2 RG approach to quantum gravity

Reuter '96

(i) Flow equ. requires graviton prop.

$\Rightarrow$ gauge fixing (of diffeomorphism)

$$S_{gf}[g,\bar{g}] = \frac{1}{2\alpha}\int d^dx \, \underbrace{\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}}\, F_\mu F_\nu$$

background-dep.

with $g = \bar{g} + h$

$$F_\mu(g,\bar{g}) = \sqrt{2}\, \kappa\, \mathcal{F}_\mu^{\alpha\beta}(\bar{g})\, h_{\alpha\beta}$$

$$F_\mu^{\alpha\beta}(\bar{g}) = \delta_\mu^\beta\, \bar{g}^{\alpha\gamma}\, \bar{\nabla}_\gamma + \lambda/2\, \bar{g}^{\alpha\beta}\, \bar{\nabla}_\mu$$

(gauge theories: $F_\mu = D_\mu(\bar{A})$)

where $\bar{\nabla} = \nabla(\bar{g})$. $\lambda = -1$: harmonic gauge

Gauge fixing parameters: α, λ

defines orthog. projection

(orthog. proj. <u>unique</u> in YM-theories)

Faddeev - Popov trick: $\Rightarrow \det \mathcal{M}(g,\bar{g})$

with $\mathcal{M}(g,\bar{g}) = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda(g_{\rho\nu}D_\sigma + g_{\sigma\nu}D_\rho)$

$- \bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda g_{\sigma\nu}D_\rho$

from '$\vec{\nabla}\frac{\delta \mathcal{F}}{\delta g}$'.

Ghosts C_μ:

$$S_{gh}[h, C, \bar{C}, \bar{g}] = -\sqrt{2}\int d^dx\sqrt{\bar{g}}\,\bar{C}_\mu \mathcal{M}^\mu{}_\nu C^\nu$$

gauge-fixed classical action

$$S[h, C, \bar{C}, \bar{g}] = S_{EH}[\bar{g}+h] + S_{gf}[h, \bar{g}] + S_{gh}[h, C, \bar{C}, \bar{g}]$$

- gauge inv. under backgr. diff. trafos
- modo STI's under dynam. diff. trafos

full diff. invariance $\Rightarrow$ $\boxed{\text{geometr. effective action}}$

Flow: in general $R_k = R_k(\bar{g})$

$$\Delta S_k = \frac{k^2}{2} \int d^dx \sqrt{\bar{g}}\, h_{\mu\nu} R_{k_g}[-\Delta_{\bar{g}}]^{\mu\nu\rho\sigma} h_{\rho\sigma}$$

$$+ \sqrt{2} \int d^dx \sqrt{\bar{g}}\, \bar{C}_\mu R_{k_c}[-\Delta_{\bar{g}}]\, C_\mu$$

($\Gamma[g,\bar{g}] \neq \Gamma[g] \Rightarrow$ Nielsen Ids)

Finally: $\bar{C}, C = 0$; $\Gamma^{(2)}_g = \frac{\delta^2 \Gamma}{\delta g^2}$, $\Gamma^{(2)}_c = \frac{\delta^2\Gamma}{\delta c \delta \bar{c}}$

$$\partial_t \Gamma[g,\bar{g}] = \frac{1}{2} \mathrm{Tr} \frac{1}{\frac{1}{k^2}\Gamma^{(2)}_g[g,\bar{g}] + R_{k_g}} \partial_t R_{k_g}$$

$$- \mathrm{Tr} \frac{1}{\Gamma^{(2)}_c(g,\bar{g}) + R_{k_c}} \partial_t R_{k_c}$$

EH - Ansatz: (truncation)

$$\boxed{\Gamma = S_{EH}(g\,;\, g_k, \lambda_k) + S_{gf} + S_{gh}}$$

$$\left(+ \int d^dx \sqrt{g}\, f(R) \right)$$

Results: (York transverse-traceless decomp)

Donkin 57, 58

optimized regulator: $\alpha = 0$, $d = 4$, $\lambda = -1$

Landau-DeWitt   harmonic gauge

$$\mathring{g}_k = 2 g_k + \dots$$

$$\mathring{\lambda}_k = -2 \lambda_k + \dots$$

$\Rightarrow$ III - 95 — III - 96

Stability matrix:

$$\begin{pmatrix} \frac{\partial \beta_g}{\partial g_k} & \frac{\partial \beta_g}{\partial \lambda_k} \\ \frac{\partial \beta_\lambda}{\partial g_k} & \frac{\partial \beta_\lambda}{\partial \lambda_k} \end{pmatrix} = \begin{pmatrix} -2.458 & -10.522 \\ 0.705 & -1.611 \end{pmatrix}$$

