symmetry?

(ii) gauge copies: Gribov problem?

(i) gauge symmetry

$Z[J, \eta, \bar{\eta}]$ is not gauge covariant

but $\langle \mathcal{O}(A^{u}) \rangle = \langle \mathcal{O}(A) \rangle$

manifests itself in

$$\int dA\, d\bar{c}\, dc\; \mathcal{O}(A)\, e^{-S[A, c, \bar{c}]}$$

$$= \int dA^{u}\, d\bar{c}^{u}\, dc^{u}\; \mathcal{O}(A^{u})\, e^{-S[A^{u}, c^{u}, \bar{c}^{u}]}$$

$$= \int dA\, d\bar{c}\, dc\; \mathcal{O}(A)\, e^{-S[A^{u}, c^{u}, \bar{c}^{u}]}$$

with

$$c^{u} = u c u^{+} = c - g[c, \omega] + \mathcal{O}(\omega^2)$$

$$\bar{c}^{u} = u \bar{c} u^{+} = \bar{c} - g[\bar{c}, \omega] + \mathcal{O}(\omega^2)$$

full δ : (see p. II-33) $\quad \delta_\omega = \int d^dx\, \omega^a(x)\, \delta^a(x)$

$$\delta^a = D_\mu^{ab} \frac{\delta}{\delta A_\mu^b} - g f^{abc} \left(c_c \frac{\delta}{\delta c_b} + \bar{c}_c \frac{\delta}{\delta \bar{c}_b} \right)$$

Example II-39a

$$\int d^d y \; \omega^a_{(y)} \delta^a_{(y)} \, c(x) = \int d^d y \left\{ -g \, \omega^a_{(y)} f^{abc} c^e_{(y)} \frac{\delta}{\delta c^b_{(y)}} c^d_{(x)} t^d \right\}$$

$$= -g \int d^d \; \omega^a_{(y)} f^{abc} c^c_{(y)} \, \delta^{(d)}(x-y) \, t^b$$

$$= -g \; \omega^a(x) \, f^{abc} \; c^c(x) \; t^b$$

$$= -g \; f^{cab} \, c^c(x) \, \omega^a(x) \; t^b$$

$$= -g \, [c , \omega]$$

$$\Rightarrow \int d^d y \; \omega^a(y) \, \delta^a(y) \, \bar{c}(x) = -g \, [\bar{c} , \omega]$$

with II-33a:

$$\delta (A, c, \bar{c}) = (-[D, \omega] , -g [c, \omega] , -g [\bar{c}, \omega])$$

or $\delta (D , c , \bar{c}) = -g \left[(D , c , \bar{c}) , \omega \right]$

Slavnov-Taylor identities: (STI's)

$$Z[J,\eta,\bar{\eta}] =$$

$$= \int dA\, d\bar{c}\, dc\; e^{-S[A,c,\bar{c}] + \int d^dx \{ J^a_\nu A^a_\nu + \bar{\eta}^a c^a - \bar{c}^a \eta^a \}}$$

$$= \int dA\, d\bar{c}\, dc\; e^{-S[A^u, c^u, \bar{c}^u] + \int d^dx \{ J^a_\nu A^{ua}_\nu + \bar{\eta}^a c^{ua} - \bar{c}^{ua} \eta^a \}}$$

Infinitesimal: $\frac{\delta}{\delta \omega^a(y)} Z[J,\eta,\bar{\eta}] = 0$

$$\int dA\, d\bar{c}\, dc\; \delta^a(y) \left\{ e^{-S[A,c,\bar{c}] + \int d^dx \{ J^b_\nu A^b + \bar{\eta}^b c^b - \bar{c}^b \eta^b \}} \right\} = 0$$

$$\Rightarrow \boxed{\left\{ D^{ab}_\nu(\langle A \rangle) J^b_\nu - g f^{abc} \left(\bar{\eta}^b \langle c^c \rangle - \langle \bar{c}^c \rangle \eta^b \right) \right\}(y) = \langle \delta^a(y) \left(S_{gf}[A] + S_{gh}[A,c,\bar{c}] \right) \rangle}$$

covariant gauge:

