

1.2 Functional RG equations for Yang-Mills

Boots trap: assume

$Z_{ren}[\mathcal{J}]$ is given

$Z_{ren}[\mathcal{J}]$ is the renormalised, finite general. fct of renormalised, finite Green fcts.

$$Z_k[\mathcal{J}] := e^{-\int d^d x \underbrace{\frac{\delta}{\delta \mathcal{J}_i} R_{kij} \frac{\delta}{\delta \mathcal{J}_j}}_{\Delta S_k[\frac{\delta}{\delta \mathcal{J}}]}} Z_{ren}[\mathcal{J}]$$

YM: $\frac{\delta}{\delta \mathcal{J}_i} R_{kij} \frac{\delta}{\delta \mathcal{J}_j} e^{-\int \mathcal{J}_i \Phi_i} = \left(\frac{\delta}{\delta \mathcal{J}_i} R_{\nu\mu}^{ab} \frac{\delta}{\delta \mathcal{J}_\nu} + 2 \frac{\delta}{\delta \mathcal{J}_i} R^{ab} \frac{\delta}{\delta \mathcal{J}_\mu} \right) e^{-\int \mathcal{J}_i \Phi_i} = A_\mu^a R_{\mu\nu}^{ab} A_\nu^b + 2 \bar{c}^a R^{ab} c^c$

\uparrow p.518 • $Z_k[\mathcal{J}]$ finite, as Z_{ren} admits

by def. infinitely many derivatives;

and $e^{-\Delta S_k[\frac{\delta}{\delta \mathcal{J}}]} < 1$.

• no path integral is needed.

Flow equation for Z_k :

$$\begin{aligned}\partial_t Z_k[\mathcal{J}] &= - \left(\partial_t \Delta S_k \left[\frac{\delta}{\delta \mathcal{J}} \right] \right) Z_k[\mathcal{J}] \\ &= - \frac{1}{2} \int d^d x \partial_t R_{kij} \frac{\delta}{\delta \mathcal{J}_j} \frac{\delta}{\delta \mathcal{J}_i} Z_k[\mathcal{J}]\end{aligned}$$

Flow equation for W_k : $W_k = \ln Z_k$

$$\partial_t W_k[\mathcal{J}] = - \frac{1}{2} \int d^d x \partial_t R_{kij} \left(W_{ji}^{(2)}[\mathcal{J}] - W_i^{(1)} W_j^{(1)} \right)$$

with $W_{i_1 \dots i_n}^{(n)} = \frac{\delta}{\delta \mathcal{J}_{i_1}} \dots \frac{\delta}{\delta \mathcal{J}_{i_n}} W$

Flow equation for $\Gamma_k = \sup_{\mathcal{J}} \left[\int d^d x \mathcal{J} \cdot \phi - W_k \right] - \Delta S_k[\phi]$

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left[\frac{1}{\Gamma^{(2)} + R_k} \right]_{ij} \partial_t R_{kij}$$

Yang-Mills theories:

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$$\Phi = (A, C, \bar{C})$$

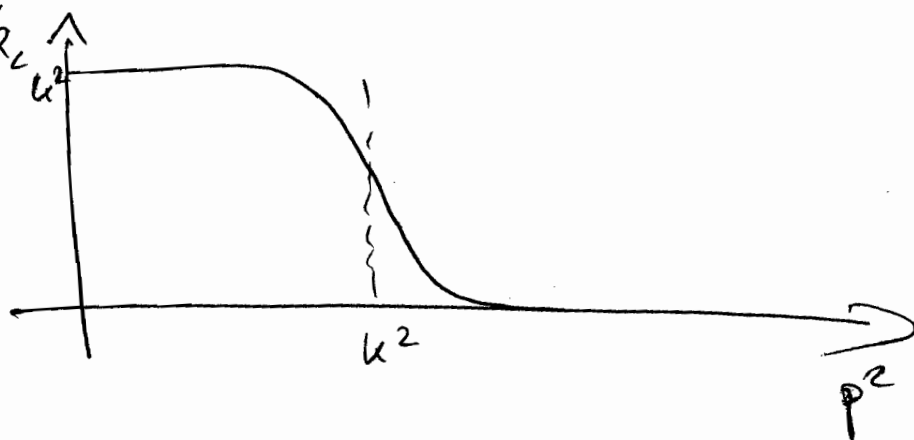
$$R_{\mu} = \begin{pmatrix} R_{A \nu}^{ab} & 0 & 0 \\ 0 & 0 & -R_C^{ab} \\ 0 & R_C^{ab} & 0 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2} \int d^d x \Phi_i R_{\mu ij} \Phi_j$$