Combining these identities we obtain the following system of coupled differential equations for λ_k and g_k:

III - 95

$$\partial_t g_k = \left(d - 2 + \frac{g_k B_1(\lambda_k)}{1 - g_k B_2(\lambda_k)} \right) g_k$$
$$\partial_t \lambda_k = A_1(\lambda_k, g_k) + \frac{g_k B_1(\lambda_k) A_2(\lambda_k, g_k)}{1 - g_k B_2(\lambda_k)} \tag{5.50}$$

with:

$$A_1(\lambda_k, g_k) = -2\lambda_k + (4\pi)^{1-\frac{d}{2}} g_k \left\{ d(d-1)\, \Phi^1_{\frac{d}{2}}(-2\lambda_k) - 2d\, \Phi^1_{\frac{d}{2}}(0) \right\}$$

$$A_2(\lambda_k, g_k) = \lambda_k - (4\pi)^{1-\frac{d}{2}} g_k \left\{ \frac{1}{2} d(d-1)\, \tilde{\Phi}^1_{\frac{d}{2}}(-2\lambda_k) + d\, \tilde{\Phi}^1_{\frac{d}{2}}(0) \right\}$$

$$B_1(\lambda_k) = 4(4\pi)^{1-\frac{d}{2}} \left\{ c_1(d)\Phi^1_{\frac{d}{2}-1}(-2\lambda_k) + c_3(d)\Phi^2_{\frac{d}{2}}(-2\lambda_k) - c_2(d)\Phi^1_{\frac{d}{2}-1}(0) + \right.$$
$$\left. + \Big(c_4(d) + c_5(d) \Big) \Phi^2_{\frac{d}{2}}(0) \right\}$$

$$B_2(\lambda_k) = -2(4\pi)^{1-\frac{d}{2}} \left\{ c_1(d)\tilde{\Phi}^1_{\frac{d}{2}-1}(-2\lambda_k) + c_3(d)\tilde{\Phi}^2_{\frac{d}{2}}(-2\lambda_k) + c_2(d)\tilde{\Phi}^1_{\frac{d}{2}-1}(0) + \right.$$
$$\left. + c_4(d)\tilde{\Phi}^2_{\frac{d}{2}}(0) \right\}$$

$$c_4(d) := \lim_{\alpha \to 0} c_4(d, \alpha) = \frac{d+1}{d}$$

$\Phi^p_n \Rightarrow$ III - 95a

5.7 Phase portrait

In order to explicitly solve the above system of differential equations we need to specify the shape function $R^{(0)}(z)$. In this thesis we shall always work with the so called optimized shape function:

$$R^{(0)}(z) = (1 - z)\Theta(1 - z) \tag{5.51}$$

In practice one usually does not work directly with the Q_n. Instead one uses the so called threshold functions $\Phi^p_n(\omega)$ and $\tilde{\Phi}^p_n(\omega)$, which we shall now introduce:

III-95a

$$\boxed{\begin{aligned}\Phi^p_n(\omega) &:= \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{R^{(0)}(z) - zR^{(0)\prime}(z)}{(z + R^{(0)}(z) + \omega)^p} \quad , \quad n > 0 \qquad (5.42)\\ \tilde{\Phi}^p_n(\omega) &:= \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{R^{(0)}(z)}{(z + R^{(0)}(z) + \omega)^p} \quad , \quad n > 0 \qquad (5.43)\end{aligned}}$$

Here $\Gamma(n) = (n-1)!$ is the well-known gamma function. The threshold functions obey the following key relations:

$$Q_n\Big[(\mathcal{D}_k + c)^{-p}N\Big] = k^{2(n-p+1)}\,\Phi^p_n\left(\frac{c}{k^2}\right) - \frac{1}{2}\eta_N(k)k^{2(n-p+1)}\,\tilde{\Phi}^p_n\left(\frac{c}{k^2}\right) \tag{5.44}$$

$$Q_n\Big[(\mathcal{D}_k + c)^{-p}N_0\Big] = k^{2(n-p+1)}\,\Phi^p_n\left(\frac{c}{k^2}\right) \tag{5.45}$$

where c is some arbitrary constant and $\eta_N(k) := -\partial_t \ln Z_{N,k}$ is the anomalous dimension of the operator $\int d^dx\,\sqrt{\bar{g}}\bar{R}$. The identities (5.44) and (5.45) allow us to reexpress everything in term of Φ^p_n and $\tilde{\Phi}^p_n$:

I. Donkin, Dipl. thesis '08

III-96

Here $\Theta(1-z)$ denotes the Heaviside step function defined according to:

$$\Theta(1-z) := \begin{cases} 0 & \text{for} \quad z > 1 \\ 1 & \text{for} \quad z \leq 1 \end{cases} \tag{5.52}$$

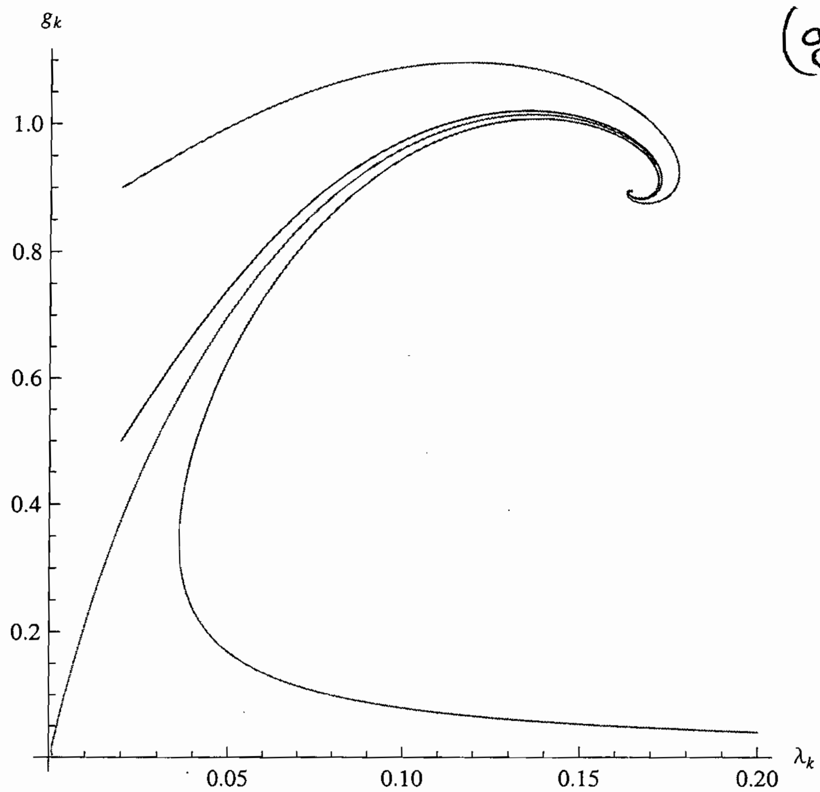
The threshold functions can be evaluated immediately:

$$\Phi^p_n(\omega) = \frac{1}{n\,\Gamma(n)}\,\frac{1}{(1+\omega)^p} \qquad\qquad \tilde{\Phi}^p_n(\omega) = \frac{1}{n(n+1)\,\Gamma(n)}\,\frac{1}{(1+\omega)^p} \tag{5.53}$$

Let us also fix the dimension of the space-time manifold at $d = 4$. After some algebraic manipulations we obtain the following system of ODEs:

$$\boxed{\begin{aligned} \partial_t g_k &= 2g_k + \frac{g_k^2\,\frac{1}{\pi}\left\{-\frac{2}{3}(1-2\lambda_k) - \frac{5}{3} - \frac{25}{24}(1-2\lambda_k)^2\right\}}{(1-2\lambda_k)^2 + g_k\,\frac{1}{2\pi}\left\{-\frac{1}{3}(1-2\lambda_k) - \frac{5}{9} + \frac{5}{12}(1-2\lambda_k)^2\right\}} \\ \partial_t \lambda_k &= \frac{g_k\,\frac{1}{\pi}\left\{-\frac{2}{3}(1-2\lambda_k) - \frac{5}{3} - \frac{25}{24}(1-2\lambda_k)^2\right\}\left\{\lambda_k - g_k\,\frac{1}{4\pi}\left\{\frac{1}{1-2\lambda_k} + \frac{2}{3}\right\}\right\}}{(1-2\lambda_k)^2 + g_k\,\frac{1}{2\pi}\left\{-\frac{1}{3}(1-2\lambda_k) - \frac{5}{9} + \frac{5}{12}(1-2\lambda_k)^2\right\}} - \\ &\quad - 2\lambda_k + g_k\,\frac{1}{4\pi}\left\{\frac{6}{1-2\lambda_k} - 4\right\} \end{aligned}}$$