$$\delta^a(y) S_{gf} = -\frac{1}{\xi} D^{ab}_\nu \partial_\nu \partial_\nu A^b_\nu(y)$$

$$\delta^a(y) S_{gh} = -f^{abc} \partial_\nu (\bar{c}^b D^{cd}_\nu c^d)$$

$$\mathcal{D}^{ab}_\nu \frac{\delta}{\delta A^b_\nu} \qquad \text{II-33a}$$

$$\shortparallel$$

$$\delta^a(y)\, S_{gf} = -\delta^a(y)\, \frac{1}{2\xi} \int d^dx\, A^b_\nu \partial_\nu \partial_\nu A^b_\nu$$

$$= -\frac{1}{2\xi} 2 \int d^dx \;\; \mathcal{D}^{ab}_{\nu(y)} \delta^{(d)}(y-x)\, \partial_\nu \partial_\nu A^b_{\nu}(x)$$

$$= -\frac{1}{\xi} \int d^dx \;\; \delta^{(d)}(y-x)\, \mathcal{D}^{ab}_{\nu(x)} \partial_\nu \partial_\nu A^b_\nu(x)$$

$$= \boxed{-\frac{1}{\xi} \mathcal{D}^{ab}_\nu(x)\, \partial_\nu \partial_\nu A^b_\nu(x)}$$

$$\delta^a(y)\, S_{gh} = -\delta^a(y) \int d^dx\, \bar{c}^b \partial_\nu \mathcal{D}^{bd}_\nu c^d$$

$$= -\frac{\delta}{\delta\omega^a(y)} \delta \int d^dx\, \bar{c}^b\, \partial_\nu \mathcal{D}^{bd}_\nu c^d$$

$\mathrm{tr}\, t^a t^b = -\frac{1}{2}\delta^{ab} \to$
$$= 2 \frac{\delta}{\delta\omega^a(y)} \delta_\omega \int d^dx\, \mathrm{tr}(\bar{c}\, \partial_\nu \mathcal{D}_\nu c)$$

$\delta(\mathcal{D}, c, \bar{c}) = -g[(\mathcal{D}, c, \bar{c}), \omega] \to$
$$= -2 \frac{\delta}{\delta\omega^a(y)} \int d^dx\, \mathrm{tr}\, \bar{c}\, [\omega, \partial_\nu]\, \mathcal{D}_\nu c$$

$$= 2 \frac{\delta}{\delta\omega^a(y)} \int d^dx\, \mathrm{tr}\, \bar{c}\, \partial_\nu \omega\, \mathcal{D}_\nu c$$

$$= 2 \int d^dx\, \mathrm{tr}\, \bar{c} t^a \partial_\nu \delta^{(d)}(x-y)\, \mathcal{D}_\nu c$$

$$\Rightarrow \delta^a S_{gh} = -2\, \partial_\nu \operatorname{tr} \bar{c}\, t^a D_\nu c$$

$$= -2\, \partial_\nu \left[\bar{c}^b (D_\nu c)^c \right] \operatorname{tr}\, t^b t^a t^c$$

$$= -\tfrac{2}{2}\, \partial_\nu \left[\bar{c}^b (D_\nu c)^c \right] \operatorname{tr} \left[t^c, t^b \right] t^a$$

$$= \partial_\nu \left[\bar{c}^b D_\nu c)^c \right] f^{cbd} \operatorname{tr} t^d t^a$$

$$= \boxed{-\frac{1}{2}\, f^{abc}\, \partial_\nu \left[\bar{c}^b D_\nu^{cd} c^d \right]}$$

General DSE's: $\phi = (A, c, \bar{c})$

$$\int d\phi \, \frac{\delta}{\delta\phi^i} \mathcal{G}^i[\phi] \, \Psi[\phi] \, e^{-S[\phi] + \int d^d x \, J_i \phi_i}$$

$$= 0$$

with $J_i = (\underset{J_1}{J^a_\nu}, \underset{J_2}{\bar{\eta}^a}, \underset{J_3}{\eta^a})$

standard DSE: $\mathcal{G}^i = \delta^{ij}$ $\quad j = 1, 2, 3$

$\Psi = 1$ translation inv. in field space

$\Rightarrow \boxed{J_j = \langle \frac{\delta S}{\delta j} \rangle}$: $J_A = \langle \frac{\delta S}{\delta A} \rangle$, $\eta = -\langle \frac{\delta S}{\delta \bar{c}} \rangle$, $\bar{\eta} = -\langle \frac{\delta S}{\delta c} \rangle$