$$= \frac{1}{2} \int d^d x A_{\nu}^a R_{A \nu r}^{ab} A_r^b$$

$$+ \int d^d x \bar{C}^a R_C^{ab} C$$

with R_A/R_C



$$R_{A \nu r}^{ab} = \nabla_{\nu r}^{ab} R_A$$

$$R_C^{ab} = \nabla_C^{ab} R_C$$

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left[\frac{1}{\frac{\delta^2 \Gamma_k}{\delta \phi^2} + R} \right] \begin{matrix} ab \\ \partial_t R_A \quad ab \\ \nu\nu \end{matrix}$$

$$- \text{Tr} \left[\frac{1}{\frac{\delta^2 \Gamma_k}{\delta \phi^2} + R} \right] \begin{matrix} ab \\ \partial_t R_c \quad ab \\ c\bar{c} \end{matrix}$$

$$= \frac{1}{2} \text{Diagram 1}$$

$$- \text{Diagram 2}$$

$$\text{---} \circ \quad \frac{1}{\frac{\delta^2 \Gamma_k}{\delta c \delta \bar{c}} + R_c}$$

$$\text{~} \circ \quad \frac{1}{\frac{\delta^2 \Gamma_k}{\delta A^2} + R_A}$$



1.3 Gauge invariance

reminder II-40 * :

$$\text{STI: } \underbrace{\left(J^i y^i \left[\frac{\delta}{\delta J} \right] + \frac{\delta y^i}{\delta \phi^i} - \frac{\delta S}{\delta \phi^i} y^i \right)}_{W_y \left[J, \frac{\delta}{\delta J} \right]} z \left[J \right] = 0 \quad \phi = \frac{\delta}{\delta J}$$

with $y = (2, g, c, g \bar{c})$

$$\Rightarrow W_y \left[J, \frac{\delta}{\delta J} \right] z \left[J \right] = W_y \left[J, \frac{\delta}{\delta J} \right] e^{\Delta S_u \left[\frac{\delta}{\delta J} \right]} e^{-\Delta S_u \left[\frac{\delta}{\delta J} \right]} z \left[J \right] = 0$$

$$= e^{\Delta S_u \left[\frac{\delta}{\delta J} \right]} \left(W_y \left[J, \frac{\delta}{\delta J} \right] - (\Delta S_u \left[\frac{\delta}{\delta J} \right] J^i) y^i \left[\frac{\delta}{\delta J} \right] \right)$$

$$\cdot z_u \left[J \right] = 0$$

$$\Rightarrow \underbrace{\left(W_y \left[J, \frac{\delta}{\delta J} \right] - R^{ji} \frac{\delta}{\delta J^i} y^j \left[\frac{\delta}{\delta J} \right] \right)}_{W_{y_u} \left[J, \frac{\delta}{\delta J} \right]} z_u \left[J \right] = 0$$

m STI , m WI

$$- R^{ij} \frac{\delta}{\delta J^i} \psi^j \left[\frac{\delta}{\delta J} \right] Z_u \left[J \right]$$

$$= - \left\langle \mathcal{D}_{uv}^{ac} R_{ik}^{cb} A_v^b \right\rangle - g \left(f^{abc} \left\langle \bar{c}^d R_c^{db} c^c \right\rangle + f^{abc} \left\langle \bar{c}^c R_c^{bd} c^d \right\rangle \right)$$

$$g \left\langle \bar{c}^b [t^a, R_c]^{bc} c^c \right\rangle$$

$$= - \left\langle \delta^a \Delta S_u \right\rangle$$

For effective action:

$$\delta^a \Gamma[\phi] = \left\langle \delta^a (S_{gf} + S_{gh}) \right\rangle$$

$$+ \underbrace{\left\langle \delta^a \Delta S_u \right\rangle}_{\text{Loop-terms}} - \delta^a \Delta S_u$$