In order to gain some insight into the qualitative behaviour of this dynamical system we will plot a couple of integral curves:



$(g^*, \lambda^*) = (0.89, 0.16)$

IV - 97

The above phase portrait possesses two distinct fixed points – a Gaussian fixed point:

$$(g_{Gauss}, \lambda_{Gauss}) = (0\,,\,0)$$

and a non-Gaussian fixed point:

$$(g_{nonGauss}, \lambda_{nonGauss}) = (0.893\,,\,0.164)$$

We will now discuss how it is possible to obtain important information about the RG flow by looking at its fixed point structure. First of all let us rewrite our system of coupled differential equations in terms of β-functions:

$$\partial_t g_k = \beta_g(g_k, \lambda_k)$$
$$\partial_t \lambda_k = \beta_\lambda(g_k, \lambda_k)$$

where

$$\beta_g(g_k, \lambda_k) = 2g_k + \frac{g_k^2 \frac{1}{\pi}\left\{-\frac{2}{3}(1-2\lambda_k) - \frac{5}{3} - \frac{25}{24}(1-2\lambda_k)^2\right\}}{(1-2\lambda_k)^2 + g_k \frac{1}{2\pi}\left\{-\frac{1}{3}(1-2\lambda_k) - \frac{5}{9} + \frac{5}{12}(1-2\lambda_k)^2\right\}}$$

$$\beta_\lambda(g_k, \lambda_k) = \frac{g_k \frac{1}{\pi}\left\{-\frac{2}{3}(1-2\lambda_k) - \frac{5}{3} - \frac{25}{24}(1-2\lambda_k)^2\right\}\left\{\lambda_k - g_k \frac{1}{4\pi}\left\{\frac{1}{1-2\lambda_k} + \frac{2}{3}\right\}\right\}}{(1-2\lambda_k)^2 + g_k \frac{1}{2\pi}\left\{-\frac{1}{3}(1-2\lambda_k) - \frac{5}{9} + \frac{5}{12}(1-2\lambda_k)^2\right\}} -$$
$$- 2\lambda_k + g_k \frac{1}{4\pi}\left\{\frac{6}{1-2\lambda_k} - 4\right\}$$

The fixed points are precisely those points, where all β-functions vanish. Such points are characterized by their stability properties and those properties can in turn be investigated by looking at the so called linearized flow. To make this idea precise we shall pick some arbitrary fixed point (g_*, λ_*) (i.e. (g_*, λ_*) could be either the Gaussian or the non-Gaussian fixed point) and we will linearize the flow about (g_*, λ_*):

$$\begin{pmatrix} \partial_t g_k \\ \partial_t \lambda_k \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial \beta_g(g_k,\lambda_k)}{\partial g_k} & \frac{\partial \beta_g(g_k,\lambda_k)}{\partial \lambda_k} \\ \frac{\partial \beta_\lambda(g_k,\lambda_k)}{\partial g_k} & \frac{\partial \beta_\lambda(g_k,\lambda_k)}{\partial \lambda_k} \end{pmatrix}\right|_{(g_*,\lambda_*)} \begin{pmatrix} g_k - g_* \\ \lambda_k - \lambda_* \end{pmatrix}$$

The matrix

$$\left.\begin{pmatrix} \frac{\partial \beta_g(g_k,\lambda_k)}{\partial g_k} & \frac{\partial \beta_g(g_k,\lambda_k)}{\partial \lambda_k} \\ \frac{\partial \beta_\lambda(g_k,\lambda_k)}{\partial g_k} & \frac{\partial \beta_\lambda(g_k,\lambda_k)}{\partial \lambda_k} \end{pmatrix}\right|_{(g_*,\lambda_*)} \equiv \begin{pmatrix} \left.\frac{\partial \beta_g(g_k,\lambda_k)}{\partial g_k}\right|_{(g_*,\lambda_*)} & \left.\frac{\partial \beta_g(g_k,\lambda_k)}{\partial \lambda_k}\right|_{(g_*,\lambda_*)} \\ \left.\frac{\partial \beta_\lambda(g_k,\lambda_k)}{\partial g_k}\right|_{(g_*,\lambda_*)} & \left.\frac{\partial \beta_\lambda(g_k,\lambda_k)}{\partial \lambda_k}\right|_{(g_*,\lambda_*)} \end{pmatrix}$$

