VI Quantum Gravity

1 Introduction

Einstein Equivalence Principle

'In small enough regions of space-time, the laws of physics reduce to those of special relativity'

Diffeomorphism invariance

Metric $g_{\mu\nu}$:

Levi-Civita connection Γ (Christoffel, Riemannian)

(1) torsion-free : $\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu} = 0$

(2) metric-compatible : $\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} = 0$

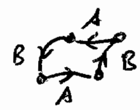
$$\boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right)}$$

Geodesic equation with geodesic parameter τ:

$$\boxed{\frac{d^2 x^\nu}{d\tau^2} + \Gamma_{\rho\sigma}{}^{\nu} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0}$$

from $S_{\text{point-part.}} = \frac{1}{2} \int ds^2 = \frac{1}{2} \int g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau$

Riemann tensor: (curvature tensor) $(1,3)$



$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma}$$

Contractions:

Ricci tensor: $(0,2)$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \qquad (= R_{\nu\mu})$$

Ricci scalar: $(0,0)$

$$R = R^{\mu}{}_{\mu}$$

Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu}$$

with

$$\boxed{\nabla^{\mu} G_{\mu\nu} = 0}$$

Bianchi identity

Einstein-Hilbert action

in general

$$S = \int \underbrace{\left[d^d x \sqrt{-g}\right]}_{\text{diff. inv. measure}} \underbrace{\mathcal{L}}_{\text{scalar}}$$

with scalar function $\mathcal{L}$ (under diffeomorph.)

Simplicity: (minimal coupling)

$$\mathcal{L} = \underbrace{\overset{\frac{1}{4g^2_{YM}}}{\frac{1}{16\pi G}}}_{\text{Newton constant}} \left(\overset{F^{a\,2}_{\mu\nu}}{R} - \underbrace{2\Lambda}_{\text{cosmological constant}}\right) + \mathcal{L}_{matter}$$

with e.g.

$$\mathcal{L}_{matter} = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi)$$

and Energy-momentum tensor

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \cdot \nabla_\rho \phi \nabla_\sigma \phi - g_{\mu\nu} V(\phi)$$

EoM: $\sqrt{-g}\,\frac{\delta S}{\delta g^{\mu\nu}} = 0$

$$\Rightarrow \quad \frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}\left[R - 2\Lambda \right] g_{\mu\nu} \right) = T_{\mu\nu}$$

(energy conditions, e.g. Carroll p. 175; nice discussion in M. Visser, Lorentzian Wormholes)

Dimensions: $[p] = 1$

$$[g_{\mu\nu}] = 0$$

$$\Rightarrow [R] = 2 \quad , \quad [\sqrt{-g}] = 0$$

Newton constant:

$$[G] = [d^d x \; R] = 2-d \quad (\sim g^2_{YM})$$

$$[\Lambda] = [R] = 2$$

$\Rightarrow$ gravitational coupling has <u>negative</u> mass-dimension

Solutions to EoM:

(1) Minkowski space: $R = 0$

(2) deSitter space

(i) maximally symmetric: $R_{\mu\nu\rho\sigma} \sim (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$

(ii) $R > 0$

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(t/\alpha)\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$$x = \alpha \cosh(t/\alpha) \sin\chi \cos\theta$$

$$y = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \cos\phi$$

$$z = \alpha \cosh(t/\alpha) \sin\chi \sin\theta \sin\phi$$

top.: $\mathbb{R} \times S^3$

(3) anti-deSitter space

(i) maximally symmetric

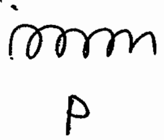
(ii) $R < 0$

top.: $\mathbb{R}^4$

Perturbative quantisation:

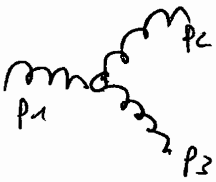
(a) Feynman rules:

graviton:

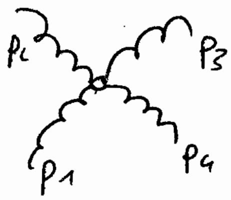


$\sim \dfrac{1}{p^2}$ $\qquad\qquad$ G/p^2

grav. vertices:



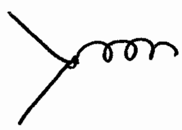
$\sim G \cdot p^2$ $\qquad\qquad$ $\frac{1}{G}\, p^2$



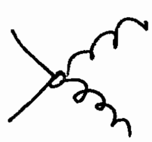
$\sim G^2 \cdot p^2$ $\qquad\qquad$ $\frac{1}{G}\, p^2$

$\vdots$

matter vertices:



$\sim G\, p^2$ $\qquad\qquad$ p^2



$\sim G^2\, p^2$ $\qquad\qquad$ p^2

$\vdots$

(1-loop) flow of g_N:

anomalous dim of g_N

$$\mu \partial_\mu g_N = \Big(d-2 + \underset{\downarrow}{\eta_N}(g_N, \lambda_c) \Big) \cdot g_N$$

$$= \beta_g (g_N, \lambda_c)$$

$$\mu \partial_\mu \lambda_c = -(2 - \eta_N)\lambda_c + \Delta(g_N, \lambda_c)\, g_N$$

$$= \beta_\lambda (g_N, \lambda_c)$$

Fixed points:

(a) Gaußian fixed point: $g_N^* = \lambda_c^* = 0$

(b) Non-Gaußian fixed point:

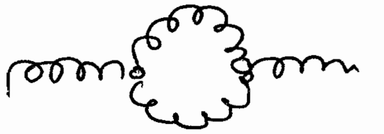
$$\beta_g (g_N^*, \lambda_c^*) = 0$$

$$\beta_\lambda (g_N^*, \lambda_c^*) = 0$$

Stability: ~~S~~tability matrix

$$B_{ij} = \frac{\partial(\beta_g, \beta_\lambda)}{\partial(g_N, \lambda_c)}$$

Vacuum polarisation: 1-loop

 $+$  $\sim c_1 G (p^2)^2 \ln p^2$

't Hooft - Veltman : $c_1 = 0$

Beyond 1-loop: all order sing. terms

$\Rightarrow$ perturbatively $\boxed{\text{non-renormalisable}}$

? non-perturbatively renormalisable ?

hints: RG - running of dimensionless Newton coupling

$$g_N = G \cdot \mu^{d-2}$$

$$\lambda_c = \Lambda / G \cdot \mu^{-d}$$

$$S_{EH} = \frac{1}{16\pi\, g_N} \int d^d \hat{x} \sqrt{-g}\, (\hat{R} - 2\lambda_c)$$

euclidean $\Rightarrow 2 \kappa^2 \int d^d x \sqrt{g} \left(-R + 2\lambda_c \right)$, $\kappa^2 = \dfrac{1}{32\pi\, g_N}$

Evidence: (see e.g. Niedermaier, Reuter, liv. reviews)
[NR]

(1) $2+\varepsilon$ - expansion

(2) pert. expansion and large N

(3) symmetry reductions: sigma models

(4) FRG - flows

Asymptotic safety scenario in QG

subtleties see NR]

basically: # of relevant couplings finite