1 R 2 loop terms in full propagators

BRST-identity : $s(A_\nu, c, \bar{c}) = (D_\nu c, \frac{1}{2} f^{abc} c^b c^c, \frac{1}{\xi} f^{abc})$

without Reg. : $\langle s \mathcal{F} \cdot \phi \rangle = 0 \iff s \mathcal{I}[\phi] = 0$

$$\int \langle s \phi \rangle \frac{\delta \mathcal{I}[\phi]}{\delta \phi} = 0$$

contains two-point functions

⇒ sources for BRST-variations

$$Z[\mathcal{J}] \rightarrow Z[\mathcal{J}, Q] = \int d\phi e^{-\mathcal{I}[\phi]} + \int d^d x Q_\phi \cdot s\phi$$

with,

$$\int d^d x Q_\phi \cdot s\phi = \int d^d x \{ K \cdot sA + L sC + \bar{L} s\bar{c} \}$$

BRST variation : $s \int d^d x Q_\phi s\phi = \int d^d x s^2 \bar{c}$

cov. gauge $\rightarrow = \int d^d x \bar{L} D_\nu sA_\nu$

with $s K \cdot sA = -K(Ds)C - K D sC = 0$ $s(Dc) = 0$

$s L sC = L \frac{s^2 C}{c^3} = 0$

$$\Rightarrow \langle s\phi \rangle = \frac{\delta W[J, Q]}{\delta Q} \quad \text{with } W[J, Q] = \ln Z[J, Q]$$

$$= - \frac{\delta \Gamma[\phi, Q]}{\delta Q}$$

BRST-identity:

$$\langle \int s J \cdot \phi + \int s Q_\phi \cdot s\phi \rangle$$

$$= \int \langle s J \cdot \phi \rangle + \frac{1}{\xi} \int \bar{L} \partial_\nu \langle s A_\nu \rangle = 0$$

Effective action:

$$\int \frac{\delta \Gamma}{\delta Q_\phi} \cdot \frac{\delta \Gamma}{\delta \phi} + \frac{1}{\xi} \int \bar{L} \partial_\nu \frac{\delta \Gamma}{\delta A_\nu} = 0$$

$$\bar{c} \rightarrow \bar{c} + f(x) : \quad \left[\partial_\nu \frac{\delta \Gamma}{\delta A_\nu} = \frac{\delta \Gamma}{\delta \bar{c}} \right]$$

BRST-identity in the presence of regulator $\Pi-53^*$

$$\int \frac{\delta \Gamma_k}{\delta Q_\phi} \cdot \frac{\delta \Gamma_k}{\delta \phi} + \frac{1}{\xi} \int \bar{L} \partial_\nu \frac{\delta \Gamma_k}{\delta u_\nu} + \underbrace{\langle s \Delta S_k \rangle - s \Delta S_k}_{\text{Loop-terms}} = 0$$

$$\frac{\delta(\Gamma_k + \Delta S_k)}{\delta Q_\phi} \frac{\delta(\Gamma_k + \Delta S_k)}{\delta \phi}$$

with $\frac{\delta \Delta S_k}{\delta Q_\phi} = 0$

Loop-terms

1-loop with full props, if using BRST-source-terms

$$\partial_\nu \frac{\delta \Gamma_k}{\delta u_\nu} = \frac{\delta \Gamma_k}{\delta \bar{c}}$$

$\langle s \Delta S_k \rangle - s \Delta S_k$: Reduction to 1-loop: $\Pi-53a^*$

$$\left\langle \varepsilon \frac{1}{2} A_\mu^a R_{\mu\nu}^{ab} A_\nu^b \right\rangle - \varepsilon \frac{1}{2} A_\mu^a R_{\mu\nu}^{ab} A_\nu^b$$

$$= \left\langle A_\mu^a R_{\mu\nu}^{ab} \varepsilon A_\nu^b \right\rangle - A_\mu^a R_{\mu\nu}^{ab} \varepsilon A_\nu^b$$

$$= \frac{\delta}{\delta K_\nu^b} \left\langle A_\mu^a R_{\mu\nu}^{ab} \right\rangle$$

$$= R_{\mu\nu}^{ab} \frac{\delta}{\delta J_\mu^a} \frac{\delta}{\delta K_\nu^b} W_u[\mathcal{J}, Q]$$

$$= - R_{\mu\nu}^{ab} \frac{\delta}{\delta J_\mu^a} \frac{\delta}{\delta K_\nu^b} \Gamma_u[\phi, Q]$$

$$= - R_{\mu\nu}^{ab} \frac{\delta \phi_i}{\delta J_\mu^a} \frac{\delta}{\delta \phi_i} \frac{\delta \Gamma_u[\phi, Q]}{\delta K_\nu^a}$$

$$= - R_{\mu\nu}^{ab} \frac{\delta^2 W_u}{\delta J_\mu^a \delta J_\nu^i} \frac{\delta^2 \Gamma_u}{\delta \phi_i \delta K_\nu^a}$$

$$= - R_{\mu\nu}^{ab} \left(\frac{1}{\Gamma^{(2)} + R} \right)_\nu^{a i} \frac{\delta^2 \Gamma_u}{\delta \phi_i \delta K_\nu^a}$$