STI:

II-39 $\longrightarrow$ $\mathcal{G}^{ai=b} = (D^{ab}_\nu, \; g f^{abc} c^c, \; g f^{abc} \bar{c}^c)$

$\Psi = 1$ gauge symmetry

Functional identities: $\Psi = 1$

$$\boxed{\Big(\underbrace{J^i \, \mathcal{G}^i}_{\text{generator of symmetry}} + \overset{\overset{\text{for STI}}{0}}{\overset{\|}{\frac{\delta \mathcal{G}^i}{\delta\phi^i}}} - \overset{\overset{0 \;\leftarrow\; S \text{ inv.}}{\|}}{\frac{\delta S}{\delta\phi^i} \mathcal{G}^i}\Big)_{\phi = \frac{\delta}{\delta J}} Z[J] = 0}$$

BRST - symmetry:

$$(s\,A_\nu)^a_\nu = \mathcal{D}^{ab}_\nu\, c^b \qquad = (\mathcal{D}_\nu c)^a$$

$$(s\,c)^a = -\frac{1}{2} g f^{abc} c^b c^c \qquad = -g(c^2)^a$$

$$(s\,\bar{c})^a = \frac{1}{\xi} \mathcal{F}^a[A]$$

with representation $\quad i_c = i_{\bar{c}} = a, \quad i_A = (a,\nu)$

$$s = \int d^dx\, (s\,\phi)^i \frac{\delta}{\delta \phi^i} \qquad \text{Grassmann-valued}$$

BRST-transformations s is an invariance of the gauge fixed action:

$$s\, S[\phi] = 0 \quad \Rightarrow \quad \boxed{\left\langle \int s\, J\cdot\phi \right\rangle = 0}$$

with $\quad S = S_{YM} + S_{gf} + S_{gh}$

$$S_{YM} = \frac{1}{2}\,\mathrm{tr} \int d^dx\, F_{\mu\nu} F_{\nu\mu}$$

$$S_{gf} = \frac{1}{2\xi} \int d^dx \left(\partial_\nu A^a_\nu\right)^2 \qquad \Rightarrow \text{Sea}$$

$$S_{gh} = -\int d^dx\, \bar{c}^a\, \partial_\nu \mathcal{D}^{ab}_\nu c^b$$

(i) $\boxed{s\, S_{YM} = 0}$

(ii)
$$s\, S_{gf} = -\frac{1}{\xi} \int d^d x\, A^a_\nu\, \partial_\nu\, \partial_\nu (sA)^a_\nu$$
$$= \frac{1}{\xi} \int d^d x\, \partial_\nu A^a_\nu\, \partial_\nu \mathcal{D}^{ab}_\nu c^b$$
$$= \frac{1}{\xi} \int d^d x\, \mathcal{F}^a(A)\, \partial_\nu \mathcal{D}^{ab}_\nu c^b$$

(iii)
$$s\, S_{gh} = -\int d^d x\, (s\bar{c})^a\, \partial_\nu \mathcal{D}^{ab}_\nu c^b$$
$$+ \int d^d x\, \bar{c}^a\, \partial_\nu (s\mathcal{D}_\nu)^{ab}\, c^b$$
$$+ \int d^d x\, \bar{c}^a\, \partial_\nu \mathcal{D}^{ab}_\nu (sc)^b$$

$$= -\frac{1}{\xi} \int d^d x\, \mathcal{F}^a(A)\, \partial_\nu \mathcal{D}^{ab}_\nu c^b$$
$$+ g \int d^d x\, \bar{c}^a\, \partial_\nu \left(\mathcal{D}^{de}_\nu c^e\right) f^{adb} c^b$$
$$- \frac{1}{2} g \int d^d x\, \bar{c}^a\, \partial_\nu \mathcal{D}^{ab}_\nu f^{bde} c^d c^e$$