is referred to as the stability matrix of the linearized system and can be used to extract information about the behaviour of the ODE around (g_*, λ_*). More precisely – it is the eigenvalues ϱ_1

and ϱ_2 of this matrix which determine the stability properties of the flow in the vicinity of the fixed point:
$\hookrightarrow$ if $\varrho_1 < 0$ and $\varrho_2 < 0$, then (g_*, λ_*) is called a stable node (informally speaking a stable node is a fixed point which attracts all nearby integral curves towards itself. This is why stable nodes are also known as attractors)
$\hookrightarrow$ if $\varrho_1 > 0$ and $\varrho_2 > 0$, then (g_*, λ_*) is called an unstable node (i.e. (g_*, λ_*) is not attracting and also not stable in the sense that integral curves, which pass near (g_*, λ_*), do not necessarily remain in the vicinity of that point). Unstable nodes are also known as repellors
$\hookrightarrow$ if $\varrho_1 > 0$ and $\varrho_2 < 0$, then (g_*, λ_*) is called a hyperbolic fixed point (or a saddle point)
$\hookrightarrow$ if $\varrho_{1,2} = -a \pm i\, b$ (with $a, b > 0$), then (g_*, λ_*) is called a stable spiral point

There are of course many other possible cases but only those four will be relevant to our discussion. Let us now go ahead and compute the stability matrix at the Gaussian fixed point. We obtain:

$$\left.\begin{pmatrix} \frac{\partial\beta_g(g_k,\lambda_k)}{\partial g_k} & \frac{\partial\beta_g(g_k,\lambda_k)}{\partial \lambda_k} \\ \frac{\partial\beta_\lambda(g_k,\lambda_k)}{\partial g_k} & \frac{\partial\beta_\lambda(g_k,\lambda_k)}{\partial \lambda_k} \end{pmatrix}\right|_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0.159 & -2 \end{pmatrix}$$

The respective eigenvalues are $\varrho_1 = 2$ and $\varrho_2 = -2$. Hence the Gaussian fixed point is a saddle point. Since $\varrho_2 < 0$, the ϱ_2-eigendirection is attractive for $k \to \infty$ (meaning that for $k \to \infty$ the integral curves are attracted towards the fixed point along this direction). Conversely, since $\varrho_1 > 0$, the ϱ_1-eigendirection is repulsive for $k \to \infty$.

Moving on to the non-Gaussian fixpoint, we obtain the following expression for the stability matrix:

$$\left.\begin{pmatrix} \frac{\partial\beta_g(g_k,\lambda_k)}{\partial g_k} & \frac{\partial\beta_g(g_k,\lambda_k)}{\partial \lambda_k} \\ \frac{\partial\beta_\lambda(g_k,\lambda_k)}{\partial g_k} & \frac{\partial\beta_\lambda(g_k,\lambda_k)}{\partial \lambda_k} \end{pmatrix}\right|_{(0.893\,,\,0.164)} = \begin{pmatrix} -2.458 & -10.522 \\ 0.705 & -1.611 \end{pmatrix}$$

The respective eigenvalues are $\varrho_1 = -2.034 + 2.691\, i$ and $\varrho_2 = -2.034 - 2.691\, i$. Hence the non-Gaussian fixed point is a stable spiral point. The behaviour of g_k and λ_k near the fixed point is described by the real parts of the eigenvalues (the imaginary parts do not influence the stability of the fixed point). Since $Re\varrho_1 = Re\varrho_2 < 0$, the non-Gaussian fixed point is an attractor. Therefore the RG flow contains all the ingredients which are necessary for the so called asymptotic safety scenario and the nonperturbative renormalizability of 4-dimensional quantum gravity.

5.8 Extracting the unphysical degrees of freedom

As we already indicated in section 5.5 the flow equation (5.50) contains unphysical degrees of freedom due to the α-dependence of the cutoff operators. In this section we will improve our analysis by eliminating those unphysical degrees of freedom. The most proper way to do this is to go all the way back to section 5.5, where the α-dependent cutoff operators were introduced, and replace them with α-independent ones. In order to accomplish this we will first have to take

Issues and applications

(1) Stability

- better truncations, e.g. $f(R)$
- matter

(2) Diffeomorphism invariance

- check of α, λ-dependence

  minimal with optimised regulator

- Diffeomorphism-invariant flow

(3) Cosmology and particle physics

- asymptotic space-time : de Sitter
- back reaction of UV-running
- extra-dimensions, relation to string theory

(4) IR-gravity

- ¡largely unsolved!