Application of
gluonic mass term

term: $R_{\mu} = 0$; STI: $\partial_{\mu} \Gamma_{\nu}^{(2)} = \frac{1}{3} P^2 P_{\nu}$
 param.: $\Gamma_{\nu}^{(2)ab}(p^2) = \delta^{ab} p^2 \left[Z_{A_{\perp}}(p^2) (\delta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2}) + (Z_{A_{\perp}}(p^2) + \frac{1}{3}) \frac{p_{\mu} p_{\nu}}{p^2} \right]$

$$-i \frac{\delta}{\delta A_{\mu}^a(p)} \frac{\delta}{\delta C^b(q)} S \Gamma_{\mu} \Big|_{\phi=0} = P_{\nu} \Gamma_{\nu}^{(2)}(p, q)$$

$$= P_{\nu} P^2 \delta^{ab} Z_{A_{\perp}}(p^2) \delta(p-q) + \frac{1}{3} P^2 P_{\nu}$$

$$\Rightarrow -i \frac{\delta}{\delta p_{\mu}} \frac{\delta}{\delta A_{\mu}^a(p)} \frac{\delta}{\delta C^b(q)} S \Gamma_{\mu} \Big|_{\phi=0} = 4 p^2 Z_{A_{\perp}}(p^2) \Big|_{p^2=0} =: m_{u_L}^2$$

e.g. one loop: $[r_c = r_A]$ $R_{\phi} = p^2 \cdot r_{\phi}(p^2/k^2)$ label

$$m_{u_L}^2 = g^2 \frac{N}{16\pi^2} \int dy y^{d-2} \frac{r(y)}{(1+r(y))^2} \left(\frac{11}{2} - d - \frac{5}{d} + 3(1-\frac{1}{d}) + (\frac{7}{2} - \frac{6}{d}) \frac{y^2 r}{1+r(y)} \right)$$

$$\geq 0$$

Comparison with $\int m_{\mu}$ from flow

gives consistency check??

Beware:

$m_{u_L}^2 \neq p^2 Z_{A_{\perp}}(p^2) \Big|_{p^2=0}$

$$\Gamma_{\nu}^{(2)}(p^2) = p^2 \left\{ Z_{A_{\perp}}(p^2) (\delta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2}) + (Z_{A_{\perp}}(p^2) + 1) \frac{p_{\mu} p_{\nu}}{p^2} \right\}$$

Gribov - problem revisited

(1) $k \rightarrow \infty$: perturbation theory near the global minimum

$$\Gamma_k[\phi] \simeq S[\phi] + \text{rel. corrections!} \\ + 1/k\text{-corrections}$$

rel. correction: e.g. mass-term for A_μ

(2) full Flow preserves convexity

Litim, Pawłowski, Vasiliev '06

i.e.

$$\frac{\delta^2}{\delta c \delta \bar{c}} \Gamma_k + R_{cc} \geq 0$$

$$\Leftrightarrow \langle -\partial_\nu \partial_\nu \rangle \geq 0$$

o Γ_k 'stays' in first Gribov region

o no preference for specific copy

$$\Gamma_n[\Phi] = \frac{1}{Z} \int \frac{d^d p}{(2\pi)^d} A_\mu^a(p) \Gamma_{\mu\nu}^{(2)ab}(p) A_\nu^b(-p)$$

$$+ \int \frac{d^d p}{(2\pi)^d} \bar{C}^a(p) \Gamma_c^{(2)ab}(p) C(-p)$$

$$+ \sum_i c_i \frac{1}{\sqrt{\pi}} \int \frac{d^d p_j}{(2\pi)^d} \phi_{i_j}(p_j) \Gamma_{i_1 i_2 i_3}^{(3)}(p_1, \dots, p_3)$$

$$+ \sum_i c_i \frac{1}{\sqrt{\pi}} \int \frac{d^d p_j}{(2\pi)^d} \phi_{i_j}(p_j) \Gamma_{i_1 \dots i_4}^{(4)}(p_1, \dots, p_4)$$