II-41b* $\longrightarrow$
$$= -\frac{1}{\xi} \int d^d x\, \mathcal{F}^a(A)\, \partial_\nu \mathcal{D}^{ab}_\nu c^b$$

(ii)+(iii) $\boxed{s\, S_{gf} + s\, S_{gh} = 0}$

$$-\frac{1}{2}\,\partial_\nu \mathcal{D}_\nu^{ab}\, f^{bde}\, c^d c^e = -\frac{1}{2}\partial_\nu\, f^{bde}\left\{(\mathcal{D}_\nu^{ab} c^d) c^e + c^d (\mathcal{D}_\nu^{ab} c^e)\right\}$$

$f^{bde} = -f^{bed}$ $\longrightarrow$

$(\mathcal{D}_\nu c)\, c = -c(\mathcal{D}_\nu c)$

$$= -\partial_\nu f^{bde} (\mathcal{D}_\nu^{ab} c^d)\, c^e$$

$$= -\partial_\nu f^{ade} (\partial_\nu c^d)\, c^e$$

$$-\partial_\nu f^{bde}\cancel{f^{acb}} A_\nu^c c^d c^e$$

$$\partial_\nu (\mathcal{D}_\nu^{de} c^e)\, f^{adb} c^b = \partial_\nu f^{adb} (\partial_\nu c^d)\, c^b$$

$$+ \partial_\nu f^{adb} \cancel{f^{dce}} A_\nu^c c^e c^b$$

$$s S_{gh} \simeq (s\mathcal{D}_\nu) c + \mathcal{D}_\nu\, s c$$

$$= \frac{1}{2}(\mathcal{D}_\nu c) c - \mathcal{D}_\nu c^2 = 0$$

Formulation for effective action:

$$\Gamma[\Phi] = \sup_{J} \left\{ \int d^dx \, J_i \Phi_i - W[J] \right\}$$

with $J = (J_A, \bar{\eta}, \eta)$, $J_{1\,\mu}^{\ a} = J_{A\mu}^{\ a}$

$\Phi = (A, c, \bar{c})$ $J_2^{\ a} = \eta^a$

mean fields $J_3^{\ a} = \bar{\eta}^a$

$$\Rightarrow \quad \Phi_i = \frac{\delta W}{\delta J_i} \, , \quad \frac{\delta \Gamma}{\delta A} = J \, , \quad \frac{\delta \Gamma}{\delta c} = -\bar{\eta}$$

$$\frac{\delta \Gamma}{\delta \bar{c}} = -\eta$$

STI (page II-40)

$$\boxed{D_\mu^{ab} \frac{\delta \Gamma}{\delta A_\mu^b} - g f^{abc} \left(c^c \frac{\delta \Gamma}{\delta c^b} + \bar{c}^c \frac{\delta \Gamma}{\delta \bar{c}} \right) = \langle \delta^a (S_{gf} + S_{gh}) \rangle}$$

or

$$\delta^a \Gamma[\Phi] = \langle \delta^a (S_{gf} + S_{gh}) \rangle$$

(ii) Gribov problem

resolution(s):

(1) choose gauge without Gribov copies

technically difficult

(2) restrict integration domain
(restriction of # of copies)

(a) $Z[J] = \int dA\Big|_{-\frac{\delta \mathcal{F}}{\delta \omega} \geq 0} d\bar{c}\, dc\, e^{-S[\phi] + \int d^dx\, J_i \phi_i}$

first Gribov region

(b) $Z[J] = \int dA\Big|_{\substack{\frac{\delta \mathcal{F}}{\delta \omega} \geq 0 \\ \|A\|_2 \text{ min}}} d\bar{c}\, dc\, e^{-S[\phi] + \int d^dx\, J_i \phi_i}$

fundamental mod. domain

$\|A\|_2 = -\int \mathrm{tr}\, A^2 \, d^4x$

Remarks:

(i) We shall come back to (1) later; here we just state that the technically simplest gauge is Landau gauge: $\mathcal{F}[A] = \partial_\nu A_\nu$ and $\xi = 0$

(ii) (2) seems to be difficult, but we shall see that the restrictions (a), (b) inflict no further terms in FRG-eqn (as well for DSEs)

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