+ o o o

Truncations: (i) $\Gamma^{(2)}(p) \Leftarrow$ full moment. dep. of propagators

(ii) approx. of $\Gamma^{(3)}$ and $\Gamma^{(4)}$

momentum dep. deduced from

$$\Gamma^{(n)}; \boxed{\Gamma^{(n>4)} \stackrel{\leq}{=} 0}$$

(iii) quarks/hadrons

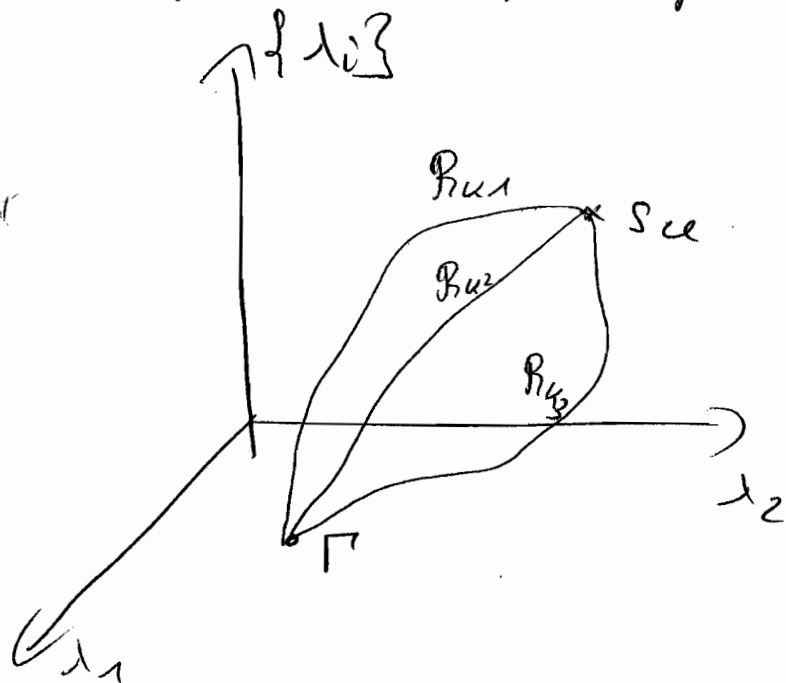
effective part.

Question: Can we accommodate
for the left-out terms?

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Are there regulators that optimise (maximise)
the physical content of a given truncation?

Theory space (space of couplings)



we shall see: optimal flow $\hat{=}$ shortest flow

Example: $O(N)$ -Theory, fixed point properties

g. critical exponents in derivative expansion

$$V = V_0 + V_1 + \dots$$

$\uparrow \quad \uparrow$
 order derivative expansion

with $v_i = v_i(R_u)$

Correlat. length $\xi \sim |T - T_c|^{-\nu}$

O(N):

$$\Gamma_u[\phi] = \int d^d x \left\{ \frac{1}{2} \vec{\phi} \overset{(-\partial^2)}{\square} \vec{\phi} + \mathcal{U}_u \left[\underbrace{\frac{1}{2} \vec{\phi} \vec{\phi}}_S \right] \right\} \quad \text{II-566}$$

$$\text{Flow: } \partial_t \mathcal{U}_u[\rho] = 2^{d+1} \pi^{d/2} \Gamma[d/2] \left((N-1) \mathcal{L}_0^d \left(\frac{u'_u(\rho)}{u^2} \right) + \mathcal{L}_0^d \left(\frac{u'_u(\rho) + 2\rho u''_u(\rho)}{u^2} \right) \right)$$

with

$$\mathcal{L}_n^d(\omega) = (\delta_{n,0+n}) \int_0^\infty dy y^{d/2-1} \frac{-y^2 r'(y)}{(y(1+r) + \omega)^{n+1}}$$

Difference of orders, eg. $v_i - v_{i+1} \sim$ differences in flow:

$$\sim \frac{1}{\Gamma_{k_i}^{(2)} + R_k} - \frac{1}{\Gamma_{k_{i+1}}^{(2)} + R_k}$$

$$= \frac{1}{\Gamma_{k_i}^{(2)} + R_k} \left(\Gamma_{k_{i+1}}^{(2)} - \Gamma_{k_i}^{(2)} \right) \frac{1}{\Gamma_{k_{i+1}}^{(2)} + R_k}$$

⇒ Every expansion is an expansion in orders of the propagator!

General series: $S = \sum a_n z^n$

Quickest convergence \triangleq maximise convergence radius

For flow:

$$R_{k_{\text{opt}}} = R_k \in \{R_k\}_{\text{compare with } k}$$

$$\Gamma^{(2)} + R_k \text{ maximal}$$

$$\Rightarrow \frac{1}{\Gamma^{(2)} + R_k} \text{ minimal}$$

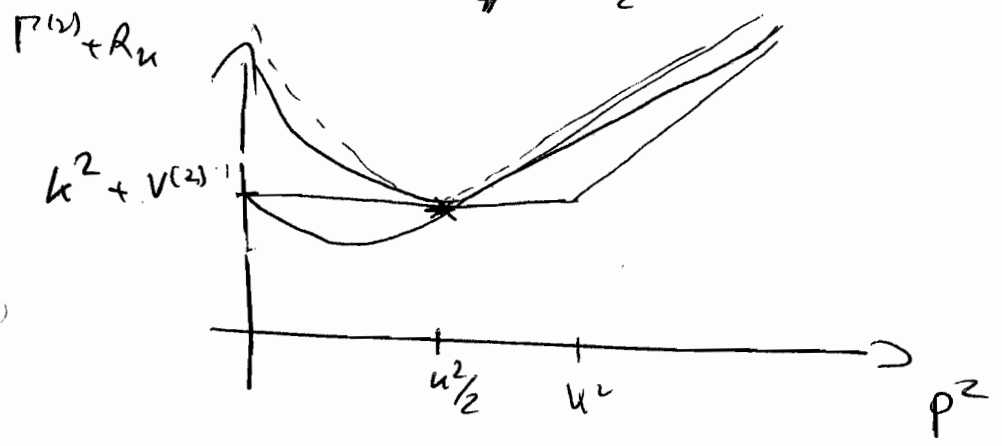
Likewise

$$\{R_k\}_{\text{compare}} = \{R_k(k_{\text{eff}}^2) = c k_{\text{eff}}^2\} \quad (c=1 \text{ standard choice})$$

Example: 0th order derivative expansion

$$\Gamma^{(2)}(p) + R_u(p^2) = \boxed{p^2 + R_u(p^2)} + V^{(2)}$$

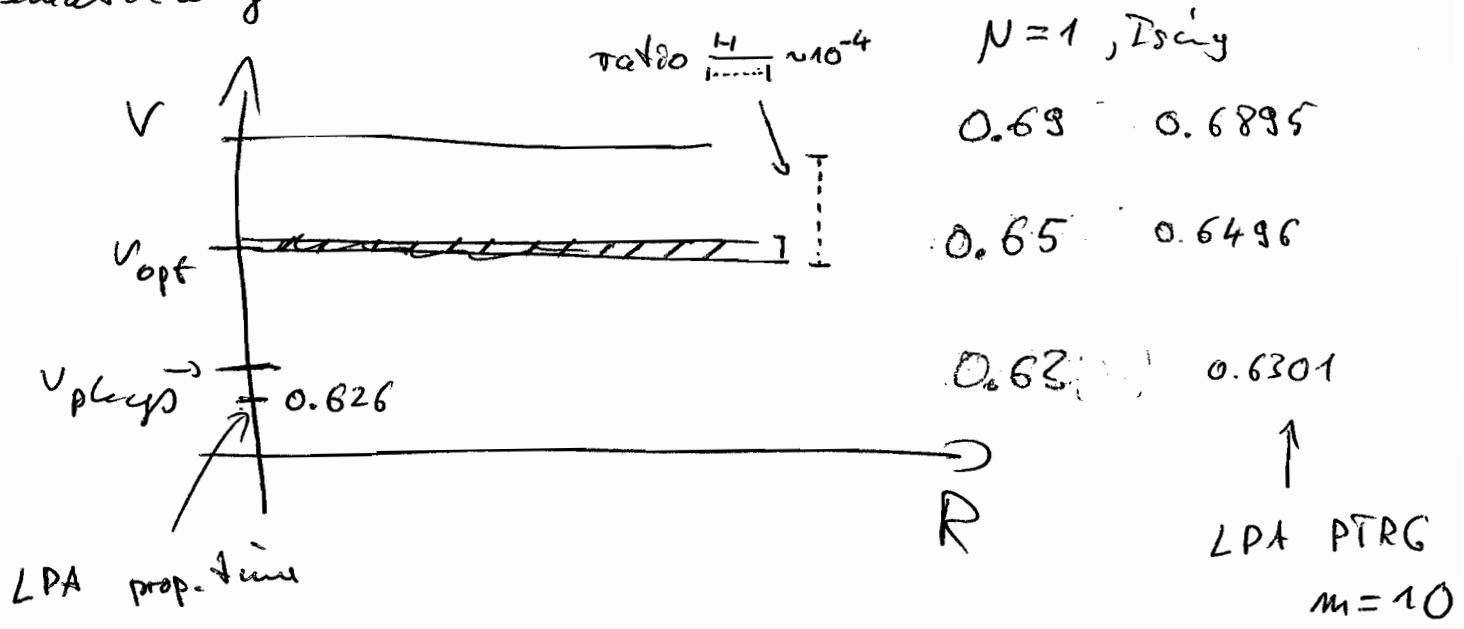
$$k_{eff}^2 = \frac{1}{2} k^2$$



⇒ optimal regulators: $p^2 + R_u(p^2) > k^2$

+ flatness: $R_{flat}(p) = (k^2 - p^2) \Theta(k^2 - p^2)$ Litim '00

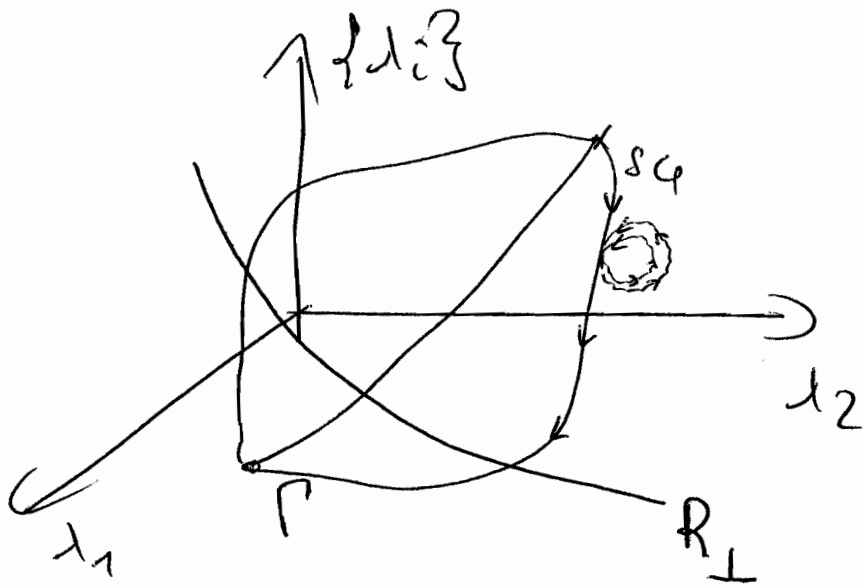
(schematically)



why is R_{flat} optimal?

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Reformulate convergence/stability idea:



• minimize the flow:

$$\frac{\delta \Gamma}{\delta \phi} = - \frac{1}{2} \underbrace{\text{Tr} \left[\frac{1}{\Gamma^{(2)}_{\perp R}} \overset{\circ}{R} \frac{1}{\Gamma^{(2)}_{\perp R}} \frac{\delta}{\delta \phi} \frac{\delta}{\delta \phi} \right]}_{\text{flow operator}} \frac{\delta \Gamma}{\delta \phi}$$

\Rightarrow minimize the flow operator

$$\boxed{\frac{\delta}{\delta R_{\perp}} \Big|_{R_{opt}} \frac{1}{\Gamma^{(2)}_{\perp R}} \overset{\circ}{R} \frac{1}{\Gamma^{(2)}_{\perp R}} = 0}$$

$\{R_{\perp}\}$: set of observables ...

$$R_{opt} \text{ minimum flow} \rightarrow \left. \frac{\partial \Gamma(\omega)}{\partial R_{\perp}} \right|_{R_{opt}} = 0$$

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$$= \left. \frac{\partial}{\partial R_{\perp}} \right|_{R_{opt}} \left. \frac{\partial}{\partial \tau} \right|_{\Gamma(\omega)} \frac{1}{\Gamma(\omega) + R} = \left. \frac{\partial}{\partial \tau} \right|_{\Gamma(\omega)} \left. \frac{\partial}{\partial R_{\perp}} \right|_{R_{opt}} \frac{1}{\Gamma(\omega) + R} = 0$$

$$\Rightarrow \boxed{\left. \frac{\partial}{\partial R_{\perp}} \right|_{R_{opt}} \frac{1}{\Gamma(\omega) + R} = 0}$$

choice of $\{R_{\perp}\}$: (i) $R(k_{eff}^2) = c k_{eff}^2$

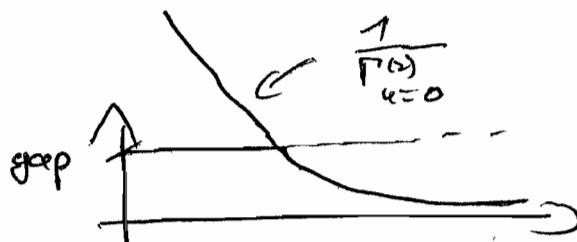
$$(ii) \max_{\phi} \frac{Z}{\Gamma(\omega) + R_{\perp}} = \max_{\phi} \frac{Z'}{\Gamma(\omega) + R_{\perp}'} = gap$$

with (ii) it follows for 0th order

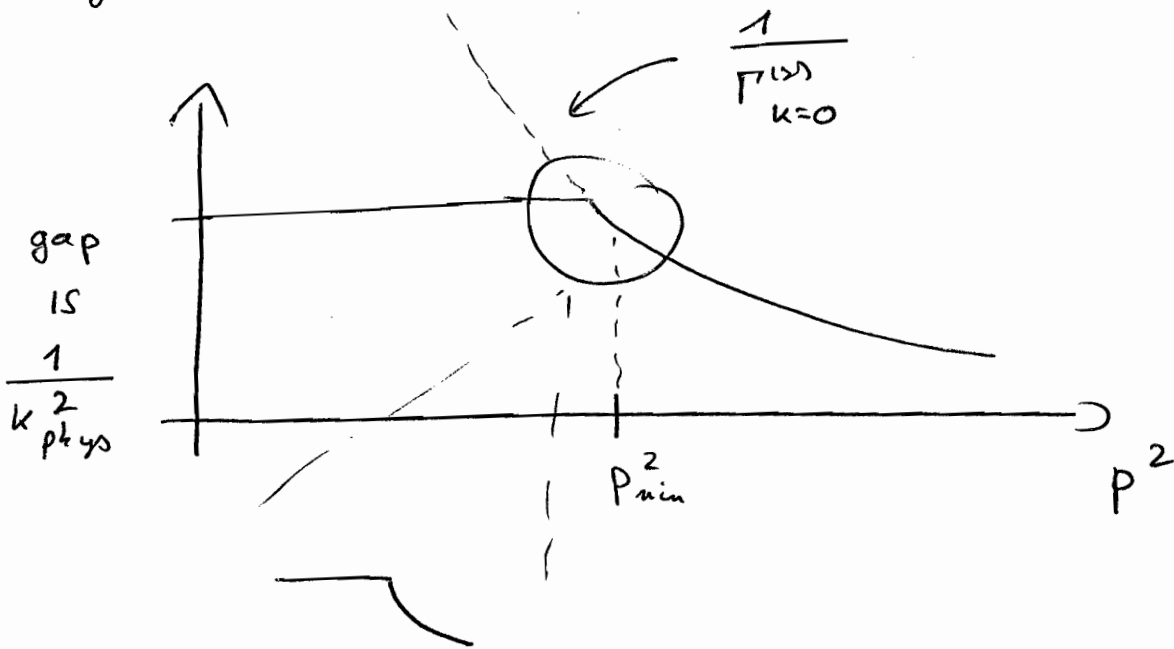
derivative expansion?

$$\left. \frac{1}{\rho^2 + R_{\perp}(\rho^2)} \right|_{R_{opt}} \text{ maximal}$$

$$\Rightarrow R_{opt} = (k^2 - \rho^2) \Theta(k^2 - \rho^2)$$



In general



$$\left\| \frac{\partial}{\partial R_1} \frac{1}{P^2 + R} \right\| = 0$$

⇒ derivative exp. : $\|\cdot\|$ sensitive to derivat.

Flow acts on \mathcal{O} with Taylor exp.